Mathematical Tripos Part 1B Dr Adrian Kent
Lent Term 2007 apak@damtp.cam.ac.uk

# QUANTUM MECHANICS

**Example Sheet 2**

**1.** Show that the probability current $j$ for a stationary state $\chi(x)$ of a particle scattering off an arbitrary potential $V(x)$ in one dimension is independent of $x$. Given that $\chi(x)$ has the asymptotic behaviour

$$\chi(x) = \begin{cases} e^{ikx} + Ae^{-ikx} & x \ll 0 \\ Be^{ikx} & x \gg 0 \end{cases},$$

show that $|A|^2 + |B|^2 = 1$. How should you interpret this?

**2.** A particle is incident on a square potential barrier of width $a$ and height $U_0$. Assuming that $U_0 = 2E$, where $E = \hbar^2 k^2/2m$ is the kinetic energy of the incident particle, find the transmission probability. [*You should work from first principles rather than quote formulae from the lectures because the algebra simplifies in this special case.*]

**3.** Consider the time-independent Schrödinger equation in Q.8, Ex.Sheet 1. Show that for any real $k$,

$$\psi(x) = e^{ikx}(\tanh x - ik)$$

is a solution, and find its (scaled) energy $\varepsilon$. Show that this is the wavefunction of a scattering state where the reflection probability vanishes. Find the transmission amplitude, and verify that the transmission probability is 1.

Consider the extension (analytic continuation) of this solution to positive imaginary values of $k$. Show that for the value of $k$ corresponding to the bound state at $\varepsilon = -1$, $\psi(x)$ reduces to the bound state wavefunction, and that the transmission amplitude has a simple pole.

**4.** Show that the energy levels of the one-dimensional harmonic oscillator of angular frequency $\omega$ are $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$. Find the wavefunctions $\chi_0(x)$,$\chi_1(x)$ and $\chi_2(x)$ corresponding, respectively, to $n = 0, 1$ and 2. Verify that $\chi_0$ and $\chi_2$ have even parity and that $\chi_1$ has odd parity. Use this to deduce that $\chi_1$ is orthogonal to both $\chi_0$ and $\chi_2$. Verify that $\chi_0$ and $\chi_2$ are also orthogonal, i.e.

$$\int_{-\infty}^{\infty} \chi_2^* \chi_0 \, dx = 0 \, .$$

**5.** What condition must an operator $A$ satisfy to be Hermitian? Show that the expectation value of a Hermitian operator is real. Show that $i[A_1, A_2]$ is Hermitian if $A_1$ and $A_2$ are.

**6.** A particle of mass $m$ is in a one-dimensional infinite square well, with $U = 0$ for $0 < x < a$ and $U = \infty$ otherwise. Show that its energy eigenstates have energies $E_n = (\hbar\pi n)^2/2ma^2$ for positive integer $n$.

The normalized wavefunction of the particle at time $t = 0$ is

$$\psi(x, 0) = Cx(a - x) \, .$$

Determine the real constant $C$. Determine $\psi(x, t)$, the wavefunction at time $t$. [*Write* $\psi(x, 0)$ *as a linear combination of normalized energy eigenstates; i.e., as a Fourier series.*] A measurement of the energy $E$ is made at time $t > 0$. Show that the probability that this yields $E_n$ for even $n$ is zero. Why is this? Show that the probability that the measurement yields $E_n$ is $960/\pi^6 n^6$ for odd $n$. Which value of $E$ is the most likely and why is its probability so close to unity?

**7.** A particle is confined to the one-dimensional box $0 < x < a$. If you have not already done this for Q.6, show that the energy levels are proportional to $n^2$ for positive integer $n$, and find the corresponding complete set of normalized stationary states $\chi_n(x)$. Let $\langle A\rangle_n$ denote the expectation value of any operator $A$ in the state $\chi_n$. Show that

$$\langle x\rangle_n = \frac{1}{2}a\,, \qquad \langle (x-\langle x\rangle_n)^2\rangle_n = \frac{a^2}{12}\left(1-\frac{6}{n^2\pi^2}\right).$$

Hence show that the classical expectation values, i.e. with the particle bouncing back and forth and equally likely to be anywhere in the box, are recovered in the $n\to\infty$ limit.

**8.** A 'two-state' quantum system has orthonormal energy eigenstates $\chi_1$ and $\chi_2$, with energy eigenvalues $E_1$ and $E_2 = E_1 + \Delta E$ ($\Delta E > 0$). These energy eigenstates form a complete set of wavefunctions for the system. Let $S$ be a linear operator such that $S\chi_1 = \chi_2$ and $S\chi_2 = \chi_1$. Show that the eigenvalues of $S$ are $\pm 1$ and write down the corresponding normalized eigenfunctions $\phi_\pm$ in terms of the energy eigenstates. Compute the expectation values $\langle E\rangle_\pm$ of the energy in the states $\phi_\pm$.

The observable corresponding to $S$ is measured and the value $+1$ is found. The system is then left undisturbed for a time $t$, after which $S$ is measured again. What is the probability that the measured value of $S$ will again be $+1$. Show that this probability vanishes when $t = T \equiv \pi\hbar/\Delta E$.

In a second run of this experiment it is decided to measure $S$ at a large number $n$ of small time intervals $T/n$. Each measurement yields either $+1$ or $-1$, with the wavefunction being reset at $\phi_+$ or $\phi_-$, respectively, by the measurement. Show that the probability amplitude for the state to be found in the $+1$ eigenstate after a time interval $T/n$, given that it started in this eigenstate, is

$$A_n = 1 - \frac{i}{n\hbar}\,T\langle E\rangle_+ + \mathcal{O}(\frac{1}{n^2})\,.$$

The probability that all $n$ measurements of $S$ will yield the value $+1$ is therefore $P_n = (|A_n|^2)^n$. Show that

$$\lim_{n\to\infty} P_n = 1\,.$$

[If you interpret $\phi_+$ and $\phi_-$ to be the 'not boiling' and 'boiling' states of a two-state 'quantum kettle' then you have just proved that *a watched kettle never boils*. This is also known as the *quantum Zeno effect*. Note that it is a real physical effect, in contrast to the ancient (so-called) Zeno paradox.]

**9.** The Hamiltonian operator for a particle in one dimension is $H = T + U$ where $T = p^2/2m$, and $U$ is any potential. Show that the expectation value $\langle T\rangle$ is positive in any (normalized) state. By considering $\langle H\rangle$, show that the energy of the lowest bound state (assuming there is one) has energy above the minimum of $U$.

Suppose $\chi$ is an eigenstate of $H$ with energy $E$. Show that, for any operator $A$, and in the state $\chi$

$$\langle [H, A]\rangle = 0\,.$$

By taking $A = x$, show that $\langle p\rangle = 0$. Now let $U(x) = kx^n$ for constants $k$ and $n$; by taking $A = xp$ derive the *virial theorem*

$$2\langle T\rangle = n\langle U\rangle\,.$$

Hence show that

$$\langle T\rangle = \frac{n}{n+2}E\,.$$

**10.** A particle of mass $m$ moves in one dimension subject to the potential $U(x) = \frac{1}{2}m\omega^2x^2$. Express the expectation value of the energy $E$ in terms of $\langle x\rangle$, $\langle p\rangle$, $\Delta x$ and $\Delta p$. Hence show, using the uncertainty relation for $x$ and $p$, that in any state

$$\langle E\rangle \geq \frac{1}{2}\hbar\omega\,.$$