Cosmology: Example Sheet 4 Michaelmas Term
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1. The energy density of a thermal bath of ultra-relativistic particles is given by

$$\rho = g_\star \frac{\pi^2}{30\hbar^3 c^3} (k_B T)^4$$

where the effective number of relativistic species is

$$g_\star(T) = \sum_{\text{bosons}} g_i + \frac{7}{8} \sum_{\text{fermions}} g_i \,.$$

Here the sum is only over particles in thermal equilibrium with $k_B T > mc^2$. When the temperature drops below a mass threshold, the corresponding species of particle is removed from the sum. When this happens, entropy considerations show that the temperature rises so that the combination $g_\star T^3$ remains constant.

At temperatures $k_B T \gg 2m_e c^2 \approx 1\,\text{MeV}$, electrons, positrons and photons are in equilibrium through the reaction $e^+ + e^- \leftrightarrow \gamma + \gamma$. Show that when electrons and positrons annihilate, the temperature of the thermal bath increases by a factor of $(11/4)^{1/3}$.

Neutrinos decoupled from the thermal bath before electron-positron annihilation. Assuming that the relic neutrino background remains relativistic today, what is its expected temperature?

Primordial gravitons are thought to decouple at $k_B T \approx M_{\text{pl}} c^2 \approx 10^{19}\,\text{GeV}$. Estimate the temperature today of the relic thermal graviton background assuming that only Standard Model particles are in thermal equilibrium with gravitons when they decouple. [The particles in the Standard Model add up to give $g_\star = 106.75$.] Should we observe this relic graviton background, why would we expect the temperature to be somewhat smaller than this?

2*. Qualitatively discuss the effects of the following scenarios on helium production in the early Universe. Do they result in a higher or lower helium-to-hydrogen ration today, or make little difference?

- The baryon-to-photon ratio was considerably larger during nucleosynthesis than today.
- There are extra species of ultra-relativistic particle during nucleosynthesis.
- The weak interaction was much stronger at the time of nucleosynthesis.
- Nucleosynthesis took place during a matter-dominated epoch rather than in the radiation era.

Since nucleosynthesis predictions are so precise, these kind of considerations give in strong constraints on what was going on in the first few minutes after the Big Bang.

3. A fluid in an expanding spacetime has energy density $\rho(\mathbf{x}, t)$, velocity $\mathbf{u}(\mathbf{x}, t) = H\,a\,\mathbf{x}(t) + \mathbf{v}(\mathbf{x}, t)$ and equation of state $P(\mathbf{x}, t) = \omega\rho(\mathbf{x}, t)$. It obeys the continuity equation

$$\frac{\partial \rho}{\partial t} + 3(1+\omega)H\rho + \frac{1}{a}(1+\omega)\nabla_{\mathbf{x}} \cdot (\rho\,\mathbf{v}) = 0\,,$$

the Euler equation

$$a\left(\frac{\partial}{\partial t} + \frac{\mathbf{v}}{a} \cdot \nabla\right)\mathbf{u} = -\frac{c^2}{\rho}\nabla P - \nabla\Phi\,,$$

and the Poisson equation

$$\nabla^2\Phi = \frac{4\pi G}{c^2}(1+3\omega)\rho\, a^2\,.$$

What is the spatially homogeneous background solution $\bar{\rho}(t)$ when $\mathbf{v} = 0$? What conditions should $\bar{P}$ and $\bar{\Phi}$ obey to ensure that the Euler and Poisson equations hold in this background? Show that these conditions are equivalent to the Raychaudhuri (acceleration) equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(1+3\omega)\rho\,.$$

The background solution is perturbed as $\rho(\mathbf{x},t) = \bar{\rho}(t)[1+\delta_\omega(\mathbf{x},t)]$ and $P(\mathbf{x},t) = \bar{P}(t)+\delta P(\mathbf{x},t)$ and $\Phi(\mathbf{x},t) = \bar{\Phi}(\mathbf{x},t)+\delta\Phi(\mathbf{x},t)$. Show that $\hat{\delta}_\omega(\mathbf{k},t)$, the Fourier transform of the contrast function $\delta_\omega(\mathbf{k},t)$, obeys the linearised equation

$$\ddot{\hat{\delta}}_\omega(\mathbf{k},t) + 2H\dot{\hat{\delta}}_\omega(\mathbf{k},t) + c_s^2(1+\omega)\left[\frac{k^2}{a^2} - (1+3\omega)k_{\rm J}^2\right]\hat{\delta}_\omega(\mathbf{k},t) = 0\,,$$

with c_s the speed of sound and the Jeans' wavenumber given by

$$k_{\rm J}^2 = \frac{4\pi G\bar{\rho}}{c_s^2 c^2}\,.$$

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4. The background energy density in matter scales as $\bar{\rho} = \rho_0/a^3$. For long wavelengths, where we may ignore pressure, the evolution of matter perturbations is governed by

$$\ddot{\delta} + 2H\dot{\delta} = \frac{4\pi G\bar{\rho}}{c^2}\delta\,.$$

By viewing $H(t)$ and $\delta(t)$ as functions of the scale factor $a(t)$, show that this equation is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}a}\left(a^3 H\frac{\mathrm{d}\delta}{\mathrm{d}a}\right) = \frac{4\pi G\rho_0}{c^2}\frac{\delta}{H\, a^2}\,. \tag{1}$$

Show that $\delta_1 = H$ is a solution provided that the expansion of the Universe is due to any combination of matter, curvature and a cosmological constant.

Show that the second solution is given by

$$\delta_2 = H\, g \quad \text{with} \quad a^3 H^3\frac{\mathrm{d}g}{\mathrm{d}a} = \text{constant}\,.$$

What is the a dependence of each of these solutions when the expansion of the universe is dominated by matter? By curvature? By a cosmological constant?

5. The purpose of this question is to solve (1) in a flat universe whose expansion is due to matter and radiation (*i.e.* with vanishing curvature and cosmological constant).

Define $x = a/a_{\rm eq}$ where $a_{\rm eq}$ is the scale factor at matter-radiation equality. Show that we can write

$$H(x) = \frac{A}{x^2}\sqrt{1+x} \quad \text{with} \quad A^2 \equiv \frac{8\pi G\,\rho_{\rm eq}}{3c^2}\,,$$

where $\rho_{\rm eq}$ is the energy density of matter at matter-radiation equality. Hence show that (1) is equivalent to

$$\delta'' + \left[\frac{1}{x} + \frac{1}{2(1+x)}\right]\delta' = \frac{3}{2}\frac{\delta}{x(1+x)}\,,$$

where the prime denotes differentiation with respect to x. Show that this equation is solved by

$$\delta_1 = 1 + \frac{3}{2}x\,.$$

The second solution to this equation is more fiddly: it turns out to be

$$\delta_2 = \left(1 + \frac{3}{2}x\right)\log\left(\frac{\sqrt{1+x}+1}{\sqrt{1+x}-1}\right) - 3\sqrt{1+x}\,.$$

Identify the growing and decaying modes at early and late times.