1. A particle of mass $m$ is confined to a one-dimensional box $0 \leq x \leq a$ (the potential $U(x)$ is zero inside the box, and infinite outside). Show that the energy eigenvalues are $E_n = \frac{\hbar^2 \pi^2 n^2}{2ma^2}$ for $n = 1, 2, \ldots$, and determine corresponding normalised energy eigenstates $\chi_n(x)$. Show that the expectation value and the uncertainty for a measurement of $\hat{x}$ in the state $\chi_n$ are given by

\[
\langle \hat{x} \rangle_n = \frac{a}{2} \quad \text{and} \quad (\Delta x)_n^2 = \frac{a^2}{12} \left(1 - \frac{6}{\pi^2 n^2}\right).
\]

Does the limit $n \to \infty$ agree with what you would expect for a classical particle in this potential?

2. Write down the time-independent Schrödinger equation for the wavefunction of a particle moving in a potential $U(x) = -U\delta(x)$, where $U$ is a positive constant and $\delta(x)$ is the Dirac delta function. Integrate the equation over the interval $-\epsilon < x < \epsilon$, for a positive constant $\epsilon$, and hence deduce that there is a discontinuity at $x = 0$ in the derivative of $\chi(x)$:

\[
\lim_{\epsilon \to 0} [\chi'(\epsilon) - \chi'(\epsilon)] = -\frac{2mU}{\hbar^2} \chi(0).
\]

By using this condition to relate appropriate solutions for $x > 0$ and $x < 0$, find the unique bound and normalisable eigenstate of the Hamiltonian, and determine its energy eigenvalue $E$ (with $E < 0$).

3. Consider a square well potential with $U(x) = -U$ for $|x| < a$ and $U(x) = 0$ otherwise ($U$ is a positive constant). Show that there are no bound states (normalisable energy eigenfunctions) which satisfy $\chi(-x) = -\chi(x)$ (i.e. which have odd parity) if $a^2 U < \left(\frac{\pi}{\hbar}\right)^2 / 8m$.

4. Sketch the potential $U(x) = -\frac{\hbar^2}{m} \sech^2 x$.

Show that the time-independent Schrödinger equation for a particle in this potential can be written

\[
\hat{A}^\dagger \hat{A} \chi = (\mathcal{E} + 1) \chi
\]

where $\mathcal{E} = 2mE/\hbar^2$ and

\[
\hat{A} = \frac{d}{dx} + \tanh x, \quad \hat{A}^\dagger = -\frac{d}{dx} + \tanh x.
\]

Show, by integrating by parts, that for any normalised wavefunction $\chi$,

\[
\int_{-\infty}^{\infty} \chi^* \hat{A}^\dagger \hat{A} \chi \, dx = \int_{-\infty}^{\infty} (\hat{A} \chi)^*(\hat{A} \chi) \, dx
\]

and deduce that the eigenvalues of $\hat{A}^\dagger \hat{A}$ are non-negative. Hence show that the ground state (with lowest energy) has $\mathcal{E} \geq -1$. Show that a wavefunction $\chi_0(x)$ is an energy eigenstate with $\mathcal{E} = -1$ iff

\[
\frac{d\chi_0}{dx} + \tanh x \chi_0 = 0.
\]

Find and sketch $\chi_0(x)$. 

5. Write down the Hamiltonian $H$ for a harmonic oscillator of mass $m$ and frequency $\omega$. Express $\langle H \rangle$ in terms of $\langle \hat{x} \rangle$, $\langle \hat{p} \rangle$, $\Delta x$ and $\Delta p$, all defined for some normalised state $\psi$. Use the Uncertainty Relation to deduce that $E \geq \frac{\hbar}{2} \omega$ for any energy eigenvalue $E$.

6. The energy levels of the harmonic oscillator are $E_n = (n+\frac{1}{2})\hbar \omega$ for $n = 0, 1, 2, \ldots$ and the corresponding stationary state wavefunctions are

$$\chi_n(x) = h_n(y)e^{-y^2/2} \quad \text{where} \quad y = (m\omega/\hbar)^{1/2}x$$

and $h_n$ is a polynomial of degree $n$ with $h_n(-y) = (-1)^n h_n(y)$. Using only the orthogonality relations

$$\langle \chi_m, \chi_n \rangle = \delta_{mn},$$

determine $\chi_2$ and $\chi_3$ up to an overall constant in each case.

Give an expression for the quantum state of the oscillator $\psi(x, t)$ if the initial state is $\psi(x, 0) = \sum_{n=0}^{\infty} c_n \chi_n(x)$, where $c_n$ are complex constants. Deducd that

$$|\psi(x, 2p\pi/\omega)|^2 = |\psi(-x, (2q+1)\pi/\omega)|^2$$

for any integers $p, q \geq 0$. Comment on this result, considering the particular case in which $\psi(x, 0)$ is sharply peaked around position $x = a$.

7. A particle of mass $m$ is in a one-dimensional infinite square well (a potential box) with $U = 0$ for $0 < x < a$ and $U = \infty$ otherwise. The normalised wavefunction for the particle at time $t = 0$ is

$$\psi(x, 0) = C x(a-x).$$

(i) Determine the real constant $C$.

(ii) By expanding $\psi(x, 0)$ as a linear combination of energy eigenfunctions (found in Example 1 above), obtain an expression for $\psi(x, t)$, the wavefunction at time $t$.

(iii) A measurement of the energy is made at time $t > 0$. Show that the probability that this yields the result $E_n = h^2\pi^2n^2/2ma^2$ is $960/\pi^6n^6$ if $n$ is odd, and zero if $n$ is even. Why should the result for $n$ even be expected? Which value of the energy is most likely, and why is its probability so close to unity?

8. Consider the Schrödinger Equation in one dimension with potential $U(x)$. Show that for a stationary state, the probability current $J$ is independent of $x$.

Now suppose that an energy eigenstate $\chi(x)$ corresponds to scattering by the potential and that $U(x) \to 0$ as $x \to \pm \infty$. Given the asymptotic behaviour

$$\chi(x) \sim e^{ikx} + Be^{-ikx} \quad (x \to -\infty) \quad \text{and} \quad \chi(x) \sim Ce^{ikx} \quad (x \to +\infty)$$

show that $|B|^2 + |C|^2 = 1$. How should this be interpreted?

9. A particle is incident on a potential barrier of width $a$ and height $U$. Assuming that $U = 2E$, where $E = h^2k^2/2m$ is the kinetic energy of the incident particle, find the transmission probability.

[Work through the algebra, which simplifies in this case, rather than quoting the general result.]

10. Consider the time-independent Schrödinger Equation with potential $U(x) = -U \delta(x)$. Show that there is a scattering solution with energy eigenvalue $E = h^2k^2/2m$ for any real $k > 0$ and find the transmission and reflection coefficients $A_t(k)$ and $A_{\text{ref}}(k)$ (that correspond to the transmission and reflection coefficients defined in the notes as $T$ and $R$ respectively). [Recall from Example 2 that the energy eigenfunction $\chi$ is continuous, but satisfies $\chi'(0+) - \chi'(0-) = -(2mU/h^2) \chi(0)$.]

Is the solution above still an eigenfunction of the Hamiltonian if $k$ is allowed to take complex values? Show that $A_t(k)$ and $A_{\text{ref}}(k)$ are singular at $k = i\kappa$ for a certain real, positive value of $\kappa$. [Further details on the behaviour of these coefficients as $k \to i\kappa$ can be found in the notes.]

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κ. By first re-scaling the scattering solution, find a bound state (normalisable) solution in the potential. What is the energy of this bound state?

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