Michaelmas term 2011 Part II COSMOLOGY

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EXAMPLES I

1. **Astronomical units**. In our region of the galaxy, stars are typically separated from their nearest neighbours by a distance of about one parsec (pc). Relative velocities of stars within the galaxy typically have magnitudes of around 10^5 m/s. Use these figures to show that the typical angular speed ω at which one star moves across the sky relative to another one is subject to the approximate upper bound

$$\omega \leq 10^{-4} \quad rad/yr.$$

[1 yr $\approx 3 \times 10^7$ s; 1pc $\approx 3 \times 10^{16}$ m.]

2. **Olbers paradox**. In a simple static cosmological model, an average cosmic mass density ρ describes an infinite number of stars, of mass M, radius R and luminosity L. The stars are evenly, but randomly, distributed throughout an infinite universe, so that any line of sight in the night sky must meet some star. Let d be the average distance to a star along a line of sight from the earth. Explain why

$$d \sim \frac{M}{\rho R^2}\,.$$

Given that the energy flux per unit area of a single star at a distance r is $\Phi = L/4\pi r^2$, show that the model predicts a total energy flux per unit area on earth of $\Phi \sim L/R^2$. Show further that this equals the energy flux per unit area that the earth would receive if each point on the sky were as bright as the nearest star, i.e. the sun. So why is the sky dark at night? This is 'Olbers' paradox'.

In our universe, the average cosmic number density of hydrogen is approximately 1 atom per cubic metre. Use this to show that

$$\frac{(d/c)}{H_0^{-1}} \sim 10^{13}\,.$$

How does this help to resolve the paradox? [$M_\odot \sim 2 \times 10^{30} kg$, $R_\odot \sim 7 \times 10^8 m$, $m_H \sim 1.7 \times 10^{-27} kg$, $cH_0^{-1} \sim 10^{26} m$.]

3. **Distance measures:**. *(i) Angular diameter distance* – The apparent angular size $\delta\theta$ of a galaxy of proper size ℓ located at a comoving distance x is

$$\delta\theta = \frac{\ell}{a(t_e)x}$$

where t_e is the time of emission from the galaxy of the light that we see now. Taking $a(t_0) = 1$, show that $x = 3ct_0[1 - (t_e/t_0)^{1/3}]$ for an Einstein-de Sitter universe ($a = (t/t_0)^{2/3}$), and hence that

$$\delta\theta = \frac{\ell}{2cH_0^{-1}} \frac{(1+z)}{[1-(1+z)^{-\frac{1}{2}}]}\,.$$

Sketch the graph of $\delta\theta$ against z and show that there is a minimum at $z = 1.25$.

(ii) Luminosity distance – A galaxy of constant luminosity L has redshift z, as seen from Earth. Show that the rate at which its radiant energy passes through a sphere that intercepts Earth, and is centred on the galaxy, is $L/(1+z)^2$. [This is the 'apparent luminosity' of a galaxy and it can be used to infer its distance, assuming its typical luminosity is well understood.]

4. **Cosmological and event horizons**. Show that the equation of state $P = -\rho c^2$ implies a *constant* mass density ρ, which we may write as

$$\rho = \left(\frac{c^2}{8\pi G}\right)\Lambda$$

where Λ is a 'cosmological' constant, with units of inverse length squared. Show that the acceleration equation now has the *de Sitter universe* solution (like an inflationary universe):

$$a(t) = a_0 e^{Ht}\,, \qquad H = c\sqrt{\Lambda/3}\,.$$

What is the value of the parameter k for this solution? Show that

$$\int_{-\infty}^{t} \frac{dt'}{a(t')} = \infty$$

and hence that the de Sitter universe has no cosmological horizon. Show, however, that the integral

$$\int_{t}^{\infty} \frac{dt'}{a(t')}$$

is *finite* (for finite t) and hence deduce that there is a *maximum* comoving distance that a signal emitted at time t can travel from its source. Thus, there are events in a de Sitter universe that an observer can never see; they are said to be over the (cosmological) *event horizon.* This is a different kind of horizon to the cosmological horizon of decelerating model universes discussed in the lectures. Show that any model universe with $a \propto t^{\alpha}$ has no event horizon as long as $\alpha < 1$ (in other words, as long as it is decelerating).

