Mathematical Tripos Part 1B — Dr A.P.A.Kent
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QUANTUM MECHANICS

Example Sheet 1

Speed of light: $c = 3.00 \times 10^8\,\mathrm{m\,s^{-1}}$
Planck's constant: $\hbar = 1.055 \times 10^{-34}\,\mathrm{J\,s}$ $\quad (h = 2\pi\hbar = 6.63 \times 10^{-34}\,\mathrm{J\,s})$
Fine-structure constant: $\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}$
Mass of electron: $m_e = 9.11 \times 10^{-31}$ kg
Mass of proton: $m_p = 1.67 \times 10^{-27}$ kg
Electron volt: 1 eV $= 1.60 \times 10^{-19}$ J
Bohr radius: $r_0 = 0.529 \times 10^{-10}$ m

1. When the surface of a sample of potassium is illuminated with light of wavelength 3×10^{-7} m it emits electrons with kinetic energy 2.1 eV. When the same sample is illuminated with light of wavelength 5×10^{-7} m it emits electrons with kinetic energy 0.5 eV. Use Einstein's explanation of this 'photoelectric' effect to obtain a value for Planck's constant $\hbar$, and find the minimum energy E_0 needed to free an electron from the surface of potassium.

2. The light from a faint star has energy flux 10^{-10} J m^{-2} s^{-1}. Assuming that the light has a wavelength of approximately 5×10^{-7} m, estimate the number of photons from this star that enter a human eye in one second.

3. Bohr's 1913 model of the hydrogen atom assumes that electrons are non-relativistic classical particles travelling with speed v in circular orbits of radius r around a point-like proton, but with an angular momentum that must be an integral multiple n of $\hbar$. By equating the attractive inverse square force $e^2/4\pi\epsilon_0 r^2$ with the repulsive centrifugal force $m_e v^2/r$, show that this model implies that the electron energy is

$$E_n = -m_e c^2 \alpha^2 / 2n^2 \,,$$

where α is the 'fine-structure' constant. An electron in the $n = 1$ 'ground state' has $r = r_0$, where r_0 is the Bohr radius; show that $r_0 = \lambda_{\rm C}/\alpha$, where $\lambda_{\rm C} = \hbar/m_e c$ is the electron's Compton wavelength. Show that the speed of the electron in the ground state is $v = \alpha c$, thereby justifying the neglect of relativistic effects.

When an electron in an excited $(n > 1)$ state decays to a lower energy state it emits a photon. What is the maximum energy that such a photon can have? What is its minimum wavelength, $\lambda_{\rm min}$? Show that $\lambda_{\rm min} \gg r_0$.

4. The muon μ^- has the same charge as an electron but a mass $m_\mu = 207 m_e$. It can be captured by a free proton to form an atom of *muonic hydrogen* (also called *mesic hydrogen*). Find the radius of the first Bohr orbit of such an atom.

5. Let $\psi_i(x)$, $i = 1, 2$, be two normalized stationary state wavefunctions. Assume that they are orthogonal, so that

$$\int_{-\infty}^{\infty} \psi_1^*(x)\psi_2(x)\,dx = 0\,.$$

Show that the linear superposition $\alpha\psi_1 + \beta\psi_2$, for complex constants α and β is normalized if and only if $|\alpha|^2 + |\beta|^2 = 1$. Suppose now that ψ_1 and ψ_2 are normalized but not orthogonal. Show that there is a unique constant γ, with $|\gamma| \le 1$, such that $\psi = \psi_1 - \gamma\psi_2$ is orthogonal to ψ_2. Given that $|\gamma| < 1$ show that $\psi/\sqrt{1 - |\gamma|^2}$ is normalized.

6. A particle with $m = \hbar$, moving freely in one dimension has wavefunction

$$\psi(x,t) = \frac{1}{\pi^{\frac{1}{4}}\,(1+it)^{\frac{1}{2}}}\ \exp\left(\frac{-x^2}{2(1+it)}\right).$$

Verify that this wavefunction is normalized. Compute the probability density and probability current and verify that they are compatible with conservation of probability.

What is the probability at time t that the particle is in the interval $-\epsilon < x < \epsilon$ for infinitesimal ϵ?

7. Show that the stationary state wavefunctions of a particle in a potential $V(x)$ with $V(-x) = V(x)$ either have definite parity or can be chosen to have definite parity. Discuss the odd-parity bound states in the one-dimensional square well with potential $V = 0$ for $|x| > a$, $V = -U$ otherwise, where U is a positive constant. Use a graphical method to show that there is no odd-parity bound state if $2mU < (\hbar\pi/2a)^2$.

8. Sketch the potential

$$V = -\frac{\hbar^2}{m}\,\mathrm{sech}^2\,x$$

and show that the time-independent Schrödinger equation for a particle in this potential can be written as

$$A^\dagger A\,\psi = (\varepsilon + 1)\psi$$

where $\varepsilon = 2mE/\hbar^2$ and

$$A = \frac{d}{dx} + \tanh x\,, \qquad A^\dagger = -\frac{d}{dx} + \tanh x\,.$$

Show, by integrating by parts, that for any normalized wavefunction ψ,

$$\int_{-\infty}^{\infty} \psi^* A^\dagger A\psi\,dx = \int_{-\infty}^{\infty} (A\psi)^*(A\psi)\,dx$$

and hence that the eigenvalues of $A^\dagger A$ are non-negative. Hence deduce that the ground state wavefunction must have $\varepsilon \geq -1$. Show that there is a wavefunction $\psi_0(x)$ with $\varepsilon = -1$, satisfying

$$\frac{d\psi_0}{dx} + \tanh x\,\psi_0 = 0\,.$$

Find and sketch $\psi_0(x)$.

9. Write down the time-independent Schrödinger equation for the wavefunction ψ of a particle moving in a potential $V = -U\delta(x)$ for positive constant U (and $\delta(x)$ the Dirac delta function). Integrate the equation over the interval $-\epsilon < x < \epsilon$, for arbitrary positive constant ϵ, and hence show that there is a discontinuity at $x = 0$ in the derivative of $\psi(x)$:

$$\lim_{\epsilon\to 0}\,[\psi'(\epsilon) - \psi'(-\epsilon)] = -\frac{2mU}{\hbar^2}\psi(0)\,.$$

Show that there is a unique bound state ($E < 0$) solution $\psi_0(x)$. Find this ground state solution, and its energy.

10. Find the energy eigenfunctions $\psi_E(x)$ for a free particle of mass m subject to the periodic boundary condition

$$\psi_E(x) = \psi_E(x + L)\,.$$

What are the allowed values of E? What are the degeneracies of the energy levels? Comment on the limit $L \to \infty$.