EXAMPLES II

1. **Dark energy**. Our universe today is believed to be flat ($k = 0$) and filled with two major components: pressure-free matter ($P_{\rm M} = 0$) and dark energy with equation of state $P_{\rm Q} = -\rho_{\rm Q}c^2$ with mass densities given respectively by $\rho_{\rm M0}$ and $\rho_{\rm Q0}$ (at $t = t_0$). Assume that each component independently satisfies the fluid conservation equation to show that the total mass density can be expressed as

$$\rho(t) = \frac{\rho_{\rm M0}}{a^3} + \rho_{\rm Q0}\,,$$

where we have set $a(t_0) = 1$.

Now consider the substitution $b = a^{3/2}$ in the Friedmann equation to show that the solution for the scalefactor can be written in the form

$$a(t) = \beta(\sinh \alpha t)^{2/3}\,,$$

where α and β are constants which you should specify in terms of $\rho_{\rm M0}$, ρ_{Q0} and t_0. [*Hint:* Recall that $\int dx/\sqrt{x^2+1} = \sinh^{-1} x$.] Show that the scale factor $a(t)$ has the expected behaviour for an Einstein-de Sitter universe at early times ($t \to 0$) and that the universe accelerates at late times ($t \to \infty$).

2. **Inflation and the flatness problem**. Suppose that the universe "reheats" instantaneously after a period of inflation ($a \propto \exp(Ht)$ with H constant), restarting the standard Hot Big Bang with an effective initial time $t_{\rm reh} \approx 10^{-35}$s (assume this universe is filled only with matter and radiation with $1 + z_{\rm eq} \approx 10^5$, $\Omega_0 \approx 1.02$ and $t_0 \approx 10^{18}$s). Show that at a time $t_{\rm reh}$, the density parameter must be fine-tuned to approximately $\Omega_{\rm reh} - 1 \approx 10^{-52}$. How much expansion during inflation is required to solve this flatness problem, that is, estimate the number N of e-folds and the time interval for inflation $\Delta t = t_{\rm reh} - t_{\rm i}$? Plot $\log \Omega$ vs $\log(t - t_i)$ to illustrate how the flatness problem is cured.

3. **Simple inflationary model***. In the very early universe, suppose that we have a homogeneous scalar field ϕ (the inflaton) with a vacuum potential energy $\epsilon_{\rm vac} = \rho_{\rm vac}c^2 = cV(\phi)/\hbar$, with

$$V(\phi) = \frac{m^2c^4}{\hbar^2}\phi^2 \equiv \mathcal{M}^2\phi^2\,.$$

(Here, ϕ has units of mass and we introduce the constant $\mathcal{M}$ with units of inverse time for convenience—so-called *fundamental units* with $\hbar = k = c = 1$ would be better for this problem, ask your supervisor about these.) The inflaton ϕ obeys the scalar wave equation (or Klein-Gordon equation),

$$\ddot{\phi} + 3H\dot{\phi} - c^2\nabla^2\phi + \frac{dV}{d\phi} = 0\,. \qquad (*)$$

However, during inflation (after starting with a large initial $\phi_{\rm i}$) we have overdamped evolution satisfying the so-called "slow-roll" conditions, $|\ddot{\phi}| \ll |3H\dot{\phi}|$ and $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$. This means (together with homogeneity) that eqn ($*$) and the Friedmann equation become

$$3H\dot{\phi} \approx -\tfrac{dV}{d\phi} = -2\mathcal{M}^2\phi\,, \qquad H^2 \approx \tfrac{8\pi G}{3}\rho_{\rm vac} = \tfrac{\mathcal{M}^2}{M_{\rm pl}^2}\phi^2\,,$$

where $M_{\rm pl} = \sqrt{3\hbar c/8\pi G}$ is the Planck mass. Solve these equations to find the inflationary solution

$$\phi(t) \;=\; \phi_{\rm i} - \tfrac{2}{3}\mathcal{M}M_{\rm pl}t\,, \qquad a(t) \;=\; \exp\left[\tfrac{\mathcal{M}}{M_{\rm pl}}\left(\phi_{\rm i}t - \tfrac{1}{3}\mathcal{M}M_{\rm pl}t^2\right)\right] \;=\; \exp\left[\tfrac{3}{4M_{\rm pl}^2}\left(\phi_{\rm i}^2 - \phi(t)^2\right)\right]\,.$$

Show that this solution only satisfies both "slow-roll" conditions while $|\phi| > \phi_{\rm reh}$, the value of which you should estimate (i.e. inflation only occurs if we choose $|\phi_{\rm i}| > \phi_{\rm reh}$ and it ends when $|\phi| \approx \phi_{\rm reh}$). How large must we take ϕ_i to solve the flatness problem (as in Q2)?

