Mathematical Biology: Example Sheet 2

David Tong, January 2025

1. Let x, y be the normalised populations of phytoplankton and zooplankton respectively. The system is modelled by the following differential equations, where the constants ϵ, b and c are positive,

$$\frac{dx}{dt} = b\,x(1-x) - y\frac{x^2}{\epsilon^2 + x^2},$$
$$\frac{dy}{dt} = c\,y\frac{x^2}{\epsilon^2 + x^2} - y.$$

Briefly explain the meaning of each term in these equations.

Assume that $c > 2$ and $\epsilon \ll 1$. By finding the nullclines and carefully considering how they intersect, show that there is one fixed point where both $x > 0$ and $y > 0$ and that it is stable.

The system is perturbed by the intrinsic growth rate b increasing as a result of a rise in the temperature of the sea. Consider how this changes the nullclines. If the system was at the stable fixed point before, what happens after the temperature rise? Deduce the possibility of excitable behaviour in which there can be a spike in the plankton population sizes.

2. Circulating red blood cells (RBCs) are destroyed in the spleen and created in the bone marrow in proportion to the number destroyed on the previous day.

a) Treat days as discrete time units, and let R_n be the number of RBCs in circulation on day n, let M_n be the number produced by marrow on day n, let f be the fraction of RBCs removed by the spleen every day, and let γ be the number produced on day n for each cell lost on day $n-1$.

Write down equations for R_{n+1} and M_{n+1} in terms of R_n and M_n, and show that

$$R_n = A\lambda_+^n + B\lambda_-^n$$

where

$$\lambda_\pm = \frac{1 - f \pm \sqrt{(1-f)^2 + 4\gamma f}}{2}\ .$$

Deduce that the RBC count may fluctuate, but that it will remain constant at large times if and only if $\gamma = 1$.

b) Start again, but this time formulate the model for RBC count as a delay differential equation for $R(t)$, where the production of new cells is proportional to $R(t-T)$. Now f should be a rate and γ is still the ratio of cells made for each one lost. Show that there exists a solution of the form $R(t) = Ce^{\lambda f t}$ if

$$\frac{1}{\gamma}(\lambda + 1) = e^{-\lambda f T}.$$

Show graphically that this equation has a positive real root if and only if $\gamma > 1$, and interpret this result.

3. Consider the difference equation with delay (note the $n-1$),

$$x_{n+1} = r x_n (1 - x_{n-1}).$$

This is the discrete equivalent of the delay differential equation for population.

Show that the fixed point $x^* = 1 - r^{-1}$ is unstable if r is sufficiently large. Show that when r takes its marginal value for instability, the linearised stability problem has a periodic solution (to be determined).

4. A discrete-time model for breathing is given by

$$V_{n+1} = \alpha\, C_{n-k},$$
$$C_{n+1} - C_n = M - \beta\, V_{n+1},$$

where V_n is the volume of each breath at time step n and C_n is the concentration of carbon dioxide in the blood at the end of time step n. (This model was presented in lectures and we found and analysed the stability of the steady state when $k = 0$ and $k = 1$.)

For general (integer) $k > 1$, by seeking parameter values when the modulus of a perturbation to the steady state is constant, show that the range of parameters where the solution is stable is

$$0 < \alpha\beta < 2\sin\left(\frac{\pi}{4k+2}\right).$$

Notice how this range shrinks as the time lag k increases. What is the periodicity of the constant-modulus solution at the upper end of this range?

5. The concentration of bacteria $C(x,t)$ in a thin channel of length L obeys the diffusion equation in one dimension with constant diffusivity D. Initially, the concentration of bacteria is $C(x,0) = C_0 + (C_1 - C_0)\, x/L$ (this was achieved by keeping the concentration at the ends $x = 0$ and $x = L$ at $C = C_0$ and $C = C_1$ respectively and waiting for the system settle to its steady state).

At $t = 0$ the ends of the channel are both suddenly sealed. There is no flux (i.e. $\partial C/\partial x = 0$) through $x = 0$ or $x = L$ for $t > 0$. Use separation of variables to find the bacterial concentration in $0 \le x \le L$ for $t > 0$, and hence show that

$$C(x,t) = C_0 + (C_1 - C_0)\left[\frac{1}{2} - \sum_{n\geq 1,\,\mathrm{odd}} \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)\exp\left(-\frac{n^2\pi^2}{L^2}Dt\right)\right] .$$

Sketch the bacterial distribution for a range of times, paying particular attention to the shape at very early and late times. Explain what is happening.

6a) In an axisymmetric cylindrical geometry, the diffusion equation is

$$\frac{\partial C}{\partial t} = \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right)$$

where $D > 0$ is a constant. Find and sketch the similarity solution of the diffusion equation which satisfies $C \to 0$ as $r \to \infty$ and

$$\int_0^\infty 2\pi r\, C(r,t)\, dr = M$$

where $M > 0$ is a constant, by assuming that the solution is of the form

$$C(r,t) = \eta F(\xi) \quad \text{with } \eta = \frac{M}{Dt} \text{ and } \xi = \frac{r}{(Dt)^{1/2}} .$$

Hence show that C is Gaussian.

b) Find the analogous similarity solution in a spherically symmetric geometry, where $C(r,t)$ satisfies

$$\frac{\partial C}{\partial t} = \frac{D}{r^2}\frac{\partial}{\partial r}\left((r^2\frac{\partial C}{\partial r}\right) \quad \text{with} \quad \int_0^\infty 4\pi r^2\, C(r,t)\, dr = M .$$

and $C \to 0$ as $r \to \infty$. [*Hint: Start by finding appropriate η (independent of r) and ξ (proportional to r)*]