Example Sheet 1

Values of some physical constants are given in the supplementary notes

1. When a sample of potassium is illuminated with light of wavelength $3 \times 10^{-7} \text{m}$, electrons are emitted with kinetic energy $2.1 \text{ eV}$. When the same sample is illuminated with light of wavelength $5 \times 10^{-7} \text{m}$ it emits electrons with kinetic energy $0.5 \text{ eV}$. Use Einstein’s explanation of this photoelectric effect to obtain a value for Planck’s constant, and find the minimum energy $W$ needed to free an electron from the surface of potassium.

2. The light from a faint star has energy flux $10^{-10} \text{Jm}^{-2}\text{s}^{-1}$. If the wavelength of the light is $5 \times 10^{-7} \text{m}$, estimate the number of photons from this star that enter a human eye in one second.

3. Consider the Bohr model of the Hydrogen atom, taking the electron to be a non-relativistic point particle of mass $m$ travelling with speed $v$ in a circular orbit of radius $r$ around a point-like proton. The inward acceleration $v^2/r$ must be provided by the Coulomb attraction $e^2/4\pi\varepsilon_0 r^2$, and the angular momentum is assumed to be quantised: $L = mvr = nh$, with $n = 1, 2, 3, \ldots$. Show that the possible energy levels for the electron are

$$E = -\frac{1}{2} \frac{mc^2\alpha^2}{n^2}$$

where $\alpha$ is the fine structure constant.

(i) Is the result for $v$ consistent with the assumption that the motion of the electron is non-relativistic?

(ii) Suppose that the electron makes a transition from one energy level to another, emitting a photon in the process. What is the smallest possible wavelength for the emitted photon, and how does this compare to the Bohr radius $r_1$ (corresponding to $n = 1$)?

4. A muon is a particle with the same charge as an electron, but with a mass about 207 times larger. A muon can be captured by a proton to form an atom of muonic Hydrogen. How does the radius of the $n = 1$ state orbit compare to that of ordinary Hydrogen?

5. The time-independent Schrödinger Equation for a one-dimensional harmonic oscillator is

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} + \frac{1}{2} K x^2 \psi = E \psi$$

where $m$ is the mass and the constant $K$ gives the strength of the restoring force. Verify that there are energy eigenfunctions of the form

$$\psi_0(x) = C_0 e^{-x^2/2\alpha}, \quad \psi_1(x) = C_1 x e^{-x^2/2\alpha}$$

for a certain value of $\alpha$, to be determined, and find the corresponding energy eigenvalues $E_0$ and $E_1$. Sketch $\psi_0$, $\psi_1$, and $V(x)$. ($C_0$ and $C_1$ are normalisation constants that you need not determine.)

6. For the harmonic oscillator in Example 5, write down the time-dependent Schrödinger Equation satisfied by $\Psi(x,t)$ and give the solution for each of the following initial conditions:

(i) $\Psi(x,0) = \psi_0(x)$, \hspace{1cm} (ii) $\Psi(x,0) = \psi_1(x)$, \hspace{1cm} (iii) $\Psi(x,0) = \frac{1}{2} (\sqrt{3} \psi_0(x) - i \psi_1(x))$.

For the solution in case (iii), what is the first time $T > 0$ at which $\Psi(x,T)$ and $\Psi(x,0)$ correspond to physically equivalent states?
7. A particle of mass $m$ moves freely in one dimension ($V = 0$). Consider the wavefunction

$$
\Psi(x, t) = C \gamma(t)^{-1/2} \exp\left( -x^2/2\gamma(t) \right),
$$

where $\gamma(t)$ is complex-valued and $C$ is a constant. By substituting into the time-dependent Schrödinger equation, find a necessary and sufficient condition on $\gamma(t)$ for $\Psi(x, t)$ to be a solution. Hence determine $\gamma(t)$ if $\gamma(0) = \alpha$, a real positive constant.

Write down and simplify an expression for the probability density for the particle at time $t$ and find a value for the constant $C$ such that $\Psi(x, t)$ is normalised. Comment briefly on the behaviour of the probability density as $t$ increases.

8. Consider a particle in one dimension in a potential $V(x)$ that tends to zero rapidly as $x \to \pm \infty$.

(i) Let $\psi_1(x)$ and $\psi_2(x)$ be normalisable energy eigenfunctions of the Hamiltonian with the same energy eigenvalue $E$. By considering $\psi_1 \psi_2' - \psi_2 \psi_1'$, show that $\psi_1$ and $\psi_2$ must be proportional to one another. What does this mean, physically?

(ii) Can the wavefunction for a normalised energy eigenstate always be chosen to be real?

(iii) Show that if $\psi(x)$ is any normalised energy eigenstate then $\langle \hat{p} \rangle \psi = 0$.

9. Write down the time-independent Schrödinger equation for the wavefunction of a particle moving in a potential $V(x) = -U \delta(x)$, where $U$ is a positive constant and $\delta(x)$ is the Dirac delta function. Integrate the equation over the interval $-\epsilon < x < \epsilon$, for a positive constant $\epsilon$, and hence deduce that there is a discontinuity at $x = 0$ in the derivative of $\psi(x)$:

$$
\lim_{\epsilon \to 0} [\psi'(\epsilon) - \psi'(-\epsilon)] = -\frac{2mU}{\hbar^2} \psi(0).
$$

By using this condition to relate appropriate solutions for $x > 0$ and $x < 0$, find the unique normalisable eigenstate of the Hamiltonian, and determine its energy eigenvalue.

10. Consider a square well potential with $V(x) = -U$ for $|x| < a$ and $V(x) = 0$ otherwise ($U$ is a positive constant). Show that there are no bound states (normalisable energy eigenfunctions) which satisfy $\psi(-x) = -\psi(x)$ (i.e. which have odd parity) if $a^2U < \left(\pi\hbar\right)^2/8m$.

11. Sketch the potential

$$
V(x) = -\frac{\hbar^2}{m} \text{sech}^2 x.
$$

Show that the time-independent Schrödinger equation for a particle in this potential can be written

$$
A^\dagger A \psi = (\mathcal{E} + 1) \psi
$$

where $\mathcal{E} = 2mE/\hbar^2$ and

$$
A = \frac{d}{dx} + \tanh x, \quad A^\dagger = -\frac{d}{dx} + \tanh x.
$$

Show, by integrating by parts, that for any normalised wavefunction $\psi$,

$$
\int_{-\infty}^{\infty} \psi^* A^\dagger A \psi \ dx = \int_{-\infty}^{\infty} (A\psi)^* (A\psi) \ dx
$$

and deduce that the eigenvalues of $A^\dagger A$ are non-negative. Hence show that the ground state (with lowest energy) has $\mathcal{E} \geq -1$. Show that a wavefunction $\psi_0(x)$ is an energy eigenstate with $\mathcal{E} = -1$ iff

$$
\frac{d\psi_0}{dx} + \tanh x \psi_0 = 0.
$$

Find and sketch $\psi_0(x)$.

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