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EXAMPLES IV

1. Electron-positron annihilation: Because of electron-positron pair creation and annihilation $\gamma \leftrightarrow e^- + e^+$, photons in the cosmic radiation background will be in thermal equilibrium with electrons and positrons at some temperature T. For $kT << m_e c^2$ the number densities of electrons and positrons is negligible, but for $kT >> m_e c^2$ their number densities are approximately those of an ideal Fermi gas of massless particles at zero chemical potential and temperature T. Discounting differences in spin degeneracies, a ideal Fermi gas of this type has an energy density that is 7/8 times that of a Bose gas of massless particles at the same temperature and chemical potential. Why must this number be less than unity? Show that the combined energy density of photons, electrons and positrons for $kT >> m_e c^2$ is $\epsilon = (2\sigma/c)\mathcal{N}T^4$ where σ is the Stephan-Boltzmann constant and $\mathcal{N}$ is an effective spin degeneracy factor which you should compute.

As the universe expands it cools adiabatically from a temperature $T >> m_e c^2/k$ to a temperature $T << m_e c^2/k$. By equating the entropy of radiation at the higher and lower temperatures show that the later temperature of the radiaton is a higher by a factor of $(11/4)^{1/3}$ than it would have been had the annihilation of electrons and positrons not occurred. How is this fact relevant to the cosmic electron-neutrino background which decoupled just before electron-positron annihilation?

2. Primordial nucleosynthesis: Discuss the effects of the following scenarios on helium production in the early universe (i.e. “higher” or “lower” or “little difference” to the helium-to-hydrogen ratio today):

(i) The weak interaction was much stronger at the time of nucleosynthesis.

(ii) The baryon-to-photon ratio was considerably larger than that estimated today.

(iii) Nucleosynthesis took place during a matter-dominated epoch rather than in the radiation era.

(iv) There is an extra species of light neutrino/antineutrino.

Since nucleosynthesis predictions are so precise, this type of consideration has resulted in strong constraints being placed on variants of the *standard model* and the *standard cosmology.*

3. Graviton decoupling: Estimate the temperature $T_{\rm g}$ today of primordial gravitons which decoupled at $kT \sim M_{pl}c^2 \approx 10^{19}$GeV in (i) the standard model with $\mathcal{N} = 106.75$ for all $kT > 1\,$TeV, and (ii) a large grand unified (GUT) model which has $\mathcal{N} \approx 10^3$ for $kT > 10^{16}$GeV. What was the graviton temperature at $kT = 1\,$MeV? [*Hint:* Compare with the neutrino decoupling calculation, noting that $T = T_\gamma$.]

4. The Jeans length – matter perturbations with non-zero pressure: The mass density perturbation equation for non-relativistic matter ($P << \rho c^2$) in the late universe ($t > t_{\rm eq}$) is

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - \left[4\pi G\bar{\rho} - c_s^2 k^2/a^2\right]\delta = 0\,, \qquad (*)$$

Suppose that a non-relativistic fluid dominates the universe with the equation of state $P \propto \rho^{4/3}$ (here, note that this polytrope has $w << 1$ which is *not* constant). Show that the sound speed can be written in the form $c_s^2(t) \equiv dP/d\rho = \bar{c}_s^2(t/t_0)^{-2/3}$ where $\bar{c}_s$ is a constant. Hence, seek power law solutions $\delta \propto t^\beta$ of eqn $(*)$ to find the growing and decaying modes ($a_0 = 1$):

$$\delta = A_k t^{n+} + B_k t^{n-} \qquad \text{with} \qquad n_\pm = -\tfrac{1}{6} \pm \left[\left(\tfrac{5}{6}\right)^2 - \bar{c}_s^2 t_0^2 k^2\right]^{1/2}.$$

Interpret your solutions in the two regimes $k < \tilde{k}_J$ and $k > \tilde{k}_J$ where $\tilde{k}_J = 5/(6\bar{c}_s t_0)$. Show that the quantity $\tilde{\lambda}_{\rm J} = 2\pi a/\tilde{k}_{\rm J}$ is effectively the sound speed horizon.

5. Cold dark matter power spectrum: Show that when a given wavenumber k crosses inside the horizon (Hubble radius), that is, $k = 2\pi aH/c$, the time $t_{\rm H}$ is given by

$$t_{\rm H} = \begin{cases} t_0\left(\frac{4\pi}{3kct_0}\right)^3 = t_0\left(\frac{2\pi}{kcH_0^{-1}}\right)^3, & \text{matter era } t > t_{\rm eq},\\ t_{\rm eq}\left(\frac{\pi(1+z_{\rm eq})^{-1}}{kct_{\rm eq}}\right)^2, & \text{radiation era } t < t_{\rm eq}.\end{cases}$$

or, equivalently, for $t > t_{\rm eq}$, we have $t_{\rm H}/t_0 = (k_0/k)^3$ where $k_0 = 2\pi/(cH_0^{-1})$.

