EXAMINATION REVISION

The following questions are based largely on past Statistical Physics and Cosmology *examination questions, but with the emphasis modified and supplementary material added.*

SECTION 1 *(Four short questions)*

Question 1

Consider a one-dimensional model universe with "stars" distributed at random on the x-axis, and choose the origin to coincide with one of the stars; call this star the "home star". Home-star astronomers have discovered that all other stars are receding from them with a velocity $v(x)$, that depends on the position x of the star. Assuming non-relativistic addition of velocities, show how the assumption of homogeneity implies that $v(x) = H_0 x$ for some constant H_0.

In attempting to understand the history of their one-dimensional universe, home-star astronomers seek to determine the velocity $v(t)$ at time t of a star at position $x(t)$. Assuming homogeneity, show how $x(t)$ is determined in terms of a scale factor $a(t)$ and hence deduce that $v(t) = H(t)x(t)$ for some function $H(t)$. What is the relation between $H(t)$ and H_0?

Question 2

The number density $n = N/V$ for a photon gas in equilibrium is given by the formula

$$n = \frac{8\pi}{c^3} \int \frac{\nu^2 d\nu}{e^{h\nu/kT} - 1} \,.$$

where ν is the photon frequency. By considering the substitution $x = h\nu/kT$ show that the photon number density can be expressed in the form

$$n = \alpha T^3 \,,$$

where the constant α need not be evaluated explicitly.

State the equation of state for a photon gas and explain why the chemical potential of the photon vanishes. Assuming the photon energy density $\epsilon = E/V = (4\sigma/c^2)T^4$, use the first law $dE = TdS - PdV + \mu dN$ to show that the entropy density is given by

$$s = S/V = \frac{16\sigma}{3c} T^3 \,.$$

Hence explain why when photons are in equilibrium at early times in our universe their temperature varies inversely with the scalefactor: $T \propto a^{-1}$.

Question 3

A spherically symmetric star has pressure $P(r)$ and mass density $\rho(r)$, where r is the distance from the star's centre. Stating without proof any theorems you may need, show that mechanical equilibrium implies the Newtonian pressure support equation

$$P' = -\frac{Gm\rho}{r^2},$$

where $m(r)$ is the mass within radius r and $P' = dP/dr$. Hence, or otherwise, show that

$$\left(\frac{r^2}{\rho}P'\right)' = -4\pi G r^2 \rho\,. \qquad (*)$$

Given an equation of state $P = P(\rho)$, specify some appropriate boundary conditions for finding solutions to $(*)$ and state reasons for your choice.

Question 4

In equilibrium, the number density of a non-relativistic particle species is given by

$$n = g_{\rm s}\left(\frac{2\pi mkT}{h^2}\right)^{3/2} e^{(\mu - mc^2)/kT},$$

where m is the mass, μ is the chemical potential and $g_{\rm s}$ is the spin degeneracy. At around $t = 100$ seconds, deuterium D forms through the nuclear fusion of nonrelativistic protons p and neutrons n via the interaction:

$$p + n \leftrightarrow D\,.$$

What is the relationship between the chemical potentials of the three species? Show that the ratio of their number densities can be expressed as

$$\frac{n_D}{n_n n_p} = \left(\frac{\pi m_p kT}{h^2}\right)^{-3/2} e^{B_D/kT},$$

where the deuterium binding energy is $B_D = m_n + m_p - m_D$ and you may take $g_D = 4$. Now consider the fractional densities $X_a = n_a/n_B$, where n_B is the baryon number of the universe, to re-express the ratio above in the form

$$\frac{X_D}{X_n X_p}$$

which incorporates the baryon-to-photon ratio η of the universe. [You may assume that the photon density is $n_\gamma = \frac{16\pi\zeta(3)}{(hc)^3}(kT)^3$.] From this expression, can you explain why deuterium does not form until well below the temperature $kT \approx B_D$?

