Mathematical Tripos Part IB **Michaelmas Term 2024**

Quantum Mechanics **Prof. Frank Verstraete**

Example Sheet 1

Values of some physical constants are given in the supplementary notes

1. When a sample of potassium is illuminated with light of wavelength 3×10^{-7}m, electrons are emitted with kinetic energy 2.1 eV. When the same sample is illuminated with light of wavelength 5×10^{-7}m it emits electrons with kinetic energy 0.5 eV. Use Einstein's explanation of this *photoelectric effect* to obtain a value for Planck's constant, and find the minimum energy W needed to free an electron from the surface of potassium.

2. The light from a faint star has energy flux $10^{-10}\,\mathrm{Joule\,m^{-2}\,s^{-1}}$. If the wavelength of the light is 5×10^{-7}m, estimate the number of photons from this star that enter a human eye in one second.

3. Consider the Bohr model of the Hydrogen atom, taking the electron to be a non-relativistic point particle of mass m travelling with speed v in a circular orbit of radius r around a point-like proton. The inward acceleration v^2/r must be provided by the Coulomb attraction $e^2/4\pi\epsilon_0 r^2$, and the angular momentum is assumed to be quantised: $L = mvr = n\hbar$, with $n = 1, 2, 3, \ldots$. Show that the possible energy levels for the electron are

$$E = -\frac{1}{2}mc^2\alpha^2\frac{1}{n^2}$$

where α is the fine structure constant.

(i) Is the result for v consistent with the assumption that the motion of the electron is non-relativistic?

(ii) Suppose that the electron makes a transition from one energy level to another, emitting a photon in the process. What is the smallest possible wavelength for the emitted photon, and how does this compare to the Bohr radius r_1 (corresponding to $n = 1$)?

4. A muon is a particle with the same charge as an electron, but with a mass about 207 times larger. A muon can be captured by a proton to form an atom of muonic hydrogen. How does the radius of the $n = 1$ state orbit compare to that of ordinary hydrogen?

Nuclei of atoms of hydrogen and its isotopes can fuse, releasing energy and creating hydrogen or helium isotope nuclei. The fusion rate is very sensitive to the distance between nuclei, increasing sharply as they get closer. What are the implications for muonic hydrogen fusion?

If you are interested, you might wish to look into the possibilities and difficulties of so-called muon-catalysed fusion.

5. Double slit experiments with electrons have been performed. Explain why neither a wave model nor a particle model of the electron adequately accounts for the results. Try to think of other simple models that might. [*Warning: theorists have been trying since 1926, and every idea to date has problems. The aim of this exercise is for you to appreciate the problems, not to solve them.*]

6. A quantum system has Hamiltonian $\hat{H}$ with normalised eigenstates χ_n and corresponding energies E_n $(n = 1, 2, 3, \ldots)$. A linear operator $\hat{Q}$ associated with the quantity Q is defined by its action on these states:

$$\hat{Q}\chi_1 = \chi_2\ , \qquad \hat{Q}\chi_2 = \chi_1\ , \qquad \hat{Q}\chi_n = 0 \quad n > 2\ .$$

Show that $\hat{Q}$ has eigenvalues ±1 (in addition to zero) and find the corresponding normalised eigenstates $\chi_\pm$, in terms of energy eigenstates. Calculate $\langle\hat{H}\rangle$ in each of the states $\chi_\pm$.

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A measurement of Q is made at time zero, and the result $+1$ is obtained. The system is then left undisturbed for a time t, at which instant another measurement of Q is made. What is the probability that the result will again be $+1$? Show that the probability is zero if the measurement is made when a time $T = \pi\hbar/(E_2 - E_1)$ has elapsed (assume $E_2 - E_1 > 0$).

7. In the previous example, suppose that an experimenter makes n successive measurements of Q at regular time intervals T/n. If the result $+1$ is obtained for one measurement, show that the amplitude for the next measurement to give $+1$ is

$$A_n = 1 - \frac{iT(E_1+E_2)}{2\hbar n} + \mathcal{O}\Big(\frac{1}{n^2}\Big) .$$

The probability that all n measurements give the result $+1$ is then $P_n = (|A_n|^2)^n$. Show that

$$\lim_{n\to\infty} P_n = 1 .$$

Interpreting $\chi_\pm$ as the 'not-boiling' and 'boiling' states of a two-state 'quantum pot', this shows that a watched quantum pot never boils (also called the Quantum Zeno Paradox).

8. The time-independent Schrödinger equation for a one-dimensional harmonic oscillator is

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\chi}{\mathrm{d}x^2} + \frac{1}{2}Kx^2\chi = E\chi$$

where m is the mass and the constant K gives the strength of the restoring force. Verify that there are energy eigenfunctions of the form

$$\chi_0(x) = C_0\, e^{-x^2/2\alpha} , \qquad \chi_1(x) = C_1\, x e^{-x^2/2\alpha}$$

for a certain value of α, to be determined, and find the corresponding energy eigenvalues E_0 and E_1. Sketch χ_0, χ_1, and $U(x)$. (C_0 and C_1 are normalisation constants that you need not determine.)

9. For the harmonic oscillator in Example **8**, write down the time-dependent Schrödinger equation satisfied by $\psi(x,t)$ and give the solution for each of the following initial conditions:

(i) $\psi(x,0) = \chi_0(x)$, (ii) $\psi(x,0) = \chi_1(x)$, (iii) $\psi(x,0) = \frac{1}{2}(\sqrt{3}\,\chi_0(x) - i\,\chi_1(x))$.

For the solution in case (iii), what is the first time $T > 0$ at which $\psi(x,T)$ and $\psi(x,0)$ correspond to physically equivalent states?
[*Hint: Remember that $\psi(x,t) = \sum_n c_n\chi_n(x)\, e^{-iE_nt/\hbar}$ and that therefore $\psi(x,0) = \sum_n c_n\chi_n(x)$.*]

10. A particle of mass m moves freely in one dimension ($U = 0$). Consider the wavefunction

$$\psi(x,t) = C\,\gamma(t)^{-1/2}\, \exp(-x^2/2\gamma(t)) ,$$

where $\gamma(t)$ is complex-valued and C is a constant. By substituting into the time-dependent Schrödinger equation, find a necessary and sufficient condition on $\gamma(t)$ for $\psi(x,t)$ to be a solution. Hence determine $\gamma(t)$ if $\gamma(0) = \alpha$, a real positive constant.

Write down and simplify an expression for the probability density for the particle at time t and find a value for the constant C such that $\psi(x,t)$ is normalised. Comment briefly on the behaviour of the probability density as t increases.

11. Consider a particle in one dimension in a potential $U(x)$ that tends to zero rapidly as $x \to \pm\infty$.

(i) Let $\chi_1(x)$ and $\chi_2(x)$ be normalisable energy eigenfunctions of the Hamiltonian with the same energy eigenvalue E. By considering $\chi_1\,\chi'_2 - \chi_2\,\chi'_1$, show that χ_1 and χ_2 must be proportional to one another. What does this mean, physically?

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(ii) Can the wavefunction for a normalised energy eigenstate always be chosen to be real?

(iii) Show that if $\chi(x)$ is any normalised energy eigenstate then $\langle \hat{p} \rangle_\chi = 0$.

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