***4.* Microstate counting and equilibrium distributions**. N equal mass particles of total energy E populate a set of degenerate energy eigenstates with energies E_i and degeneracies g_i ($i = 1, 2, 3, \ldots, \infty$). The set $\{n\}$ of numbers n_i of particles with energy E_i is assigned a weight of the form

$$\Omega(\{n\}) = \prod_i W(n_i, g_i)\,. \qquad (*)$$

The most probable distribution $\{\bar{n}\}$ is obtained by maximising $\log \Omega$ subject to the constraints of fixed particle number N and fixed total energy E. Show that $\bar{n}_i$ is found by solving the equation

$$\frac{\partial \log W(n_i, g_i)}{\partial n_i} = \alpha + \beta E_i$$

where α and β are constants such that $\sum_i \bar{n}_i = N$ and $\sum_i \bar{n}_i E_i = E$. Write out this equation for each of the following three choices of the function W:

$$(i)\ W(n,g) = \frac{(g+n-1)!}{n!(g-1)!}\,, \qquad (ii)\ W(n,g) = \frac{g!}{n!(g-n)!}\,, \qquad (iii)\ W(n,g) = \frac{g^n}{n!}\,.$$

Assuming $g \gg 1$, $n \gg 1$, and $g \geq n$, use Stirling's formula $[\log n! = n \log n - n + \mathcal{O}(\log n)]$ to simplify your result. Hence show that if α and β are appropriately related to the chemical potential μ and temperature T then $\{\bar{n}\}$ is the equilibrium distribution found in the lectures for a gas of (i) Bose-Einstein, (ii) Fermi-Dirac, and (iii) Maxwell-Boltzmann type. [The gas particles are said to obey BE, FD or MB 'statistics', respectively.]

***5.* Indistinguishable particles**. Assuming that $\Omega(\{n\})$ of the previous question equals the number of microstates available to the N particles for a given occupation number distribution $\{n\}$, explain why $\Omega(\{n\})$ must take the form $(*)$ if the N particles are identical.

Show that Ω *is* equal to the number of available microstates in cases (i) and (ii) assuming Bose-Einstein statistics and Fermi-Dirac statistics, respectively. [*Hint*: Consider how many different ways there are of painting n identical balls in g colours assuming (i) no restriction on the number of times each colour is used or (ii) that no colour may be used more than once.]

Show that in case (iii) Ω is $1/N!$ times the number of microstates available to N *distinguishable* particles. [This fact is related to the 'Gibbs paradox' of classical statistical mechanics.]

***6.* Relativistic pressure**. (i) Evaluate the expression for the pressure,

$$P = \frac{1}{3V}\int_0^\infty pE'(p)\bar{n}(p)dp\,, \qquad (*)$$

in the case of a massless particle $E = pc$ with $\mu = 0$ to find

$$P = \begin{cases} \frac{2\sigma}{3c} g_{\rm s} T^4 & \text{(bosons)} \\ \frac{7}{8}\frac{2\sigma}{3c} g_{\rm s} T^4 & \text{(fermions)} \end{cases}$$

where $\sigma = \pi^2 k^4 / 60\hbar^3 c^2$ is the Stefan-Boltzmann constant.
[*Hint:* Make the substitution $z = \beta E$ and note that the Riemann zeta function $\zeta(n+1) = \frac{1}{n!}\int_0^\infty \frac{z^n}{e^z - 1}dz$ with $\zeta(4) = \pi^4/90$ (also $\zeta(2) = \pi^2/6$ - see below).]
(ii)** Consider a highly relativistic particle $E = c\sqrt{p^2 + m^2c^2}$ with mass $m \ll T$ (and $\mu = 0$). Expand the pressure in powers of m/T to show that the first correction is

$$P = \frac{2\sigma}{3c} g_{\rm s} T^4 - \frac{15}{6\pi^2}\frac{c^3\sigma}{k^2} g_{\rm s} m^2 T^2\,. \qquad \textit{(bosons)}$$

This expression is important for understanding the nature of phase transitions, notably those occurring in the early universe.

***7.* Pressure support equation**. Let r be the radial distance from the centre of a spherically symmetric star of pressure $P(r)$, and let $m(r)$ be the mass within a sphere of radius r. Use the pressure-support equations to show that the function

$$F(r) = P + \frac{Gm^2}{8\pi r^4}$$

is a decreasing function of r. Let M be the mass of the star and R its radius. Derive the lower bound $P_c > GM^2/(8\pi R^4)$ on the central pressure P_c.

8. **Thermostatic equilibrium**. A star is assumed to be a spherically-symmetric ball of ideal gas held together by gravity. Assuming that the number density $n(r)$, pressure $P(r)$ and temperature $T(r)$ are functions only of radial distance r from the centre, use the ideal gas law (Boyle-Charles law) to show that their gradients n', P' and T' are related by

$$\frac{n'}{n} = \frac{P'}{P} - \frac{T'}{T}\,.$$

**Please send any corrections to acd@damtp.cam.ac.uk*