SECTION 2 *(Two long questions)*

Question 5

(a) Consider a homogeneous and isotropic universe with mass density $\rho(t)$, pressure $P(t)$ and scale factor $a(t)$. As the universe expands its energy E decreases according to the thermodynamic relation $dE = -PdV$ where V is the volume. Deduce the fluid conservation law

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + P/c^2\right) .$$

Apply the conservation of total energy (kinetic plus gravitational potential) to a test particle on the edge of a spherical region in this universe to obtain the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - kc^2 ,$$

where k is a constant. State clearly any assumptions you have made.

(b) Our universe is believed to be flat ($k = 0$) and filled with two major components: pressure-free matter ($P_{\rm M} = 0$) and dark energy with equation of state $P_{\rm Q} = -\rho_{\rm Q}$ where the mass densities today ($t = t_0$) are given respectively by $\rho_{\rm M0}$ and $\rho_{\rm Q0}$. Assume that each component independently satisfies the fluid conservation equation to show that the total mass density can be expressed as

$$\rho(t) = \frac{\rho_{\rm M0}}{a^3} + \rho_{\rm Q0} ,$$

where we have set $a(t_0) = 1$.

Now consider the substitution $b = a^{3/2}$ in the Friedmann equation to show that the solution for the scalefactor can be written in the form

$$a(t) = \beta(\sinh \alpha t)^{2/3} ,$$

where α and β are constants. [*Hint:* Recall that $\int dx/\sqrt{x^2+1} = \sinh^{-1} x$.] For $a(t_0) = 1$, specify α and β in terms of $\rho_{\rm M0}$, $\rho_{\rm Q0}$ and t_0. Show that the scale factor $a(t)$ has the expected behaviour for an Einstein-de Sitter universe at early times ($t \to 0$) and that the universe accelerates at late times ($t \to \infty$).

Question 6

In the Zel'dovich approximation, matter trajectories can be described by $\mathbf{r}(\mathbf{q},t) = a(t)[\mathbf{q} + \vec{\psi}(\mathbf{q},t)]$, where $\mathbf{q}$ is the unperturbed comoving position in a homogeneous flat universe and $\vec{\psi}$ is the comoving displacement. The particle equation of motion is then

$$\ddot{\mathbf{r}} = -\nabla\Phi - \frac{1}{\rho}P\,,$$

where ρ is the perturbed density, P is the pressure ($P \ll \rho c^2$), and the Newtonian potential Φ satisfies the Poisson equation $\nabla^2\Phi = 4\pi G\rho$.

For small comoving displacements $|\vec{\psi}| \ll |\mathbf{q}|$, show that the fractional density perturbation and the pressure gradient respectively are given by

$$\delta \equiv \frac{\rho(\mathbf{r},t) - \bar{\rho}}{\bar{\rho}(t)} \approx -\nabla_q \cdot \vec{\psi}\,, \qquad \nabla P \approx -\bar{\rho}\frac{c_{\rm s}^2}{a}\nabla_q^2\vec{\psi}\,,$$

where $\bar{r}ho(t)$ is the uniform background density, $c_{\rm s}^2 = dP/d\rho$ is the sound speed and you may assume that the Jacobian $|\partial r_k/\partial q_j|^{-1} = |\delta_{jk} + \partial\psi_k/\partial q_j|^{-1} \approx 1 - \nabla_q \cdot \vec{\psi}$.

Use these expressions to obtain an evolution equation for $\vec{\psi}$ by integrating the Poisson equation and substituting the result into the particle equation of motion. [*Hint: You may assume the Raychaudhuri equation for the homogeneous background* $\ddot{a}/a = -4\pi G\bar{\rho}/3$ ($P \ll \rho c^2$).]

Consider the Fourier expansion $\delta(\mathbf{x},t) = \sum_{\mathbf{k}} \delta_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{x})$ (with $\mathbf{r} = a\mathbf{x}$) and hence obtain the evolution equation for the density perturbation mode $\delta_{\mathbf{k}}$:

$$\ddot{\delta}_{\mathbf{k}} + 2\frac{\dot{a}}{a}\dot{\delta}_{\mathbf{k}} - \left(4\pi G\bar{\rho} - \frac{c_{\rm x}^2 k^2}{a^2}\right)\delta_{\mathbf{k}} = 0\,.$$

Discuss the qualitative behaviour of solutions to this equation as a function of the comoving wavenumber $k = |\mathbf{k}|$.