EXAMPLES III

1. Maxwell-Boltzmann distribution A non-relativistic, quantum mechanical particle sits in a box whose sides have length $a(t) \times L$. Write down the wavefunctions when $a(t)$ is constant.

It can be shown that if $a(t)$ changes suitably slowly then the system remains in a given energy eigenstate (Ehrenfest's Principle). Show that the momentum "redshifts" as $\mathbf{p}(t) = \mathbf{p}_0/a(t)$ where $\mathbf{p}_0$ is the momentum when $a(t_0) = 1$.

A gas of non-relativistic particles at temperature T is described by the Maxwell-Boltzmann distribution at time $t = t_0$. Assuming the momentum-redshift above, show that the gas retains the Maxwell-Boltzmann form as the Universe expands, but with a temperature that scales as $T(t) = T_0/a(t)^2$. (This means that a non-relativistic particle species maintains its equilibrium distribution after decoupling, that is, when there are no interactions to maintain equilibrium (just like photons do in the ultrarelativistic limit).

2. CMB dipole The thermal cosmic microwave background is assumed to be isotropic with a temperature T in an inertial frame S. The same radiation is detected in another inertial frame S', moving with velocity v with respect to S.

The Lorentz transformation relating the energy E and 3-momentum $\mathbf{p}$ of a particle in the two frames is

$$E = \gamma(E' - \mathbf{v}\cdot\mathbf{p}') \quad \text{with} \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} .$$

A photon has $E = p\,c$. Show that the microwave background will also appear thermal in S', but with an anisotropic temperature

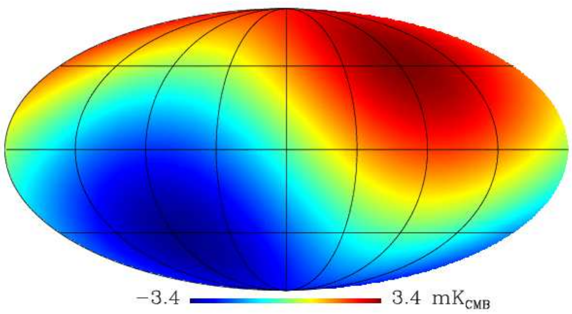
$$T'(\theta') = \frac{T}{\gamma\left(1 - \frac{v}{c}\cos\theta'\right)} \approx T\left[1 + \frac{v}{c}\cos\theta' + \mathcal{O}\left(\frac{v^2}{c^2}\right)\right] ,$$

where θ' is the angle between the velocity $\mathbf{v}$ and the momentum $\mathbf{p}'$ of the photon arriving at the detector.

Let T'_+ and T'_- be the maximum and minimum temperatures seen in the inertial frame S'. Show that

$$T = \sqrt{T'_+ T'_-} .$$

The observed CMB, shown in the figure, has $T'_+ - T'_- \approx 6.5 \times 10^{-3}$K, with $T = \sqrt{T'_+ T'_-} \approx 2.7\,$K. How fast are we travelling with respect to the Universe's preferred inertial frame?



It is believed that there exists a yet-to-be-observed thermal cosmic neutrino background that is isotropic in the same frame S as the CMB. The neutrino has a small mass and so $E^2 = p^2c^2 + m^2c^4$. Today, the neutrinos are travelling at non-relativistic speeds. Show that when (if!) we finally observe the cosmic neutrino background, we do not expect the energy density to be thermal, even at a fixed angle.

3. Planck thermal black-body spectrum and recombination. The Planck blackbody formula states that the number of photons with frequency between ν and $\nu + \mathrm{d}\nu$ is

$$n(\nu)\mathrm{d}\nu = \frac{8\pi}{c^3}\frac{\nu^2}{e^{\beta\,h\,\nu} - 1} .$$

Show that the total number of photons is

$$n_\gamma = \frac{4\pi\,\zeta(3)}{(h\,c)^3}(k_B\,T)^3 \tag{1}$$

[You will need the integral $\int_0^{+\infty} \mathrm{d}y\, y^2/(e^y - 1) = 2\zeta(3)$ with $\zeta(3) \approx 1.2$.]

