

Example Sheet 1

1. The population of a certain insect, $N(t)$, is modelled by the ODE

$$\frac{dN}{dt} = rN \left(1 - \frac{N}{K} \right) - p(N)$$

where $p(N)$ is the sigmoidal function $p(N) = BN/(A + N)$.

- (a) Give suggestions as to the meaning of the terms in this equation.
 - (b) Show by rescaling that the dynamics depends only on the two parameters $\alpha = A/K, \beta = B/rK$ [Hint: focus on simplifying the logistic terms first].
 - (c) Investigate how many *positive* steady states there are, i.e. fixed points with $N > 0$. Sketch the (α, β) plane, dividing it into regions where there are zero, one and two positive steady states.
 - (d) What is the number of *stable* solutions, including the fixed point at $N = 0$, in each region? [Hint: investigating $N = 0$ stability will be enough to deduce the rest].
2. Consider an infectious disease in some population where immunity wanes with time (and ignore births and deaths). This can be captured by the SIRS model of an infectious disease: start with the SIR model from lectures, but recovered individuals (R) can lose their immunity and become susceptible again at a rate γR . Using that the total population size remains constant (N), reduce the system of equations to two, for S and I (the populations of susceptibles and infecteds respectively).
- (a) Give an expression for the basic reproduction ratio R_0 and show that when $R_0 > 1$ the system has a stable fixed point where both $S > 0$ and $I > 0$.
 - (b) Find the nullclines and sketch trajectories in the $S - I$ plane. What happens in the long term?
3. A fungal disease is introduced into an isolated population of frogs. Without disease, the population size n would obey the (normalised) logistic equation $\dot{n} = n(1 - n)$, where the dot denotes differentiation with respect to time. The disease causes death at rate d and there is no recovery. The disease transmission rate is β and in addition, offspring of infected frogs are also infected from birth.
- (a) Briefly explain why the population sizes of the uninfected n_U and infected n_I frogs now satisfy

$$\begin{aligned} \dot{n}_U &= n_U [1 - n_U - (1 + \beta)n_I] \\ \dot{n}_I &= n_I [(1 - d) - (1 - \beta)n_U - n_I] . \end{aligned}$$

- (b) The population starts at the disease-free population size ($n_U = 1$) and a small number of infected frogs are introduced. Show that the disease will successfully invade iff $\beta > d$.
- (c) By finding all the equilibria in $n_U, n_I \geq 0$ and considering their stability, find the long term outcome for the frog population. Specify d as a function of β at any boundaries.
- (d) Plot the long-term steady *total* population size as a function of d for fixed β , and note that an intermediate mortality rate is actually the most harmful for overall population numbers. Explain why this is the case.

4. Let n_P, n_Z be the normalised populations of phytoplankton and zooplankton respectively. The system is modelled by the following differential equations, where the constants ϵ , b and c are positive:

$$\begin{aligned}\frac{dn_P}{dt} &= b n_P(1 - n_P) - n_Z \frac{n_P^2}{\epsilon^2 + n_P^2} \\ \frac{dn_Z}{dt} &= c n_Z \frac{n_P^2}{\epsilon^2 + n_P^2} - n_Z.\end{aligned}$$

- Briefly explain the meaning of each term in these equations.
- Assume that $c > 2$ and $\epsilon \ll 1$. By finding the nullclines and carefully considering how they intersect show that there is one fixed point where both $n_P > 0$ and $n_Z > 0$ and that it is stable.
- The system is perturbed by the intrinsic growth rate b increasing as a result of a rise in the temperature of the sea. Consider how this changes the nullclines. If the system was at the stable fixed point before, what happens after the temperature rises? Deduce the possibility of excitable behaviour in which there can be a spike in the plankton population sizes.