1. A particle of mass $m$ is confined to a one-dimensional box $0 \leq x \leq a$ (the potential $V(x)$ is zero inside the box, and infinite outside). Show that the energy eigenvalues are $E_n = \hbar^2 \pi^2 n^2 / 2ma^2$ for $n = 1, 2, \ldots$, and determine corresponding normalised energy eigenstates $\psi_n(x)$. Show that the expectation value and the uncertainty for a measurement of $\hat{x}$ in the state $\psi_n$ are given by

$$\langle \hat{x} \rangle_n = \frac{a}{2} \quad \text{and} \quad (\Delta x)^2_n = \frac{a^2}{12} \left( 1 - \frac{6}{\pi^2 n^2} \right).$$

Does the limit $n \to \infty$ agree with what you would expect for a classical particle in this potential?

2. Write down the Hamiltonian $H$ for a harmonic oscillator of mass $m$ and frequency $\omega$. Express $\langle H \rangle$ in terms of $\langle \hat{x} \rangle$, $\langle \hat{p} \rangle$, $\Delta x$ and $\Delta p$, all defined for some normalised state $\psi$. Use the Uncertainty Relation to deduce that $E \geq \frac{1}{2} \hbar \omega$ for any energy eigenvalue $E$.

3. Let $\Psi(x, t)$ be a solution of the time-dependent Schrödinger Equation with zero potential (corresponding to a free particle). Show that

$$\Phi(x, t) = \Psi(x-ut, t) e^{i k x} e^{-i \omega t}$$

is also a solution if the constants $k$ and $\omega$ are chosen suitably, in terms of $u$. Express $\langle \hat{x} \rangle_\Phi$ and $\langle \hat{p} \rangle_\Phi$ in terms of $\langle \hat{x} \rangle_\psi$ and $\langle \hat{p} \rangle_\psi$. Are the results consistent with Ehrenfest’s Theorem?

4. The energy levels of the harmonic oscillator are $E_n = (n+\frac{1}{2}) \hbar \omega$ for $n = 0, 1, 2, \ldots$ and the corresponding stationary state wavefunctions are

$$\psi_n(x) = h_n(y) e^{-y^2/2} \quad \text{where} \quad y = (m \omega / h)^{1/2} x$$

and $h_n$ is a polynomial of degree $n$ with $h_n(-y) = (-1)^n h_n(y)$. Using only the orthogonality relations

$$(\psi_m, \psi_n) = \delta_{mn},$$

determine $\psi_2$ and $\psi_3$ up to an overall constant in each case.

Give an expression for the quantum state of the oscillator $\Psi(x, t)$ if the initial state is $\Psi(x, 0) = \sum_{n=0}^{\infty} c_n \psi_n(x)$, where $c_n$ are complex constants. Deduce that

$$|\Psi(x, 2p\pi/\omega)|^2 = |\Psi(-x, (2q+1)\pi/\omega)|^2$$

for any integers $p, q \geq 0$. Comment on this result, considering the particular case in which $\Psi(x, 0)$ is sharply peaked around position $x = a$.

5. Consider the Schrödinger Equation in one dimension with potential $V(x)$. Show that for a stationary state, the probability current $J$ is independent of $x$.

Now suppose that an energy eigenstate $\psi(x)$ corresponds to scattering by the potential and that $V(x) \to 0$ as $x \to \pm \infty$. Given the asymptotic behaviour

$$\psi(x) \sim e^{ikx} + Be^{-ikx} \quad (x \to -\infty) \quad \text{and} \quad \psi(x) \sim Ce^{ikx} \quad (x \to +\infty)$$

show that $|B|^2 + |C|^2 = 1$. How should this be interpreted?

6. A particle is incident on a potential barrier of width $a$ and height $U$. Assuming that $U = 2E$, where $E = \hbar^2 k^2 / 2m$ is the kinetic energy of the incident particle, find the transmission probability. [Work through the algebra, which simplifies in this case, rather than quoting the general result.]
7. Consider the time-independent Schrödinger Equation with potential \( V(x) = -U\delta(x) \). Show that there is a scattering solution with energy eigenvalue \( E = \hbar^2 k^2 / 2m \) for any real \( k > 0 \) and find the transmission and reflection amplitudes \( A_t(k) \) and \( A_r(k) \). [Recall from Example 9 on Sheet 1 that the wavefunction \( \psi \) is continuous, but satisfies \( \psi'(0+) - \psi'(0-) = -(2mU/\hbar^2) \psi(0) \).]

