Cosmology: Example Sheet 3 Michaelmas Term

J. E. Santos November 2022

1. A non-relativistic, quantum mechanical particle sits in a box whose sides have length $a(t) \times L$. Write down the wavefunctions when $a(t)$ is constant.

   It can be shown that if $a(t)$ changes suitably slowly then the system remains in a given energy eigenstate. Show that the momentum "redshifts" as $\mathbf{p}(t) = \mathbf{p}_0/a(t)$ where $\mathbf{p}_0$ is the momentum when $a(t_0) = 1$.

   A gas of non-relativistic particles at temperature T is described by the Maxwell-Boltzmann distribution at time $t = t_0$. Assuming the momentum-redshift above, show that the gas retains the Maxwell-Boltzmann form as the Universe expands, but with a temperature that scales as $T(t) = T_0/a(t)^2$.

2. The thermal cosmic microwave background is assumed to be isotropic with a temperature T in an inertial frame S. The same radiation is detected in another inertial frame S', moving with velocity v with respect to S.

   The Lorentz transformation relating the energy E and 3-momentum $\mathbf{p}$ of a particle in the two frames is

$$E = \gamma(E' - \mathbf{v} \cdot \mathbf{p}') \quad \text{with} \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} .$$

   A photon has $E = p\,c$. Show that the microwave background will also appear thermal in S', but with an anisotropic temperature

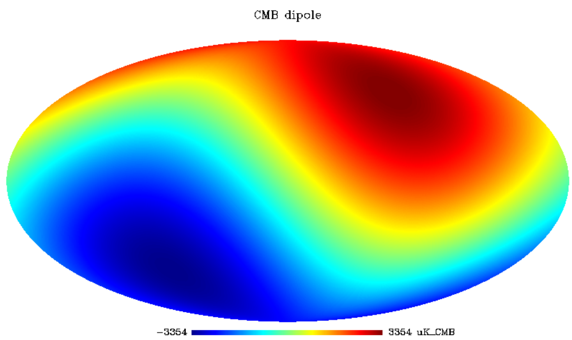
$$T'(\theta') = \frac{T}{\gamma\left(1 - \frac{v}{c}\cos\theta'\right)} \approx T\left[1 + \frac{v}{c}\cos\theta' + \mathcal{O}\left(\frac{v^2}{c^2}\right)\right] ,$$

   where θ' is the angle between the velocity $\mathbf{v}$ and the momentum $\mathbf{p}'$ of the photon arriving at the detector.

   Let T'_+ and T'_- be the maximum and minimum temperatures seen in the inertial frame S'. Show that

$$T = \sqrt{T'_+ T'_-} .$$

   The observed CMB, shown in the figure, has $T'_+ - T'_- \approx 6.5 \times 10^{-3}$K, with $T = \sqrt{T'_+ T'_-} \approx 2.7\,$K . How fast are we travelling with respect to the Universe's preferred inertial frame?



   It is believed that there exists a yet-to-be-observed thermal cosmic neutrino background that is isotropic in the same frame S as the CMB. The neutrino has a small mass and so $E^2 = p^2c^2 + m^2c^4$.

Today, the neutrinos are travelling at non-relativistic speeds. Show that when (if!) we finally observe the cosmic neutrino background, we do not expect the energy density to be thermal, even at a fixed angle.

3. The Planck blackbody formula states that the number of photons with frequency between ω and $\omega + \mathrm{d}\omega$ is

$$n(\omega)\mathrm{d}\omega = \frac{1}{\pi^2 c^3} \frac{\omega^2}{e^{\beta\hbar\omega} - 1} .$$

Show that the total number of photons is

$$n_\gamma = \frac{2\,\zeta(3)}{\pi^2 \hbar^3 c^3} (k_B\, T)^3 \tag{1}$$

[You will need the integral $\int_0^{+\infty} \mathrm{d}y\, y^2/(e^y - 1) = 2\zeta(3)$ with $\zeta(3) \approx 1.2$.]

Define $n_{\text{energetic}}$ to be the number of energetic photons with energy greater than the hydrogen binding energy $E_{\text{bind}} \approx 13.6\,\text{eV}$. Show that, when $k_B T \ll E_{\text{bind}}$,

$$\frac{n_{\text{energetic}}}{n_\gamma} \approx \frac{(\beta E_{\text{bind}})^2}{2\zeta(3)} e^{-\beta E_{\text{bind}}} .$$

As a naïve diagnostic, suppose that recombination occurs when there is less than a single energetic photon per baryon. Use the baryon-to-photon ratio $\eta = n_B/n_\gamma \approx 10^{-9}$ to determine the temperature and redshift of recombination according to this criterion.

4. Assume that electrons, protons and hydrogen are in chemical equilibrium during recombination, with the chemical potentials related by $\mu_e + \mu_p = \mu_H$. Show that the number of electrons is related to the number of hydrogen atoms by

$$n_e^2 \approx n_H \left(\frac{m_e k_B T}{2\pi\hbar^2} \right)^{3/2} e^{-\beta\, E_{\text{bind}}} ,$$

with E_{bind} the binding energy of the hydrogen ground state. What assumptions did you make along the way?

The ionization fraction is defined as $X_e = n_e/n_B$ with $n_B \approx n_p + n_H$, the number of baryons. Use the expression (1) to show that

$$\frac{1 - X_e}{X_e^2} = \eta\, \frac{2\zeta(3)}{\pi^2} \left(\frac{2\pi k_B T}{m_e c^2} \right)^{3/2} e^{\beta\, E_{\text{bind}}} ,$$

with η the baryon-to-photon ratio. Consider the limiting regimes $k_B T \ll E_{\text{bind}}$ and $E_{\text{bind}} \ll k_B T \ll m_e c^2$ to roughly sketch X_e as a function of temperature.

5. Recombination is not instantaneous, but happens over a period of time. Some photons in the CMB come from earlier times, when the universe was hotter, and some from later times.

   Why does the observed CMB exhibit a perfect blackbody spectrum at a single temperature?

6. At temperature T, and vanishing chemical potential, the expected number of particles with momentum $\mathbf{p}$ is given by

$$n(\mathbf{p}) = \frac{1}{e^{\beta\, E(\mathbf{p})} \mp 1}$$

where the minus sign is for bosons and the plus sign for fermions.

For ultra-relativistic particles, with $E(\mathbf{p}) \approx pc$, show that the total number of fermions, n_F , is related to the total number of bosons, n_B , by $n_F = 3n_B/4$. Show that the total energy density of fermions, ρ_F , is related to the total energy density of bosons, ρ_B, by $\rho_F = 7\rho_B/8$.

[Note: you need not evaluate any integral to do this question.]

7. Consider a gas of electrons and positrons in the ultra-relativistic limit $k_B T \gg m_e c^2$. In the early Universe, there must have been a slight imbalance of electrons over positrons. This is modelled by introducing a small chemical potential $\mu_e \ll k_B T$ for electrons, with an equal and opposite chemical potential for positrons. Show that this results in a small excess Δn of electrons over positrons, given by

$$\Delta n = \frac{g(k_B T)^2}{6\hbar^3 c^3}\,\mu_e \left[1 + \mathcal{O}\left(\frac{\mu_e^2}{k_B^2 T^2}\right)\right] .$$

[You will need to use the integral $\int_0^{+\infty} \mathrm{d}y\, y/(e^y + 1) = \pi^2/12$.]