(b) Assume (i) that the growing mode solutions for the cold matter perturbations $\delta_k(t)$ in the radiation and matter eras are respectively, $\delta_{\mathbf{k}}(t) \propto$ const. $(t < t_{\rm eq})$ and $\delta_{\mathbf{k}}(t) \propto (t/t_0)^{2/3}$ $(t > t_{\rm eq})$, and (ii) that the amplitude of *scale-invariant* perturbations $\delta_{\mathbf{k}}(t)$ from inflation at horizon-crossing is given by

$$\langle|\delta_{\mathbf{k}}(t_{\rm H})|^2\rangle = \mathcal{V}^{-1}\frac{2\pi^2}{k^3}\left(\frac{\hbar H_{\rm inf}}{M_{\rm pl}c^2}\right)^4 \equiv \mathcal{V}^{-1}A\left(\frac{k_0}{k}\right)^3 .$$

Show that the cold dark matter power spectrum for large-scale structure is predicted today to be

$$P(k) = \mathcal{V}\langle|\delta_{\mathbf{k}}(t_0)|^2\rangle = A \times \begin{cases} k\,, & k < k_{\rm eq}\,, \\ k_{\rm eq}\left(\frac{k_{\rm eq}}{k}\right)^3, & k > k_{\rm eq}\,, \end{cases} \qquad \text{where} \qquad \begin{aligned} k_{\rm eq} &= 2\pi a(t_{\rm eq})/\left(cH^{-1}(t_{\rm eq})\right) \\ &= \sqrt{1+z_{\rm eq}}\;k_0\,. \end{aligned}$$

6. Towards nonlinear evolution – pancake formation*: Suppose that matter above and below the xy-plane was given a uniform inward velocity boost $\pm v_z$ at a time $t_{\rm i} > t_{\rm eq}$ (as would happen if a straight cosmic string passed by). Using the Zel'dovich approximation with these initial conditions,

$$\ddot{\vec{\psi}} + 2\frac{\dot a}{a}\dot{\vec{\psi}} - 4\pi G\bar\rho\vec{\psi} = 0\,,$$

show that the solution for the comoving displacement $\psi(z,t)$ is (taking $a(t_{\rm i}) = 1$)

$$\psi(z,t) = -\frac{3}{5}v_z t_{\rm i}\left[\left(\frac{t}{t_{\rm i}}\right)^{2/3} - \left(\frac{t}{t_{\rm i}}\right)^{-1}\right],$$

Locate the comoving 'turnaround surface' $z(t)$ at which $\dot r_z = 0$, that is, the comoving distance in the z-direction at which matter stops expanding and breaks away from the Hubble flow. Given a thickness d_0 defined by this turnaround, show that the surface density of the 'pancake' or 'wake' today is

$$\sigma_0 \approx \rho_0 d_0 = \frac{v_z}{5\pi G t_0}\left(\frac{t_i}{t_0}\right)^{1/3} .$$

Why are such pancakes generic (and also filaments) – consider the superclusters in the SDSS survey?

7. The cosmic microwave sky and radiation perturbations: The *Sachs-Wolfe formula* for temperature anisotropies in the CMB is

$$\frac{\delta T}{T}(\mathbf{n}) = \tfrac{1}{4}\delta_\gamma + \frac{\mathbf{v.n}}{c} + \frac{\Phi}{3c^2}\,,$$

where each of these terms is evaluated at photon decoupling $t = t_{\rm dec}$, that is, on the surface of last scattering in the direction of the unit vector $\mathbf{n}$. Discuss the physical origin of each term. Why is there a factor of 1/4 multiplying δ_γ? As recombination is approached $t < t_{\rm dec}$, perturbation modes in the radiation fluid δ_γ below the horizon scale $(t < t_{\rm H})$ oscillate because of radiation pressure as

$$\delta_\gamma \approx A_{\mathbf{k}}\,\cos(k\tau/\sqrt{3})\,\exp(-k/k_{\rm D})\,,$$

where τ is conformal time $(a d\tau = dt)$ and $k_{\rm D}$ is a time-dependent damping scale due to photon diffusion. Use this approximate solution to qualitatively explain the series of *acoustic peaks* in the CMB temperature power spectrum observed by the WMAP satellite.

8. Density perturbation equation derived from fluid dynamics:** An alternative derivation of the density perturbation equation for Fourier modes $(*)$ begins with the equations of fluid dynamics in a gravitational potential:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0\,, \qquad \text{continuity}$$

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathrm{v} + \frac{1}{\rho}\nabla P + \nabla\Phi = 0\,, \qquad \text{Euler}$$

$$\nabla^2\Phi = 4\pi G\rho\,, \qquad \text{Poisson.}$$

The idea is to consider linear perturbations about the homogeneous expanding universe solution

$$\bar\rho(t) = \frac{\bar\rho(t_0)}{a(t)^3}\,, \qquad \mathbf{v} = \frac{\dot a}{a}\mathbf{r}\,, \qquad \nabla\Phi = \frac{4\pi G}{3}\bar\rho\mathbf{r}\,,$$

and obtain equations for the perturbed Fourier modes from which $(*)$ follows immediately,

$$\dot\delta_{\mathbf{k}} - \frac{i\mathbf{k}}{a}\cdot\mathbf{v}_{\mathbf{k}} = 0\,, \qquad \dot{\mathbf{v}}_{\mathbf{k}} + \frac{\dot a}{a}\mathbf{v}_{\mathbf{k}} - \frac{i\mathbf{k}}{a}(c_s^2\delta_{\mathbf{k}} + \phi_{\mathbf{k}}) = 0\,, \qquad \phi_{\mathbf{k}} = -\frac{4\pi G\bar\rho}{k^2}a^2\delta_{\mathrm{k}}\,.$$