EXAMPLES I

1. Olbers' paradox In a simple static cosmological model, an average cosmic mass density ρ describes an infinite number of stars, of mass M, radius R and luminosity L (i.e. total power emitted). Assuming the stars are evenly, but randomly, distributed throughout an infinite universe, note that an observer at $r = 0$ will observe an average number $(\rho/M)4\pi r^2 dr$ stars on a spherical shell at r of thickness dr. By considering the fraction of the sky covered by these stars, argue that the entire night sky will be covered if we can integrate out to the distance:

$$d \;\approx\; \frac{M}{\rho\,\pi R^2}\,.$$

Show that the total number of visible stars will be $N = 4\pi M^2/(3\rho^2(\pi R^2)^3)$ (though in fact it is a larger number because the stars would partially obscure each other (occultation)).

Given that the energy flux per unit area of a single star at a distance r is $\Phi = L/4\pi r^2$, show that the model predicts a total energy flux per unit area on earth of $\Phi \sim L/\pi R^2$. Show further that this equals the energy flux per unit area that the earth would receive if each point on the sky were as bright as the nearest star, i.e. the sun, so not very habitable. This is 'Olbers' paradox': Why is the sky dark at night?

In our universe, the average cosmic number density of hydrogen is approximately 1 atom per cubic metre (which we will assume is mainly concentrated in stars like our Sun). Use this to show that

$$\frac{d}{cH_0^{-1}} \;\sim\; 10^{13}\,.$$

How does this resolve the paradox? [$M_\odot \sim 2\times10^{30}$kg, $R_\odot \sim 7\times10^{8}$m, $m_H \sim 1.7\times10^{-27}$kg, $cH_0^{-1} \sim 10^{26}$m.]

2. Newtonian gravitational collapse: (i) *Instability of Newton's 'static' universe* – A spherical cloud of mass M and initial radius R contains material with uniform density, ρ, and zero pressure. Ignoring the cosmological constant, show that if it is initially at rest at $t = 0$ when the radius $r(t)$ is R, then the subsequent gravitational collapse governed by Newton's law $\ddot{r} = -GM/r^2$ is described by a parametric solution for $r(t)$, with

$$r(\theta) = R\cos^2\theta\,, \qquad\qquad \theta + \frac{1}{2}\sin 2\theta = \left(\frac{2GM}{R^3}\right)^{1/2} t\,,$$

where we define $\theta(t{=}0) = 0$. Sketch the behaviour of $r(t)$ as it collapses from $r = R$ to $r = 0$ and show that the cloud reaches $r = 0$ after a time

$$t_{\rm col} = \sqrt{\frac{3\pi}{32G\rho_0}},$$

where $\rho_0 = M/(\frac{4\pi}{3}R^3)$ is the initial density. [*Note:* This is the same behaviour as the evolution of a closed dust-filled homogeneous and isotropic universe from the time of expansion maximum to the final singularity (see Q5 for $w = 0$). The parametric solution for $r(t)$ describes a cycloid.]

(ii) *Astronomical units* – Estimate this collapse timescale $\sim(G\rho_0)^{-1/2}$ in years, assuming a typical star has one solar mass with an interstellar separation of one parsec. [$M_\odot \approx 2\times10^{30}$ kg, 1pc $\approx 3\times10^{16}$ m, 1 yr $\approx 3\times10^{7}$ s.]

In fact, our galaxy does not collapse and its structure has been sustained over billions of years by 'random' orbital velocities of about $v \sim 220\,$km/s (see virial theorem later). Show that the typical angular speed ω at which one star moves across the sky relative to another is subject to the approximate upper bound $\omega \lesssim 10^{-4}$ rad/yr. How long can we expect Proxima Centauri to be our nearest neighbour star?

3. Distance measures: *(i) Angular diameter distance* – The apparent angular size $\delta\theta$ of a galaxy of fixed proper (physical) size ℓ located at a comoving distance x is

$$\delta\theta = \frac{\ell}{a(t_e)x}\,,$$

where t_e is the time of emission from the galaxy of the light that we see now. Taking $a(t_0) = 1$ in an Einstein-de Sitter universe ($\Omega_{\rm M}$=1, $a = (t/t_0)^{2/3}$), show that $x = 3ct_0[1-(t_e/t_0)^{1/3}]$ and hence that

$$\delta\theta = \frac{\ell}{2cH_0^{-1}}\frac{(1+z)}{[1-(1+z)^{-\frac{1}{2}}]}\,.$$

Sketch the graph of $\delta\theta$ against z and show that there is a minimum at $z = 1.25$. [*Note:* This behaviour could be used to distinguish cosmological models, e.g., verify there is no minimum size in a de Sitter universe (Ω_Λ=1).]

(ii) Luminosity distance – A galaxy of constant intrinsic luminosity L has redshift z, as seen from Earth. Show that the rate at which its radiant energy passes through a sphere that intercepts Earth, and is centred on the galaxy, is $L/(1+z)^2$. [This is the 'apparent luminosity' of a galaxy and it can be used to infer its distance, assuming its intrinsic luminosity is well understood.]