Define $n_{\text{energetic}}$ to be the number of energetic photons with energy greater than the hydrogen binding energy $E_{\text{bind}} \approx 13.6\,\text{eV}$. Show that, when $k_B T \ll E_{\text{bind}}$,

$$\frac{n_{\text{energetic}}}{n_\gamma} \approx \frac{(\beta E_{\text{bind}})^2}{2\zeta(3)} e^{-\beta E_{\text{bind}}} .$$

As a naïve diagnostic, suppose that recombination occurs when there is less than a single energetic photon per baryon. Use the baryon-to-photon ratio $\eta = n_B/n_\gamma \approx 10^{-9}$ to determine the temperature and redshift of recombination according to this criterion.

4. Saha's equation. Assume that electrons, protons and hydrogen are in chemical equilibrium during recombination, with the chemical potentials related by $\mu_e + \mu_p = \mu_H$. Show that the number of electrons is related to the number of hydrogen atoms by

$$n_e^2 \approx n_H \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} e^{-\beta\, E_{\text{bind}}} ,$$

with E_{bind} the binding energy of the hydrogen ground state. What assumptions did you make along the way?

The ionization fraction is defined as $X_e = n_e/n_B$ with $n_B \approx n_p + n_H$, the number of baryons. Use the expression (1) to show that

$$\frac{1 - X_e}{X_e^2} = \eta\, \frac{2\zeta(3)}{\pi^2} \left(\frac{2\pi k_B T}{m_e c^2}\right)^{3/2} e^{\beta\, E_{\text{bind}}} ,$$

with η the baryon-to-photon ratio. Consider the limiting regimes $k_B T \ll E_{\text{bind}}$ and $E_{\text{bind}} \ll k_B T \ll m_e c^2$ to roughly sketch X_e as a function of temperature.

5. Photon Planck spectrum and decoupling. Recombination is not instantaneous, but happens over a period of time. Some photons in the CMB come from earlier times, when the universe was hotter, and some from later times.

Why does the observed CMB exhibit a perfect blackbody spectrum at a single temperature?

6. Boson and fermion densities. At temperature T, and vanishing chemical potential, the expected number of particles with momentum $\mathbf{p}$ is given by

$$n(\mathbf{p}) = \frac{1}{e^{\beta\, E(\mathbf{p})} \mp 1}$$

where the minus sign is for bosons and the plus sign for fermions.

For ultra-relativistic particles, with $E(\mathbf{p}) \approx pc$, show that the total number of fermions, n_F , is related to the total number of bosons, n_B , by $n_F = 3n_B/4$. Show that the total energy density of fermions, ρ_F , is related to the total energy density of bosons, ρ_B, by $\rho_F = 7\rho_B/8$.

[Note: you need not evaluate any integral to do this question.]

7. Electron–positron annihilation and lepton asymmetry. Consider a gas of electrons and positrons in the ultra-relativistic limit $k_B\, T \gg m_e c^2$. In the early Universe, there must have been a slight imbalance of electrons over positrons. This is modelled by introducing a small chemical potential $\mu_e \ll k_B T$ for electrons, with an equal and opposite chemical potential for positrons. Show that this results in a small excess Δn of electrons over positrons, given by

$$\Delta n = \frac{4\pi^3 g\,(k_B T)^2}{3h^3 c^3}\, \mu_e \left[1 + \mathcal{O}\left(\frac{\mu_e^2}{k_B^2 T^2}\right)\right] .$$

[You will need to use the integral $\int_0^{+\infty} \mathrm{d}y\, y/(e^y + 1) = \pi^2/12$.]

**Please send any corrections to epss@damtp.cam.ac.uk*