Is the solution above still an eigenfunction of the Hamiltonian if \( k \) is allowed to take complex values? Show that \( A_t(k) \) and \( A_r(k) \) are singular at \( k = ik \) for a certain real, positive value of \( \kappa \). By first re-scaling the scattering solution, find a bound state (normalisable) solution in the potential. What is the energy of this bound state?

8. A particle of mass \( m \) is in a one-dimensional infinite square well (a potential box) with \( V = 0 \) for \( 0 < x < a \) and \( V = \infty \) otherwise. The normalised wavefunction for the particle at time \( t = 0 \) is

\[
\Psi(x,0) = Cx(a-x). 
\]

(i) Determine the real constant \( C \).

(ii) By expanding \( \Psi(x,0) \) as a linear combination of energy eigenfunctions (found in Example 1 above), obtain an expression for \( \Psi(x,t) \), the wavefunction at time \( t \).

(iii) A measurement of the energy is made at time \( t > 0 \). Show that the probability that this yields the result \( E_n = \hbar^2 n^2 / 2ma^2 \) is \( 960/\pi^6 n^6 \) if \( n \) is odd, and zero if \( n \) is even. Why should the result for \( n \) even be expected? Which value of the energy is most likely, and why is its probability so close to unity?

9. A quantum system has Hamiltonian \( H \) with normalised eigenstates \( \psi_n \) and corresponding energies \( E_n \) (\( n = 1, 2, 3, \ldots \)) . A linear operator \( Q \) is defined by its action on these states:

\[
Q \psi_1 = \psi_2, \quad Q \psi_2 = \psi_1, \quad Q \psi_n = 0 \quad n > 2.
\]

Show that \( Q \) has eigenvalues \( \pm 1 \) (in addition to zero) and find the corresponding normalised eigenstates \( \chi_{\pm} \), in terms of energy eigenstates. Calculate \( \langle H \rangle \) in each of the states \( \chi_{\pm} \).

A measurement of \( Q \) is made at time zero, and the result +1 is obtained. The system is then left undisturbed for a time \( t \), at which instant another measurement of \( Q \) is made. What is the probability that the result will again be +1? Show that the probability is zero if the measurement is made when a time \( T = \pi \hbar / (E_2 - E_1) \) has elapsed (assume \( E_2 - E_1 > 0 \)).

10. In the previous example, suppose that an experimenter makes \( n \) successive measurements of \( Q \) at regular time intervals \( T/n \). If the result +1 is obtained for one measurement, show that the amplitude for the next measurement to give +1 is

\[
A_n = 1 - \frac{i T (E_1 + E_2)}{2\hbar n} + O\left(\frac{1}{n^2}\right). 
\]

The probability that all \( n \) measurements give the result +1 is then \( P_n = (|A_n|^2)^n \). Show that

\[
\lim_{n \to \infty} P_n = 1.
\]

Interpreting \( \chi_{\pm} \) as the ‘not-boiling’ and ‘boiling’ states of a two-state ‘quantum kettle’, this shows that a watched quantum kettle never boils (also called the Quantum Zeno Paradox).

11. Let \( H \) be a Hamiltonian and \( \psi \) any normalised eigenstate with energy \( E \). Show that, for any operator \( A \),

\[
\langle [H, A] \rangle_{\psi} = 0.
\]

For a particle in one dimension, let \( H = T + V \) where \( T = \hat{p}^2 / 2m \) is the kinetic energy and \( V(\hat{x}) \) is any (real) potential. By setting \( A = \hat{x} \) in the result above and using the canonical commutation relation between position and momentum, show that \( \langle \hat{p} \rangle_{\psi} = 0 \).

Now assume further that \( V(\hat{x}) = k \hat{x}^n \) (with \( k \) and \( n \) constants). By taking \( A = \hat{x} \hat{p} \), show that

\[
\langle T \rangle_{\psi} = \frac{n}{n+2} E \quad \text{and} \quad \langle V \rangle_{\psi} = \frac{2}{n+2} E.
\]

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