4. Cosmological horizons and event horizons: Show that the equation of state $P{=}-\rho c^2$ implies a *constant* mass density ρ, which we may write as

$$\rho = \left(\frac{c^2}{8\pi G}\right)\Lambda\,,$$

where Λ is the 'cosmological constant', with units of inverse length squared. Show that the acceleration equation for $\ddot{a}/a$ has the *de Sitter universe* solution (like an inflationary universe):

$$a(t) = a_0 e^{Ht}\,, \qquad H = c\sqrt{\Lambda/3}\,.$$

What is the value of the parameter k for this solution? Show that

$$\int_{-\infty}^{t} \frac{dt'}{a(t')} = \infty$$

and hence that the de Sitter universe has no cosmological ('particle') horizon. Show, however, that the integral

$$\int_{t}^{\infty} \frac{dt'}{a(t')}$$

is *finite* (for finite t) and hence deduce that there is a *maximum* comoving distance that a signal emitted at time t can travel from its source. Thus, there are events in a de Sitter universe that an observer can never see; they are said to be beyond the cosmological *event horizon*. This is a different kind of horizon to the cosmological particle horizon of decelerating model universes discussed in the lectures. Show that any model universe with $a \propto t^\alpha$ has no event horizon as long as $\alpha < 1$ (in other words, as long as it is decelerating).

5. Open universe: Consider an empty universe, with $\rho = 0$. Find the general solution of the acceleration (Raychaudhuri) equation, and then show that a solution with non-constant scale factor $a(t)$ solves the Friedmann equation only if $k < 0$. You have just found the Milne universe.

Show that the age of the Milne universe equals the Hubble time H_0^{-1}. Show further that the general universe with this property has $\rho \propto t^{-2}$ and an equation of state $P = -\frac{1}{3}\rho c^2$.

6. Closed universes: A homogeneous and isotropic model universe has pressure $P(t)$ and energy density $E/V = \rho(t)c^2$ such that $P = w\rho c^2$ where w is a constant. Assuming that the universe is expanding adiabatically, such that $dE = -PdV$, show that $\rho = \rho_0 a^{-3(w+1)}$ for constant ρ_0, where $a(t)$ is the scale factor of the universe. Let

$$\tau(t) = \int^{t} \frac{dt'}{a(t')}$$

be a new time parameter (conformal time), and define the new function $y(\tau)$ by $y = a^{(3w+1)/2}$. Show that the Friedmann equation for $a(t)$ implies that $y(\tau)$ satisfies

$$y'' + \frac{kc^2}{4}\,(3w+1)^2\, y = 0\,.$$

Fix the zero of time by $a(0) = 0$ and show that for a radiation-dominated universe ($w = 1/3$) with $k = 1$ the graph of $a(t)$ against t is a semi-circle. Find the total time duration from Big Bang to Big Crunch as a function of ρ_0.

7. Matter-radiation transition: Consider the evolution of a flat ($k = 0$) universe containing both a matter density $\rho_{\rm M}$ (pressure $P_{\rm M}$=0) and a radiation density $\rho_{\rm R}$ (pressure $P_{\rm R}{=}\frac{1}{3}\rho_{\rm R} c^2$). Show that these densities are equal $\rho_{\rm M} = \rho_{\rm R}$ when the scale factor is given by

$$a = a_{\rm eq} \equiv \frac{\rho_{\rm R0}}{\rho_{\rm M0}} = \frac{\Omega_{\rm R0}}{\Omega_{\rm M0}}\,,$$

where $\Omega_{\rm M0}$ and $\Omega_{\rm R0}$ are the respective density parameters today. Use the conformal time parameter $d\tau = dt/a$ (see previous question), to show that the Friedmann equation takes the form

$$a'^2 = A(a + a_{\rm eq})\,,$$

where $A = H_0^2\Omega_{\rm M_0}$. Hence find the parametric solution for the transition between radiation and matter domination:

$$a(\tau) = \frac{1}{4}A\tau^2 + B\tau\,, \qquad t(\tau) = \frac{1}{12}A\tau^3 + \frac{1}{2}B\tau^2\,,$$

where $B^2 = H_0^2\Omega_{\rm R0}$. Find the asymptotic solutions for $a(t)$ when $a \gg a_{\rm eq}$ and $a \ll a_{\rm eq}$.

8*. Time-varying G: The expansion scale factor $r(t)$ of a Newtonian universe with zero pressure matter and $\Lambda = 0$ is governed by the gravitation law $\ddot{r} = -GM/r^2$, where $M = 4\pi\rho r^3/3$ is constant and $\rho(t)$ is the matter density. If the gravitation 'constant' is assumed to vary in time with $G(t) = G_0(t_0/t)^n$, where G_0, n and t_0 are positive constants, then show that there is an exact solution with

$$r(t) \propto t^{(2-n)/3}\,.$$

Why is there no counterpart of the Friedmann equation in this problem?

This solution reduces to the Einstein-de Sitter universe when $n = 0$. This is the Newtonian analogue of the exact zero-curvature cosmological solution with $P = 0$ in what is known as the Brans-Dicke theory of gravity, a theory which generalises Einstein's theory of general relativity to include a varying G.

[*Note:* Observations indicate that any variation must be tiny today with $\left|\dot{G}/G\right| < 10^{-12}$ per year.]

**Please send any corrections to epss@damtp.cam.ac.uk*