

## QUANTUM MECHANICS

### Example Sheet 2

1. Show that the probability current  $j$  for a stationary state  $\psi(x)$  of a particle scattering off an arbitrary potential  $V(x)$  in one dimension is independent of  $x$ . Given that  $\psi(x)$  has the asymptotic behaviour

$$\psi(x) = \begin{cases} e^{ikx} + Ae^{-ikx} & x \ll 0 \\ Be^{ikx} & x \gg 0 \end{cases},$$

show that  $|A|^2 + |B|^2 = 1$ . How should you interpret this?

2. A particle is incident on a square potential barrier of width  $a$  and height  $U_0$ . Assuming that  $U_0 = 2E$ , where  $E = \hbar^2 k^2 / 2m$  is the kinetic energy of the incident particle, find the transmission probability. [You should work from first principles rather than quote formulae from the lectures because the algebra simplifies in this special case.]

3. Consider the time-independent Schrödinger equation in Q.8, Ex.Sheet 1. Show that for any real  $k$ ,

$$\psi(x) = e^{ikx}(\tanh x - ik)$$

is a solution, and find its (scaled) energy  $\varepsilon$ . Show that this is the wavefunction of a scattering state where the reflection probability vanishes. Find the transmission amplitude, and verify that the transmission probability is 1.

\*Consider the extension (analytic continuation) of this solution to positive imaginary values of  $k$ . Show that for the value of  $k$  corresponding to the bound state at  $\varepsilon = -1$ ,  $\psi(x)$  reduces to the bound state wavefunction, and that the transmission amplitude has a simple pole.

4. Show that the energy levels of the one-dimensional harmonic oscillator of angular frequency  $\omega$  are  $E_n = (n + \frac{1}{2}) \hbar\omega$ . Find the wavefunctions  $\psi_0(x), \psi_1(x)$  and  $\psi_2(x)$  corresponding, respectively, to  $n = 0, 1$  and 2. Verify that  $\psi_0$  and  $\psi_2$  have even parity and that  $\psi_1$  has odd parity. Use this to deduce that  $\psi_1$  is orthogonal to both  $\psi_0$  and  $\psi_2$ . Verify that  $\psi_0$  and  $\psi_2$  are also orthogonal, i.e.

$$\int_{-\infty}^{\infty} \psi_2^* \psi_0 dx = 0.$$

5. What condition must an operator  $A$  satisfy to be Hermitian? Show that the expectation value of a Hermitian operator is real. Show that  $i[A_1, A_2]$  is Hermitian if  $A_1$  and  $A_2$  are.

6. A particle of mass  $m$  is in a one-dimensional infinite square well, with  $U = 0$  for  $0 < x < a$  and  $U = \infty$  otherwise. Show that its energy eigenstates have energies  $E_n = (\hbar\pi n)^2 / 2ma^2$  for positive integer  $n$ .

The normalized wavefunction of the particle at time  $t = 0$  is

$$\psi(x, 0) = Cx(a - x).$$

Determine the real constant  $C$ . Determine  $\psi(x, t)$ , the wavefunction at time  $t$ . [Write  $\psi(x, 0)$  as a linear combination of normalized energy eigenstates; i.e., as a Fourier series.] A measurement of the energy  $E$  is made at time  $t > 0$ . Show that the probability that this yields  $E_n$  for even  $n$  is zero. Why is this? Show that the probability that the measurement yields  $E_n$  is  $960/\pi^6 n^6$  for odd  $n$ . Which value of  $E$  is the most likely and why is its probability so close to unity?

7. A particle is confined to the one-dimensional box  $0 < x < a$ . If you have not already done this for Q.6, show that the energy levels are proportional to  $n^2$  for positive integer  $n$ , and find the corresponding complete set of normalized stationary states  $\psi_n(x)$ . Let  $\langle A \rangle_n$  denote the expectation value of any operator  $A$  in the state  $\psi_n$ . Show that

$$\langle x \rangle_n = \frac{1}{2}a, \quad \langle (x - \langle x \rangle_n)^2 \rangle_n = \frac{a^2}{12} \left( 1 - \frac{6}{n^2\pi^2} \right).$$

Hence show that the classical expectation values, i.e. with the particle bouncing back and forth and equally likely to be anywhere in the box, are recovered in the  $n \rightarrow \infty$  limit.

