1. In special and general relativity, the energy density $\rho$ and pressure $P$ of a perfect fluid combine to form the *energy-momentum* tensor. In the rest frame of the fluid, this is given by

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}.$$ 

After a Lorentz boost by velocity $v$ in the $x$-direction, the energy-momentum tensor transforms to

$$\tilde{T}^{\mu\nu} = \Lambda^{\mu}_{\lambda} \Lambda^{\nu}_{\sigma} T^{\lambda\sigma} \quad \text{with} \quad \Lambda^{\mu}_{\lambda} = \begin{bmatrix} \gamma & \frac{-\gamma v}{c} & 0 & 0 \\ \frac{-2v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$ 

What equation of state $P = \omega \rho$ gives rise to a fluid that looks the same to all inertial observers?

2. The *deceleration parameter* is defined by

$$q(t) \equiv -\frac{a\ddot{a}}{a^2}.$$ 

Compute $q$ for a flat universe dominated by a single fluid with equation of state $P = \omega \rho$.

For a universe with arbitrary curvature, filled with matter, radiation and a cosmological constant, show that today

$$q_0 = \frac{1}{2} \Omega_m + \Omega_{rad} - \Omega_{\Lambda}.$$ 

What is the deceleration parameter for our universe?

Show that the Taylor expansion of the scale factor about the present day (with $a(t_0) = 1$) can be written as

$$\frac{1}{1 + z} = a(t) = 1 + H_0(t - t_0) - \frac{1}{2} q_0 H_0^2(t - t_0)^2 + \ldots.$$ 

Hence show that the light from a nearby galaxy at redshift $z \ll 1$ was emitted at time

$$t_0 - t = H_0^{-1} z - \frac{1}{2} H_0^{-1} (2 + q_0) z^2 + \ldots.$$ 

3. A flat universe, containing matter and a cosmological constant, expands according to the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \left(\rho_\Lambda + \frac{\rho_0}{a^3}\right).$$ 

Solve this to find

$$a(t) = \beta (\sinh \alpha t)^{2/3} \quad (1)$$

with $\alpha$ and $\beta$ constant. [Hint: you may find the substitution $b = a^{3/2}$ helpful.] Verify the expected asymptotic behaviour for early and late times.

Our universe today is flat, with $\Omega_m \approx 0.3$ and $\Omega_{\Lambda} \approx 0.7$ and $H_0^{-1} \approx 14$ Gyr.
Determine:

- The age of the universe.
- The age at which the expansion of the universe first started to accelerate.
- The age at which the energy density in matter and the cosmological constant were equal.

A cosmologically ignorant civilisation measures $H_0^{-1} \approx 14$ Gyr but refuses to countenance the possibility of a cosmological constant. How old would they believe their universe to be if they assumed it was flat with $\Omega_m = 1$?

4. The jerk is defined as

$$J(t) = \frac{\dddot{a}}{\dot{a}^2}.$$ 

14 billion years of cosmic expansion can be roughly characterised as: the jerk is one. To see this, consider the Friedmann equation with matter, a cosmological constant and curvature. Show that the curvature can be written as

$$k c^2 R^2 = a^2 H^2 (J - 1).$$

Confirm that the solution (1) indeed has unit jerk.

[Fun fact: the 4th, 5th and 6th derivatives of the scale factor are called snap, crackle and pop respectively.]

5. A flat universe, containing radiation and a cosmological constant, expands according to the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3 c^2} \left(\rho_\Lambda + \frac{\rho_0}{a^4}\right).$$

Solve for $a(t)$. Verify the expected asymptotic behaviour for early and late times.

6. Consider a collection of particles, each of which interacts through the potential

$$V(x_i - x_j) = \alpha |x_i - x_j|^n.$$ 

Prove the virial theorem,

$$2\bar{T} = n \bar{V},$$

where $\bar{T}$ is the time-averaged kinetic energy and $\bar{V}$ the time-averaged total potential energy.

7. The equations of motion for inflation can be written as

$$H^2 = \frac{1}{3 M_{\text{pl}}^2} \left[ \frac{1}{2} \dot{\phi}^2 + V(\phi) \right]$$ (2)

and

$$\ddot{\phi} + 3H \dot{\phi} + \frac{\partial V}{\partial \phi} = 0,$$ (3)

where $M_{\text{pl}}^2 = c^2/(8\pi G)$ is related to the reduced Planck mass. Consider the potential

$$V(\phi) = V_0 e^{-\frac{\lambda \phi}{M_{\text{pl}}}}$$
Show that the inflationary equations have the exact solution
\[ a(t) = \left( \frac{t}{t_0} \right)^{\frac{1}{2}} \quad \text{and} \quad \phi(t) = \phi_0 + \alpha \log t, \]

for some choice of \( \phi_0 \) and \( \alpha \).

For what values of \( \lambda \) does inflation occur? Define the pressure-energy density ratio \( \omega_\phi = P_\phi/\rho_\phi \). For what range of \( \omega_\phi \) does inflation occur?

8. Assume that an inflationary solution \( \phi(t) \) is monotonic and view the Hubble parameter \( H(t) \) as a function \( H(\phi(t)) \). Show that the equations of inflation (2) and (3) can be written in “Hamilton-Jacobi” form
\[ \dot{\phi} = -2M_{\text{pl}}^2 \frac{\partial H}{\partial \phi} \]

and
\[ V(\phi) = 3M_{\text{pl}}^2 H^2 - 2M_{\text{pl}}^4 \left( \frac{\partial H}{\partial \phi} \right)^2. \]

[Hint: you might start by differentiating the Friedmann equation (2) with respect to time.]

The Hamilton-Jacobi formalism can be used to construct exact solutions in which one specifies a choice of \( H(\phi) \) and then works backwards to read off the corresponding potential \( V(\phi) \), the scalar evolution \( \phi(t) \) and, hence the scale factor \( a(t) \). Find these three quantities for
\[ H(\phi) = \exp \left( -\sqrt{\frac{\alpha}{2M_{\text{pl}}}} \phi \right). \]

9. Under the slow roll conditions, the inflationary equations become
\[ H^2 \approx \frac{1}{3M_{\text{pl}}^2} V(\phi) \quad \text{and} \quad 3H\dot{\phi} \approx -\frac{\partial V}{\partial \phi}. \]

Solve these equations to find the scale factor \( a(t) \) for the choice of potentials
i) \( V(\phi) = \frac{1}{2} m^2 \phi^2 \);
ii) \( V(\phi) = \frac{1}{4} \lambda \phi^4 \).