5. **Accelerating universe**. The 'deceleration parameter' q_0 is defined as $q(t_0)$, where

$$q(t) = -\frac{a\ddot{a}}{\dot{a}^2}\,.$$

Show that $q = \frac{1}{2}(3w+1)\Omega$ for a universe with equation of state $P = w\rho c^2$. One might expect $q_0 = \frac{1}{2}\Omega_0$ for a pressure-free universe, but recent observations of distant supernovae suggest $q_0 \approx -\Omega_0$. Discuss.

6. **Open universe**. Consider an empty universe, with $\rho = 0$. Find the general solution of the Raychaudhuri equation, and then show that a solution with non-constant scale factor $a(t)$ solves the Friedmann equation only if $k < 0$. You have just found the Milne universe. Show that the age of the Milne universe equals the Hubble time H_0^{-1}. Show further that the general universe with this property has $\rho \propto t^{-2}$ and an equation of state $P = -\frac{1}{3}\rho c^2$.

7. **Closed universe**. A homogeneous and isotropic model universe has pressure $P(t)$ and energy density $E/V = \rho(t)c^2$ such that $P = w\rho c^2$ where w is a constant. Assuming that the universe is expanding adiabatically, such that $dE = -PdV$, show that $\rho = \rho_0 a^{-3(w+1)}$ for constant ρ_0, where $a(t)$ is the scale factor of the universe. Let

$$\tau(t) = \int^{t} \frac{dt'}{a(t')}$$

be a new time parameter (conformal time), and define the new function $y(\tau)$ by $y = a^{(3w+1)/2}$. Show that the Friedmann equation for $a(t)$ implies that $y(\tau)$ satisfies

$$y'' + \frac{kc^2}{4}\left(3w+1\right)^2 y = 0\,.$$

Hence show that for a radiation-dominated universe ($w = 1/3$) with $k = 1$ the graph of $a(t)$ against t is a semi-circle. Find the total time duration from Big Bang to Big Crunch as a function of ρ_0.

8. **Matter-radiation transition**. Consider the evolution of a flat ($k = 0$) universe containing both a matter density $\rho_{\rm M}$ (pressure $P_{\rm M}$=0) and a radiation density $\rho_{\rm R}$ (pressure $P_{\rm R}$=$\frac{1}{3}\rho_{\rm R}c^2$). Show that these densities are equal $\rho_{\rm M} = \rho_{\rm R}$ when the scalefactor is given by

$$a = a_{\rm eq} \equiv \frac{\rho_{\rm R0}}{\rho_{\rm M0}} = \frac{\Omega_{\rm R0}}{\Omega_{\rm M0}}\,,$$

where $\Omega_{\rm M0}$ and $\Omega_{\rm R0}$ are the respective density parameters today. Use the conformal time parameter $d\tau = dt/a$ (see previous question), to show that the Friedmann equation takes the form

$$a'^2 = A(a + a_{\rm eq})\,,$$

where $A = H_0^2\Omega_{\rm M_0}$. Hence find the parametric solution for the transition between radiation and matter domination:

$$a(\tau) = \frac{1}{4}A\tau^2 + B\tau\,, \qquad t(\tau) = \frac{1}{12}A\tau^3 + \frac{1}{2}B\tau^2\,,$$

where $B^2 = H_0^2\Omega_{\rm R0}$. Show that the expected asymptotic solutions are obtained for $a \gg a_{\rm eq}$ (Einstein-de Sitter) and $a \ll a_{\rm eq}$ (Tolman).