Mathematical Tripos Part 1B
Michaelmas Term 2011

Dr. Eugene. A. Lim
eal40@damtp.cam.ac.uk

QUANTUM MECHANICS

Example Sheet 1

Speed of light: $c = 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Planck's constant: $\hbar = 1.055 \times 10^{-34}\,\mathrm{J\,s}$ $(h = 2\pi\hbar = 6.63 \times 10^{-34}\,\mathrm{J\,s})$
Fine-structure constant: $\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}$
Mass of electron: $m_e = 9.11 \times 10^{-31}$ kg
Mass of proton: $m_p = 1.67 \times 10^{-27}$ kg
Electron volt: 1 eV $= 1.60 \times 10^{-19}$ J
Bohr radius: $r_0 = 0.529 \times 10^{-10}$ m

1. When the surface of a sample of potassium is illuminated with light of wavelength 3×10^{-7} m it emits electrons with kinetic energy 2.1 eV. When the same sample is illuminated with light of wavelength 5×10^{-7} m it emits electrons with kinetic energy 0.5 eV. Use Einstein's explanation of this 'photoelectric' effect to obtain a value for Planck's constant $\hbar$, and find the minimum energy E_0 needed to free an electron from the surface of potassium.

2. The light from a faint star has energy flux 10^{-10} J m^{-2} s^{-1}. Assuming that the light has a wavelength of approximately 5×10^{-7} m, estimate the number of photons from this star that enter a human eye in one second.

3. Bohr's 1913 model of the hydrogen atom assumes that electrons are non-relativistic classical particles travelling with speed v in circular orbits of radius r around a point-like proton, but with an angular momentum that must be an integral multiple n of $\hbar$. By equating the attractive inverse square force $e^2/4\pi\epsilon_0 r^2$ with the repulsive centrifugal force $m_e v^2/r$, show that this model implies that the electron energy is

$$E_n = -m_e c^2 \alpha^2 / 2n^2 \,,$$

where α is the 'fine-structure' constant. An electron in the $n = 1$ 'ground state' has $r = r_0$, where r_0 is the Bohr radius; show that $r_0 = \lambda_C/\alpha$, where $\lambda_C = \hbar/m_e c$ is the electron's Compton wavelength. Show that the speed of the electron in the ground state is $v = \alpha c$, thereby justifying the neglect of relativistic effects.

When an electron in an excited $(n > 1)$ state decays to a lower energy state it emits a photon. What is the maximum energy that such a photon can have? What is its minimum wavelength, $\lambda_{\min}$. Show that $\lambda_{\min} \gg r_0$.

4. Consider a double-slit experiment where a single electron passes through the apparatus and is detected on a vertical screen of width $2L$ parallel to the y-axis. The slits are located at $x = 0$, $y = \pm a$ (with $a < L$) and the screen lies at $x = l$, extending between $y = -L$ and $y = +L$. An approximate form (valid for $L << l$) for the normalised electron wavefunction, $\psi(x, y, t)$, evaluated at the screen $x = l$ for $|y| \leq L$, is given as follows. When only the slit at $y = a$ is open, the wavefunction is $\psi(l, y, t) = A \exp(ikay/l) \exp(iky^2/2l) \exp(i\omega t)$. When only the other slit at $y = -a$ is open the corresponding result is $\psi(l, y, t) = A \exp(-ikay/l) \exp(iky^2/2l) \exp(i\omega t)$. Here A is an undetermined normalisation constant which is the same in both cases. The wave function vanishes identically for $|y| > L$.

Now assume both slits are open. What is the new wavefunction? Determine the probability distribution for the y-coordinate of an electron detected at the screen. How would the distribution change if we use an additional detector next to the slits to determine which slit the electron passes through?

5. Let $\psi_i(x)$, $i = 1, 2$, be two normalized stationary state wavefunctions. Assume that they are orthogonal, so that

$$\int_{-\infty}^{\infty} \psi_1^*(x)\psi_2(x)\, dx = 0\,.$$

Show that the linear superposition $\alpha\psi_1 + \beta\psi_2$, for complex constants α and β is normalized if and only if $|\alpha|^2 + |\beta|^2 = 1$. Suppose now that ψ_1 and ψ_2 are normalized but not orthogonal. Show that there is a unique constant γ, with $|\gamma| \leq 1$, such that $\psi = \psi_1 - \gamma\psi_2$ is orthogonal to ψ_2. Given that $|\gamma| < 1$ show that $\psi/\sqrt{1 - |\gamma|^2}$ is normalized.

6. A particle with $m = \hbar$, moving freely in one dimension has wavefunction

$$\psi(x,t) = \frac{1}{\pi^{\frac{1}{4}}\,(1+it)^{\frac{1}{2}}}\, \exp\left(\frac{-x^2}{2(1+it)}\right)\,.$$

Verify that this wavefunction is normalized. Compute the probability density and probability current and verify that they are compatible with conservation of probability.

Consider the probability of finding the particle in an arbitrary finite interval $a \leq x \leq b$. Show that this probability vanishes in the limit $t \to \infty$ (with a and b held fixed).

7. Show that the stationary state wavefunctions of a particle in a potential $V(x)$ with $V(-x) = V(x)$ either have definite parity or can be chosen to have definite parity. Discuss the odd-parity bound states in the one-dimensional square well with potential $V = 0$ for $|x| > a$, $V = -U$ otherwise, where U is a positive constant. Use a graphical method to show that there is no odd-parity bound state if $2mU < (\hbar\pi/2a)^2$.

8. Sketch the potential

$$V = -\frac{\hbar^2}{m}\,\mathrm{sech}^2\, x$$

and show that the time-independent Schrödinger equation for a particle in this potential can be written as

$$A^\dagger A\, \psi = (\varepsilon + 1)\psi$$

where $\varepsilon = 2mE/\hbar^2$ and

$$A = \frac{d}{dx} + \tanh x\,, \qquad A^\dagger = -\frac{d}{dx} + \tanh x\,.$$

Show, by integrating by parts, that for any normalized wavefunction ψ,

$$\int_{-\infty}^{\infty} \psi^* A^\dagger A\psi\, dx = \int_{-\infty}^{\infty} (A\psi)^*(A\psi)\, dx$$

and hence that the eigenvalues of $A^\dagger A$ are non-negative. Hence deduce that the ground state wavefunction must have $\varepsilon \geq -1$. Show that there is a wavefunction $\psi_0(x)$ with $\varepsilon = -1$, satisfying

$$\frac{d\psi_0}{dx} + \tanh x\, \psi_0 = 0\,.$$

Find and sketch $\psi_0(x)$.

9. Write down the time-independent Schrödinger equation for the wavefunction ψ of a particle moving in a potential $V = -U\delta(x)$ for positive constant U (and $\delta(x)$ the Dirac delta function). Integrate the equation over the interval $-\epsilon < x < \epsilon$, for arbitrary positive constant ϵ, and hence show that there is a discontinuity at $x = 0$ in the derivative of $\psi(x)$:

$$\lim_{\epsilon \to 0} \left[\psi'(\epsilon) - \psi'(-\epsilon)\right] = -\frac{2mU}{\hbar^2}\psi(0)\,.$$

Show that there is a unique bound state ($E < 0$) solution $\psi_0(x)$. Find this ground state solution, and its energy.

10. Find the energy eigenfunctions $\psi_E(x)$ for a free particle of mass m subject to the periodic boundary condition

$$\psi_E(x) = \psi_E(x+L)\,.$$

What are the allowed values of E? What are the degeneracies of the energy levels? Recover the spectrum of a free particle on an infinite line by taking an appropriate limit.