8. A ‘two-state’ quantum system has orthonormal energy eigenstates  $\psi_1$  and  $\psi_2$ , with energy eigenvalues  $E_1$  and  $E_2 = E_1 + \Delta E$  ( $\Delta E > 0$ ). These energy eigenstates form a complete set of wavefunctions for the system. Let  $S$  be a linear operator such that  $S\psi_1 = \psi_2$  and  $S\psi_2 = \psi_1$ . Show that the eigenvalues of  $S$  are  $\pm 1$  and write down the corresponding normalized eigenfunctions  $\phi_{\pm}$  in terms of the energy eigenstates. Compute the expectation values  $\langle E \rangle_{\pm}$  of the energy in the states  $\phi_{\pm}$ .

The observable corresponding to  $S$  is measured and the value  $+1$  is found. The system is then left undisturbed for a time  $t$ , after which  $S$  is measured again. What is the probability that the measured value of  $S$  will again be  $+1$ . Show that this probability vanishes when  $t = T \equiv \pi\hbar/\Delta E$ .

\*In a second run of this experiment it is decided to measure  $S$  at a large number  $n$  of small time intervals  $T/n$ . Each measurement yields either  $+1$  or  $-1$ , with the wavefunction being reset at  $\phi_+$  or  $\phi_-$ , respectively, by the measurement. Show that the probability amplitude for the state to be found in the  $+1$  eigenstate after a time interval  $T/n$ , given that it started in this eigenstate, is

$$A_n = 1 - \frac{i}{n\hbar} T \langle E \rangle_+ + \mathcal{O}\left(\frac{1}{n^2}\right).$$

The probability that all  $n$  measurements of  $S$  will yield the value  $+1$  is therefore  $P_n = (|A_n|^2)^n$ . Show that

$$\lim_{n \rightarrow \infty} P_n = 1.$$

[If you interpret  $\phi_+$  and  $\phi_-$  to be the ‘not boiling’ and ‘boiling’ states of a two-state ‘quantum kettle’ then you have just proved that *a watched kettle never boils*. This is also known as the *quantum Zeno effect*. Note that it is a real physical effect, in contrast to the ancient (so-called) Zeno paradox.]

9. The Hamiltonian operator for a particle in one dimension is  $H = T + U$  where  $T = p^2/2m$ , and  $U$  is any potential. Show that the expectation value  $\langle T \rangle$  is positive in any (normalized) state. By considering  $\langle H \rangle$ , show that the energy of the lowest bound state (assuming there is one) has energy above the minimum of  $U$ .

Suppose  $\psi$  is an eigenstate of  $H$  with energy  $E$ . Show that, for any operator  $A$ , and in the state  $\psi$

$$\langle [H, A] \rangle = 0.$$

By taking  $A = x$ , show that  $\langle p \rangle = 0$ . Now let  $U(x) = kx^n$  for constants  $k$  and  $n$ ; by taking  $A = xp$  derive the *virial theorem*

$$2\langle T \rangle = n\langle U \rangle.$$

Hence show that

$$\langle T \rangle = \frac{n}{n+2} E.$$

10. A particle of mass  $m$  moves in one dimension subject to the potential  $U(x) = \frac{1}{2}m\omega^2 x^2$ . Express the expectation value of the energy  $E$  in terms of  $\langle x \rangle$ ,  $\langle p \rangle$ ,  $\Delta x$  and  $\Delta p$ . Hence show, using the uncertainty relation for  $x$  and  $p$ , that in any state

$$\langle E \rangle \geq \frac{1}{2}\hbar\omega.$$

11. (Optional) *The Beam Interpretation.* Instead of interpreting the plane-wave solution, with  $k^2 \equiv |\mathbf{k}|^2$

$$\psi(\mathbf{x}, t) = A \exp[i\mathbf{k} \cdot \mathbf{x}] \exp\left[-\frac{i\hbar k^2 t}{2m}\right]$$

as describing the wavefunction of a particle, we can interpret it as describing a beam of particles with momentum  $\mathbf{p} = \hbar\mathbf{k}$  and energy  $E = |\mathbf{p}|^2/2m$  instead. In this interpretation, we think of the probability density function  $\rho(\mathbf{x}, \mathbf{t})$  as the *average* density of particles. Show that the density current  $\mathbf{j}$  is then given by

$$\mathbf{j}(\mathbf{x}, t) = \text{average density} \times \text{velocity}$$

i.e. the number of particles that flow through a unit area per time (flux).