

QUANTUM MECHANICS

Example Sheet 1

Speed of light: $c = 3.00 \times 10^8 \text{ m s}^{-1}$

Planck's constant: $\hbar = 1.055 \times 10^{-34} \text{ J s}$ ($h = 2\pi\hbar = 6.63 \times 10^{-34} \text{ J s}$)

Fine-structure constant: $\alpha \equiv \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}$

Mass of electron: $m_e = 9.11 \times 10^{-31} \text{ kg}$

Mass of proton: $m_p = 1.67 \times 10^{-27} \text{ kg}$

Electron volt: $1 \text{ eV} = 1.60 \times 10^{-19} \text{ J}$

Bohr radius: $r_1 = 0.529 \times 10^{-10} \text{ m}$

1. When the surface of a sample of potassium is illuminated with light of wavelength $3 \times 10^{-7} \text{ m}$ it emits electrons with kinetic energy 2.1 eV . When the same sample is illuminated with light of wavelength $5 \times 10^{-7} \text{ m}$ it emits electrons with kinetic energy 0.5 eV . Use Einstein's explanation of this 'photoelectric' effect to obtain a value for Planck's constant \hbar , and find the minimum energy E_0 needed to free an electron from the surface of potassium.
2. The light from a faint star has energy flux $10^{-10} \text{ J m}^{-2} \text{ s}^{-1}$. Assuming that the light has a wavelength of approximately $5 \times 10^{-7} \text{ m}$, estimate the number of photons from this star that enter a human eye in one second.
3. Bohr's 1913 model of the hydrogen atom assumes that electrons are non-relativistic classical particles travelling with speed v in circular orbits of radius r around a point-like proton, but with an angular momentum that must be an integral multiple n of \hbar . By equating the attractive inverse square force $e^2/4\pi\epsilon_0 r^2$ with the repulsive centrifugal force $m_e v^2/r$, show that this model implies that the electron energy is

$$E_n = -m_e c^2 \alpha^2 / 2n^2,$$

where α is the 'fine-structure' constant. An electron in the $n = 1$ 'ground state' has $r = r_1$, where r_1 is the Bohr radius; show that $r_1 = \lambda_C/\alpha$, where $\lambda_C = \hbar/m_e c$ is the electron's Compton wavelength. Show that the speed of the electron in the ground state is $v = \alpha c$, thereby justifying the neglect of relativistic effects.

When an electron in an excited ($n > 1$) state decays to a lower energy state it emits a photon. What is the maximum energy that such a photon can have? What is its minimum wavelength, λ_{\min} . Show that $\lambda_{\min} \gg r_1$.

4. The muon μ^- has the same charge as an electron but a mass $m_\mu = 207m_e$. It can be captured by a free proton to form an atom of *muonic hydrogen* (also called *mesic hydrogen*). Find the radius of the first Bohr orbit of such an atom.

5. Let $\psi_i(x)$, $i = 1, 2$, be two normalized stationary state wavefunctions. Assume that they are orthogonal, so that

$$\int_{-\infty}^{\infty} \psi_1^*(x) \psi_2(x) dx = 0.$$

Show that the linear superposition $\alpha\psi_1 + \beta\psi_2$, for complex constants α and β is normalized if and only if $|\alpha|^2 + |\beta|^2 = 1$. Suppose now that ψ_1 and ψ_2 are normalized but not orthogonal. Show that there is a unique constant γ , with $|\gamma| \leq 1$, such that $\psi = \psi_1 - \gamma\psi_2$ is orthogonal to ψ_2 . Given that $|\gamma| < 1$ show that $\psi/\sqrt{1-|\gamma|^2}$ is normalized.

6. A particle with $m = \hbar$, moving freely in one dimension has wavefunction

$$\psi(x, t) = \frac{1}{\pi^{\frac{1}{4}} (1 + it)^{\frac{1}{2}}} \exp\left(\frac{-x^2}{2(1 + it)}\right).$$

Verify that this wavefunction is normalized. Compute the probability density and probability current and verify that they are compatible with conservation of probability.

What is the probability at time t that the particle is in the interval $-\epsilon < x < \epsilon$ for infinitesimal ϵ ?

7. Show that the stationary state wavefunctions of a particle in a potential $V(x)$ with $V(-x) = V(x)$ either have definite parity or can be chosen to have definite parity. Discuss the odd-parity bound states in the one-dimensional square well with potential $V = 0$ for $|x| > a$, $V = -U$ otherwise, where U is a positive constant. Use a graphical method to show that there is no odd-parity bound state if $2mU < (\hbar\pi/2a)^2$.

8. Sketch the potential

$$V = -\frac{\hbar^2}{m} \operatorname{sech}^2 x$$

and show that the time-independent Schrödinger equation for a particle in this potential can be written as

$$A^\dagger A \psi = (\varepsilon + 1)\psi$$

where $\varepsilon = 2mE/\hbar^2$ and

$$A = \frac{d}{dx} + \tanh x, \quad A^\dagger = -\frac{d}{dx} + \tanh x.$$

Show, by integrating by parts, that for any normalized wavefunction ψ ,

$$\int_{-\infty}^{\infty} \psi^* A^\dagger A \psi dx = \int_{-\infty}^{\infty} (A\psi)^* (A\psi) dx$$

and hence that the eigenvalues of $A^\dagger A$ are non-negative. Hence deduce that the ground state wavefunction must have $\varepsilon \geq -1$. Show that there is a wavefunction $\psi_0(x)$ with $\varepsilon = -1$, satisfying

$$\frac{d\psi_0}{dx} + \tanh x \psi_0 = 0.$$

Find and sketch $\psi_0(x)$.

9. Write down the time-independent Schrödinger equation for the wavefunction ψ of a particle moving in a potential $V = -U\delta(x)$ for positive constant U (and $\delta(x)$ the Dirac delta function). Integrate the equation over the interval $-\epsilon < x < \epsilon$, for arbitrary positive constant ϵ , and hence show that there is a discontinuity at $x = 0$ in the derivative of $\psi(x)$:

$$\lim_{\epsilon \rightarrow 0} [\psi'(\epsilon) - \psi'(-\epsilon)] = -\frac{2mU}{\hbar^2} \psi(0).$$

Show that there is a unique bound state ($E < 0$) solution $\psi_0(x)$. Find this ground state solution, and its energy.

10. Find the energy eigenfunctions $\psi_E(x)$ for a free particle of mass m subject to the periodic boundary condition

$$\psi_E(x) = \psi_E(x + L).$$

What are the allowed values of E ? What are the degeneracies of the energy levels? Comment on the limit $L \rightarrow \infty$.