Michaelmas term 2011 Part II COSMOLOGY Prof A.C. Davis
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EXAMPLES III

1. **Endpoints of stellar evolution**. The formation of a neutron star involves the removal of electrons, and hence their degeneracy pressure, by inverse beta-decay, $p + e^- \to n + \nu_e$. Why are white dwarf stars stable against inverse beta-decay? [You will need to use the fact that $m_n - m_p \approx 2.6 m_e$.]

2. **Fermi degeneracy pressure**. Starting from the Fermi-Dirac distribution, obtain the relation

$$n = \frac{4\pi g_s}{3c^3\hbar^3}\left(\mu^2 - m^2c^4\right)^{\frac{3}{2}}$$

for the number density of fully degenerate relativistic fermions of mass m and spin degeneracy g_s at chemical potential μ.

At densities much higher than those available in white dwarfs, inverse beta decay allows a star composed of protons and electrons to 'neutronize', i.e. to turn into neutrons with the neutrinos escaping from the star. The equilibrium concentration of protons and neutrons in the star is determined by the equation.

$$\mu_p + \mu_e = \mu_n \, .$$

Why is it reasonable to suppose that the neutrinos have zero chemical potential? Assuming that the nucleons (protons and neutrons) are non-relativistic, and that the electrons are ultra-relativistic, show that the fraction $\alpha = n_p/n_N$, where $n_N = n_p + n_n$ is the nucleon number density, satisfies

$$\alpha^{\frac{2}{3}} - \left(\frac{m_p}{m_n}\right)(1-\alpha)^{\frac{2}{3}} = \frac{2}{\lambda_p(3\pi^2 n_N)^{\frac{1}{3}}}\left[\frac{(m_n - m_p)}{m_p\lambda_p(3\pi^2 n_N)^{\frac{1}{3}}} - \alpha^{\frac{1}{3}}\right]$$

where $\lambda_p = \hbar/m_p c$ is the proton's Compton wavelength. [This equation determines α as a function of n_N. A typical nucleon number density is $n_N \sim 10^{44} m^{-3}$, which yields $\alpha \approx 1/200$. Thus, 'neutron' stars are indeed composed mostly of neutrons. Note that typically they have a radius of about 17km and a central density $\rho_c \approx 2 \times 10^{17}\mathrm{kg\,m^{-3}}$.]

3. **General relativistic corrections**. Why are GR effects significant for neutron stars but not for white dwarfs? [You will need to consider the dimensionless quantity GM/c^2R.]

4. **CMB isotropy**. A thermal (Planckian) cosmic radiation background is assumed to be isotropic with temperature T in an inertial frame S. The same radiation is detected in another (laboratory) inertial frame S' moving with velocity $\mathbf{v}$ with respect to S. The Lorentz transformation relating the energy-momentum 4-vector in the two frames is

$$E = \gamma\left(E' - \mathbf{v}\cdot\mathbf{p}'\right) \, , \qquad \mathbf{p} = \gamma\left(\mathbf{p}' - \mathbf{v}E'/c^2\right) \, ,$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. Use this to show that the background will still be thermal in S' but with an anisotropic temperature

$$T'(\theta') = \frac{T}{\gamma\left[1 - (v/c)\cos\theta'\right]} = T\left[1 + \frac{v}{c}\cos\theta'\right] + \mathcal{O}(v^2/c^2) \, ,$$

where θ' is the angle between the velocity $\mathbf{v}$ of the frame S' and the momentum $\mathbf{p}'$ of the photon arriving at the detector. Given that T'_+ and T'_- are the maximum and minimum temperatures seen in the inertial frame S', show that

$$(i)\ T = \sqrt{T'_+ T'_-} \, , \qquad (ii)\ \frac{v}{c} = \frac{T'_+ - T'_-}{T'_+ + T'_-} \, .$$

What is the significance of these results to observations of the CMB?

It is believed that there is a thermal cosmic electron-neutrino background that is isotropic in the same 'isotropy frame' S as the CMB, but with a slightly lower temperature. The above analysis will still apply if the neutrino is massless, but there is now evidence that the neutrino may have a small mass. If it does, show that the energy density in the cosmic neutrino background will not be thermal, even at fixed angle, when measured in any other inertial frame S' with non-zero velocity relative to S.

***5.* <u>Entropy conservation</u>**. Let the internal energy U of a gas be related to its pressure P and volume V by the formula $PV = (\gamma - 1)U$. Assuming either fixed particle number or vanishing chemical potential, use the first law of thermodynamics to show that

$$(\gamma - 1)TdS = \gamma PdV + VdP$$

Hence show that PV^{γ} is constant for an isentropic ($dS = 0$) change of state. Show also that if S is proportional to V, at fixed pressure, then $TS = \gamma U$. [The constant γ is called the 'adiabatic index' because a 'quasi-static', i.e. slow, adiabatic change of state is isentropic.]

How are these results applicable to the CMB? Use them, and the Stephan-Boltzmann law for blackbody radiation, to show that $s \propto T^3$ where s is the entropy density of the CMB.

***6.* <u>CMB black body spectrum</u>**. Let $a(t)$ be the scale factor of an expanding universe. Assuming that the expansion is too slow to cause transitions between energy eigenstates with different energy, show that a particle of momentum p_0 at time t_0 will have momentum $p = p_0/a(t)$ at time t.

Use this to show that a thermal distribution of photons with temperature T_0 at time $t = t_0$ will still be thermal at time t, but with a temperature $T(t) = T_0/a(t)$. Hence show, using the result for the entropy density of the previous question, that the total entropy of the CMBR is conserved during the expansion.

Show that a thermal distribution of particles of a *non-relativistic* ideal gas will also remain thermal but with a temperature $T = T_0/a^2(t)$. Assuming that $PV = (\gamma - 1)U$ with $\gamma \neq 1$, and using the results of the previous question, deduce that $\gamma = 5/3$.

***7.* <u>Recombination</u>**. Neutral hydrogen atoms can be ionized by collisions with sufficiently energetic photons via the photo-ionization reaction $\gamma + H \to e^- + p^+$. For simplicity we assume that only the ground state of the hydrogen atom is relevant, so that the minimum energy that the photon must have to ionize the atom is the ground-state binding energy I. The reverse reaction is called 'recombination' and at equilibrium the forward and reverse reactions balance. Let n_H, n_e and n_p be the equilibrium number densities of hydrogen atoms, electrons and protons, respectively. In equilibrium the chemical potentials must balance. Since $\mu_\gamma = 0$ this requires

$$\mu_H = \mu_e + \mu_p \,.$$

Assuming charge neutrality, and that all particles other than the photons are non-relativistic, show that the equilibrium electron number density is given by

$$n_e^2 \approx n_H \left(\frac{2\pi m_e kT}{h^2}\right)^{\frac{3}{2}} e^{-I/kT} \,.$$

[N.B. A NR approx is adequate since $I \approx 13.6\,\mathrm{eV} << m_e c^2 = 511\mathrm{MeV}$. Note also that $g_H = 4$.] Consider the fractional ionization $X_e = n_e/n_B$, where $n_B = n_p + n_H = \eta n_\gamma$ (with the baryon-to-photon ratio $\eta \approx 10^{-9}$), to find Saha's equation

$$\frac{1 - X_e}{X_e^2} = \eta\, 16\pi\zeta(3) \left(\frac{kT}{2\pi m_e c^2}\right)^{3/2} e^{I/kT} \,.$$

Consider the limiting regimes (i) $kT \ll I$ and (ii) $I < kT < m_e c^2$ to roughly sketch X_e as a function of temperature T.