

EXAMPLES III

CMB isotropy A thermal (Planckian) cosmic radiation background is assumed to be isotropic with temperature T in an inertial frame S . The same radiation is detected in another (laboratory) inertial frame S' moving with velocity \mathbf{v} with respect to S . The Lorentz transformation relating the energy-momentum 4-vector in the two frames is

$$E = \gamma(E' - \mathbf{v} \cdot \mathbf{p}'), \quad \mathbf{p} = \gamma(\mathbf{p}' - \mathbf{v}E'/c^2),$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. Use this to show that the background will still be thermal in S' but with an anisotropic temperature

$$T'(\theta') = \frac{T}{\gamma[1 - (v/c)\cos\theta']} = T \left[1 + \frac{v}{c} \cos\theta' \right] + \mathcal{O}(v^2/c^2),$$

where θ' is the angle between the velocity \mathbf{v} of the frame S' and the momentum \mathbf{p}' of the photon arriving at the detector. Given that T'_+ and T'_- are the maximum and minimum temperatures seen in the inertial frame S' , show that

$$(i) \quad T = \sqrt{T'_+ T'_-}, \quad (ii) \quad \frac{v}{c} = \frac{T'_+ - T'_-}{T'_+ + T'_-}.$$

What is the significance of these results to observations of the CMB?

It is believed that there is a thermal cosmic electron-neutrino background that is isotropic in the same 'isotropy frame' S as the CMB, but with a slightly lower temperature. The above analysis will still apply if the neutrino is massless, but there is now evidence that the neutrino may have a small mass. If it does, show that the energy density in the cosmic neutrino background will not be thermal, even at fixed angle, when measured in any other inertial frame S' with non-zero velocity relative to S .

Entropy conservation Let the internal energy U of a gas be related to its pressure P and volume V by the formula $PV = (\gamma - 1)U$. Assuming either fixed particle number or vanishing chemical potential, use the first law of thermodynamics to show that

$$(\gamma - 1)TdS = \gamma PdV + VdP$$

Hence show that $PV\gamma$ is constant for an isentropic ($dS = 0$) change of state. Show also that if S is proportional to V , at fixed pressure, then $TS = \gamma U$. [The constant γ is called the 'adiabatic index' because a 'quasi-static', i.e. slow, adiabatic change of state is isentropic.]

How are these results applicable to the CMB? Use them, and the Stephan-Boltzmann law for blackbody radiation, to show that $s \propto T^3$ where s is the entropy density of the CMB.

CMB black body spectrum Let $a(t)$ be the scale factor of an expanding universe. Assuming that the expansion is too slow to cause transitions between energy eigenstates with different energy, show that a particle of momentum p_0 at time t_0 will have momentum $p = p_0/a(t)$ at time t .

Use this to show that a thermal distribution of photons with temperature T_0 at time $t = t_0$ will still be thermal at time t , but with a temperature $T(t) = T_0/a(t)$. Hence show, using the result for the entropy density of the previous question, that the total entropy of the CMBR is conserved during the expansion.

Show that a thermal distribution of particles of a *non-relativistic* ideal gas will also remain thermal but with a temperature $T = T_0/a^2(t)$. Assuming that $PV = (\gamma - 1)U$ with $\gamma \neq 1$, and using the results of the previous question, deduce that $\gamma = 5/3$.

Recombination Neutral hydrogen atoms can be ionized by collisions with sufficiently energetic photons via the photo-ionization reaction $\gamma + H \rightarrow e^- + p^+$. For simplicity we assume that only the ground state of the hydrogen atom is relevant, so that the minimum energy that the photon must have to ionize the atom is the ground-state binding energy I . The reverse reaction is called 'recombination' and at equilibrium the forward and reverse reactions balance. Let n_H , n_e and n_p be the equilibrium number densities of hydrogen atoms, electrons and protons, respectively. In equilibrium the chemical potentials must balance. Since $\mu_\gamma = 0$ this requires

$$\mu_H = \mu_e + \mu_p.$$

Assuming charge neutrality, and that all particles other than the photons are non-relativistic, show that the equilibrium electron number density is given by

$$n_e^2 \approx n_H \left(\frac{2\pi m_e kT}{h^2} \right)^{\frac{3}{2}} e^{-I/kT}.$$

[N.B. A NR approx is adequate since $I \approx 13.6 \text{ eV} \ll m_e c^2 = 511 \text{ MeV}$. Note also that $g_H = 4$.] Consider the fractional ionization $X_e = n_e/n_B$, where $n_B = n_p + n_H = \eta n_\gamma$ (with the baryon-to-photon ratio $\eta \approx 10^{-9}$), to find Saha's equation

$$\frac{1 - X_e}{X_e^2} = \eta 16\pi\zeta(3) \left(\frac{kT}{2\pi m_e c^2} \right)^{3/2} e^{I/kT}.$$

Consider the limiting regimes (i) $kT \ll I$ and (ii) $I < kT < m_e c^2$ to roughly sketch X_e as a function of temperature T .

Electron-positron annihilation Because of electron-positron pair creation and annihilation $\gamma \leftrightarrow e^- + e^+$, photons in the cosmic radiation background will be in thermal equilibrium with electrons and positrons at some temperature T . For $kT \ll m_e c^2$ the number densities of electrons and positrons is negligible, but for $kT \gg m_e c^2$ their number densities are approximately those of an ideal Fermi gas of massless particles at zero chemical potential and temperature T . Discounting differences in spin degeneracies, a ideal Fermi gas of this type has an energy density that is 7/8 times that of a Bose gas of massless particles at the same temperature and chemical potential. Why must this number be less than unity? Show that the combined energy density of photons, electrons and positrons for $kT \gg m_e c^2$ is $\epsilon = (2\sigma/c)\mathcal{N}T^4$ where σ is the Stephan-Boltzmann constant and \mathcal{N} is an effective spin degeneracy factor which you should compute.

As the universe expands it cools adiabatically from a temperature $T \gg m_e c^2/k$ to a temperature $T \ll m_e c^2/k$. By equating the entropy of radiation at the higher and lower temperatures show that the later temperature of the radiation is a higher by a factor of $(11/4)^{1/3}$ than it would have been had the annihilation of electrons and positrons not occurred. How is this fact relevant to the cosmic electron-neutrino background which decoupled just before electron-positron annihilation?

Primordial nucleosynthesis* Discuss the effects of the following scenarios on helium production in the early universe (i.e. "higher" or "lower" or "little difference" to the helium-to-hydrogen ratio today):

- (i) The weak interaction was much stronger at the time of nucleosynthesis.
- (ii) The baryon-to-photon ratio was considerably larger than that estimated today.
- (iii) Nucleosynthesis took place during a matter-dominated epoch rather than in the radiation era.
- (iv) There is an extra species of light neutrino/antineutrino.

Since nucleosynthesis predictions are so precise, this type of consideration has resulted in strong constraints being placed on variants of the *standard model* and the *standard cosmology*.

Graviton decoupling Estimate the temperature T_g today of primordial gravitons which decoupled at $kT \sim M_{pl}c^2 \approx 10^{19} \text{ GeV}$ in (i) the standard model with $\mathcal{N} = 106.75$ for all $kT > 1 \text{ TeV}$, and (ii) a large grand unified (GUT) model which has $\mathcal{N} \approx 10^3$ for $kT > 10^{16} \text{ GeV}$. What was the graviton temperature at $kT = 1 \text{ MeV}$? [Hint: Compare with the neutrino decoupling calculation, noting that $T = T_\gamma$.]

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