

QUANTUM MECHANICS

Example Sheet 2

1. Show that the probability current j for a stationary state $\psi(x)$ describing a beam of particles scattering off an arbitrary potential $V(x)$ in one dimension is independent of x . Given that $\psi(x)$ has the asymptotic behaviour

$$\psi(x) = \begin{cases} e^{ikx} + Ae^{-ikx} & x \ll 0 \\ Be^{ikx} & x \gg 0 \end{cases},$$

show that $|A|^2 + |B|^2 = 1$. How should you interpret this?

2. A beam of particles is incident on a square potential barrier of width a and height U_0 . Assuming that $U_0 = 2E$, where $E = \hbar^2 k^2 / 2m$ is the kinetic energy of the incident particles, find the transmission probability. [You should work from first principles rather than quote formulae from the lectures because the algebra simplifies in this special case.]

3. Consider the time-independent Schrödinger equation in Q.8, Ex.Sheet 1. Show that for any real k ,

$$\psi(x) = e^{ikx}(\tanh x - ik)$$

is a solution, and find its (scaled) energy ε . Show that this is the wavefunction of a scattering state where the reflection probability vanishes. Find the transmission amplitude, and verify that the transmission probability is 1.

*Consider the extension (analytic continuation) of this solution to positive imaginary values of k . Show that for the value of k corresponding to the bound state at $\varepsilon = -1$, $\psi(x)$ reduces to the bound state wavefunction, and that the transmission amplitude has a simple pole.

4. Show that the energy levels of the one-dimensional harmonic oscillator of angular frequency ω are $E_n = (n + \frac{1}{2})\hbar\omega$. Find the wavefunctions $\psi_0(x), \psi_1(x)$ and $\psi_2(x)$ corresponding, respectively, to $n = 0, 1$ and 2. Verify that ψ_0 and ψ_2 have even parity and that ψ_1 has odd parity. Use this to deduce that ψ_1 is orthogonal to both ψ_0 and ψ_2 .

5. What condition must an operator A satisfy to be Hermitian? Show that the expectation value of a Hermitian operator is real. Show that $i[A_1, A_2]$ is Hermitian if A_1 and A_2 are.

6. A particle of mass m is in a one-dimensional infinite square well, with $U = 0$ for $0 < x < a$ and $U = \infty$ otherwise. Show that its energy eigenstates have energies $E_n = (\hbar\pi n)^2 / 2ma^2$ for positive integer n .

The normalized wavefunction of the particle at time $t = 0$ is

$$\psi(x, 0) = Cx(a - x).$$

Determine the real constant C . Determine $\psi(x, t)$, the wavefunction at time t . [Write $\psi(x, 0)$ as a linear combination of normalized energy eigenstates; i.e., as a Fourier series.] A measurement of the energy E is made at time $t > 0$. Show that the probability that this yields E_n for even n is zero. Why is this? Show that the probability that the measurement yields E_n is $960/\pi^6 n^6$ for odd n . Which value of E is the most likely and why is its probability so close to unity? You may use the following definite integrals:

$$\int_0^\pi y \sin(ny) dy = -\frac{\pi}{n}(-1)^n \quad \int_0^\pi y^2 \sin(ny) dy = -\frac{1}{n^3} [2 - (-1)^n(2 - n^2\pi^2)]$$

where n is a non-zero integer.

7. A particle is confined to the one-dimensional box $0 < x < a$. If you have not already done this for Q.6, show that the energy levels are proportional to n^2 for positive integer n , and find the corresponding complete set of normalized stationary states $\psi_n(x)$. Let $\langle A \rangle_n$ denote the expectation value of any operator A in the state ψ_n . Show that

$$\langle x \rangle_n = \frac{1}{2}a, \quad \langle (x - \langle x \rangle_n)^2 \rangle_n = \frac{a^2}{12} \left(1 - \frac{6}{n^2\pi^2} \right).$$

Hence show that the classical expectation values, i.e. with the particle bouncing back and forth and equally likely to be anywhere in the box, are recovered in the $n \rightarrow \infty$ limit.

8. A ‘two-state’ quantum system has orthonormal energy eigenstates ψ_1 and ψ_2 , with energy eigenvalues E_1 and $E_2 = E_1 + \Delta E$ ($\Delta E > 0$). These energy eigenstates form a complete set of wavefunctions for the system. Let S be a linear operator such that $S\psi_1 = \psi_2$ and $S\psi_2 = \psi_1$. Show that the eigenvalues of S are ± 1 and write down the corresponding normalized eigenfunctions ϕ_{\pm} in terms of the energy eigenstates. Compute the expectation values $\langle E \rangle_{\pm}$ of the energy in the states ϕ_{\pm} .

The observable corresponding to S is measured and the value $+1$ is found. The system is then left undisturbed for a time t , after which S is measured again. What is the probability that the measured value of S will again be $+1$. Show that this probability vanishes when $t = T \equiv \pi\hbar/\Delta E$.

*In a second run of this experiment it is decided to measure S at a large number n of small time intervals T/n . Each measurement yields either $+1$ or -1 , with the wavefunction being reset at ϕ_+ or ϕ_- , respectively, by the measurement. Show that the probability amplitude for the state to be found in the $+1$ eigenstate after a time interval T/n , given that it started in this eigenstate, is

$$A_n = 1 - \frac{i}{n\hbar} T \langle E \rangle_+ + \mathcal{O}\left(\frac{1}{n^2}\right).$$

The probability that all n measurements of S will yield the value $+1$ is therefore $P_n = (|A_n|^2)^n$. Show that

$$\lim_{n \rightarrow \infty} P_n = 1.$$

[If you interpret ϕ_+ and ϕ_- to be the ‘not boiling’ and ‘boiling’ states of a two-state ‘quantum kettle’ then you have just proved that *a watched kettle never boils*. This is also known as the *quantum Zeno effect*. Note that it is a real physical effect, in contrast to the ancient (so-called) Zeno paradox.]

9. The Hamiltonian operator for a particle in one dimension is $H = T + U$ where $T = p^2/2m$, and U is any potential. Show that the expectation value $\langle T \rangle$ is positive in any (normalized) state. By considering $\langle H \rangle$, show that the energy of the lowest bound state (assuming there is one) has energy above the minimum of U .

Suppose ψ is an eigenstate of H with energy E . Show that, for any operator A , and in the state ψ

$$\langle [H, A] \rangle = 0.$$

By taking $A = x$, show that $\langle p \rangle = 0$. Now let $U(x) = kx^n$ for constants k and n ; by taking $A = xp$ derive the *virial theorem*

$$2\langle T \rangle = n\langle U \rangle.$$

Hence show that

$$\langle T \rangle = \frac{n}{n+2} E.$$

10. A particle of mass m moves in one dimension subject to the potential $U(x) = \frac{1}{2}m\omega^2 x^2$. Express the expectation value of the energy E in terms of $\langle x \rangle$, $\langle p \rangle$, Δx and Δp . Hence show, using the uncertainty relation for x and p , that in any state

$$\langle E \rangle \geq \frac{1}{2}\hbar\omega.$$