Examples Sheet 1

1. The population of a certain insect, $N(t)$, is modelled by the ODE

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - p(N)$$

where $p(N)$ is the sigmoidal function $p(N) = BN/(A+N)$.

(a) Give suggestions as to the meaning of the terms in this equation.

(b) Show by rescaling that the dynamics depends only on the two parameters $\alpha = A/K, \beta = B/rK$ [Hint: focus on simplifying the logistic terms first].

(c) Investigate how many **positive** steady states there are, i.e. fixed points with $N > 0$. Sketch the (α, β) plane, dividing it into regions where there are zero, one and two positive steady states.

(d) What is the number of *stable* solutions, including the fixed point at $N = 0$, in each region? [Hint: investigating $N = 0$ stability will be enough to deduce the rest]

2. A variant of the Hutchinson-Wright equation given by this equation:

$$\frac{dx(t)}{dt} = \alpha\left[x(t-T) - x(t)^2\right] ,$$

where $\alpha, T > 0$. Give a brief interpretation of what this might represent in terms of population dynamics. Show that the constant solution with $x(t) = 1$ is stable for all $\alpha, T > 0$. [Hint: show that any 's' must have negative real part (as in lectures, the growth exponent of a small perturbation).]

3. Circulating red blood cells (RBCs) are destroyed in the spleen and created in the bone marrow in proportion to the number destroyed on the previous day.

(a) Treat days as discrete time units, and let R_n be the number of RBCs in circulation on day n, let M_n be the number produced by marrow on day n, let f be the fraction of RBCs removed by the spleen every day, and let γ be the number produced on day n for each cell lost on day $n-1$.

Write down equations for R_{n+1} and M_{n+1} in terms of R_n and M_n, and show that

$$R_n = A\lambda_1^n + \beta\lambda_2^n,$$

where

$$\lambda_{1,2} = \frac{1 - f \pm \sqrt{(1-f)^2 + 4\gamma f}}{2}.$$

Deduce that the RBC count may fluctuate, but that it will remain constant at large times if and only if $\gamma = 1$.

(b) Start again, but this time formulate the model for RBC count as a delay differential equation for $R(t)$, where the production of new cells is proportional to $R(t-T)$. Now f should be a rate and γ is still the ratio of cells made for each one lost. Show that there exists a solution of the form $R(t) = Ce^{\lambda f t}$ if

$$\frac{1}{\gamma}(\lambda + 1) = e^{-\lambda f T}.$$

Show graphically that this equation has a positive real root if and only if $\gamma > 1$, and interpret this result.

4. The population density $n(a,t)$ of individuals of age a at time t satisfies

$$\frac{\partial n(a,t)}{\partial t} + \frac{\partial n(a,t)}{\partial a} = -\mu(a)n(a,t), \quad \text{with} \quad n(0,t) = \int_0^\infty b(a)n(a,t)da,$$

where $\mu(a)$ is the age-dependent death rate and $b(a)$ is the birth rate per individual of age a.

Using the standard similarity solution $n(a,t) = e^{\gamma t}\, r(a)$ for each of the examples below, give (i) the mean number of offspring, (ii) the population growth rate γ (solve where possible otherwise give an implicit expression) (iii) the value of the birth rate parameter B for which there is neither growth nor decay and sketch the age-profile of the population in this case.

(a) The birth rate $b(a)$ is a constant B for $a_1 < a < a_2$ and zero otherwise. The death rate $\mu(a)$ is a constant d for $a > a_2$ and zero otherwise.

(b) Individuals give birth only at age a^*: $b(a) = B\,\delta(a - a^*)$. The death rate $\mu(a)$ is a constant d for all ages.

(c) The birth rate $b(a)$ is a constant B for all ages. All individuals die at age A. [Hint: in this extreme case, rather than using $\mu(a)$, just reformulate the standard approach slightly.]

5. Consider the difference equation with delay (note the $n-1$):

$$x_{n+1} = r x_n (1 - x_{n-1}).$$

This is the discrete equivalent of the delay differential equation for population discussed in lectures. Show that the fixed point $x^* = 1 - r^{-1}$ is unstable if r is sufficiently large. [Hint: cannot use standard results for stability of 1D maps as this is NOT a 1D map!] Show that when r takes its marginal value for instability, the linearised stability problem has a periodic solution (to be determined).

6. A discrete-time model for breathing is given by

$$\begin{aligned} V_{n+1} &= \alpha\, C_{n-k} \\ C_{n+1} - C_n &= \gamma - \beta\, V_{n+1} \end{aligned}$$

where V_n is the volume of each breath at time step n and C_n is the concentration of carbon dioxide in the blood at the end of time step n. (This model was presented in lectures and we found and analysed the stability of the steady state when $k = 0$ and $k = 1$.)

For general (integer) $k > 1$, by seeking parameter values when the modulus of a perturbation to the steady state is constant, show that the range of parameters where the solution is stable is

$$0 < \alpha\beta < 2\sin\left(\frac{\pi}{4k+2}\right).$$

Notice how this range shrinks as the time lag k increases. What is the periodicity of the constant-modulus solution at the upper end of this range?

7. A simple model of two competing populations eating the same food takes the form

$$\dot{N}_1 = r_1 N_1 \left(1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_2}\right),$$

$$\dot{N}_2 = r_2 N_2 \left(1 - \frac{N_2}{K_2} - b_{21}\frac{N_1}{K_1}\right).$$

Rescale the equations to simplify them, and show that the solutions depend only on $\rho = r_2/r_1, b_{12}$ and b_{21}.

Now assume that $\rho, b_{12}, b_{21} > 0$. Find all the physically relevant fixed points and determine their stability. Give conditions on the coefficients such that there is a stable state of *coexistence*, with $N_1, N_2 > 0$.

8. Consider this 'harvesting' model:

$$\dot{u} = u(1-v) - \epsilon u^2 - f$$
$$\dot{v} = -\alpha v(1-u),$$

with constants $\alpha > 0, f > 0$ and $0 < \epsilon < 1/2$.

Find all the biologically relevant fixed points of this system, and investigate their stability, distinguishing between different ranges of f:

(a) $0 < f < \epsilon$,
(b) $\epsilon < f < 1 - \epsilon$
(c) $1 - \epsilon < f < 1/(4\epsilon)$
(d) $1/(4\epsilon) < f$.

In each case, sketch trajectories in the $u - v$ phase-plane, and discuss what would happen to the predator and prey populations in practice.

Note that for this model something odd happens at $u = 0$. Comment on this, and discuss how the model might be improved in this respect.

In addition to the examples sheets, students are encouraged to do the exercises given in lectures.