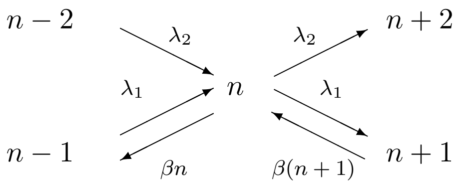
C6b Lent Term 2011

Mathematical Tripos Part II P.H. Haynes

Mathematical Biology – Examples Sheet 2

[Please communicate any errors in this sheet to `phh@damtp.cam.ac.uk`. Some further material for this course is available at `http://www.damtp.cam.ac.uk/user/phh/mathbio.html`.]

1. Consider a birth and death process in which births can give rise to either one or two offspring, with probability λ_1 and λ_2 respectively, while the probability of death per individual is β; i.e. consider the system



Write down the master equation (or probability balance equation) for this system, and show that the generating function

$$\phi(s,t) = \sum_{n=0}^{\infty} s^n P(n,t) \quad \text{satisfies the equation:}$$

$$\frac{\partial \phi}{\partial t} = \left[\lambda_1(s-1) + \lambda_2(s^2-1)\right]\phi - \beta(s-1)\frac{\partial \phi}{\partial s}.$$

Use this equation in the steady state to show that

$$\langle n \rangle = \frac{1}{\beta}(\lambda_1 + 2\lambda_2)$$

$$\text{and} \quad \sigma^2 = \frac{1}{\beta}(\lambda_1 + 3\lambda_2).$$

2. Consider a birth-death process described by the following master equation:

$$\frac{\partial P}{\partial t} = \lambda(P(n-1) - P(n)) + \beta(f(n+1)P(n+1) - f(n)P(n)), f(n) = n(n-1).$$

(i) Give an explanation of the terms on the right hand side.

(ii) Show that the equation satisfied by the generating function $\phi(s,t) \equiv \sum_{n\geq 0} s^n P(n,t)$ is

$$\frac{\partial \phi}{\partial t} = \lambda(s-1)\phi + \beta\left((s-s^2)\frac{\partial^2 \phi}{\partial s^2}\right).$$

(iii) Use the equation for ϕ in the steady state, or the master equation directly, to obtain equations for $\langle n^2 \rangle$ and $\langle n^3 \rangle$, in terms of $\langle n \rangle$ and $r = \lambda/\beta$ (do not try to evaluate $\langle n \rangle$ itself).

(iv) With $\langle n \rangle$ unknown this system of equations is not closed. Nonetheless show that the variance $\sigma^2 = \langle n^2 \rangle - \langle n \rangle^2 \leq r + \frac{1}{4}$. Show also, using the inequality $\langle n^2 \rangle \geq \langle n \rangle^2$, that $\langle n \rangle \leq (1 + \sqrt{1+4r})/2$.

3. Consider the following Fokker-Planck equation (FPE) for a certain random process:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}(x(1-x)P) + \frac{1}{\alpha}\frac{\partial^2}{\partial x^2}(x^2 P).$$

(i) Show directly from the equation that

$$\frac{\partial}{\partial t}\left\langle x^k \right\rangle = k\left[(1 + (k-1)\alpha^{-1})\left\langle x^k \right\rangle - \left\langle x^{k+1} \right\rangle\right].$$

(ii) Solve the FPE in the steady state, showing that $P(x) = Cx^{\alpha-2}e^{-\alpha x}$ for some constant C. Give C in terms of an integral (which need not be evaluated). What restriction must be placed on α for the solution to make sense?

(iii) Use the steady-state solution for $P(x)$ to verify that for $k = 0, 1, 2, ...$

$$\left\langle x^{k+1} \right\rangle = \frac{k+\alpha-1}{\alpha}\left\langle x^k \right\rangle.$$

Use this result to determine $\left\langle x^k \right\rangle, k = 1, 2, 3$.

4. A two-population dynamic model has the following transition probabilities:

$$(m,n) \rightarrow (m+1,n) : a + \lambda_1 n; \quad (m,n) \rightarrow (m-1,n) : \beta_1 m;$$
$$(m,n) \rightarrow (m,n+1) : \lambda_2 m; \quad (m,n) \rightarrow (m,n-1) : \beta_2 n.$$

(i) Construct a master equation for $P(m,n,t)$ and use it to derive equations for $\langle m \rangle, \langle n \rangle$. Show that, under appropriate conditions on the coefficients a, λ_i, β_i, there is a fixed point with $\langle m \rangle, \langle n \rangle > 0$, and give the conditions for this to be stable.

(ii) Write down a Fokker-Planck equation describing small perturbations from this fixed point using the method in lectures, with the matrix A the jacobian at the fixed point and the matrix B evaluated at the fixed point. Write down the equation for the covariance matrix C.

(iii) Show that the equation for C has a stable fixed point, satisfying $AC + CA^T + fB = 0$, and calculate C in this case.

5. For the one-dimensional random walk problem discussed in lectures, i.e. the process

$$n \xrightarrow{\lambda} n+1$$

use the master equation and the Fokker-Planck equation to calculate the time development of the fourth moment $(\langle n^4 \rangle \; or \; \langle x^4 \rangle)$. Explain carefully why the two answers are not the same, and why the ratio of the two answers tends to unity at large times.

6. (This question concerns a random walk in two dimensions.)

(i) A particle starts at the origin (0, 0) at time $t = 0$. It moves (a) on a square grid from (m, n) with equal probability λ per unit time to one of the four adjacent points $(m \pm 1, n \pm 1)$, (b) on a triangular grid from $X_{m,n} = (m + \frac{n}{2}, \frac{n\sqrt{3}}{2})$ with the same probability λ to one of the six adjacent points

$$X_{m\pm1,n}, X_{m,n\pm1}, X_{m-1,n+1}, X_{m+1,n-1}.$$

Write down the master equation for the probability distribution in each case and derive the corresponding Fokker-Planck equation in the $x - y$ plane.

(ii) For the square grid, the isotropy is broken due to the existence of an additional probability λ per unit time to jump from (m, n) to $(m + 1, n + 1)$and $(m - 1, n - 1)$. Find the Fokker-Planck equation in this case and calculate $\langle m^2 + n^2 \rangle$ directly from the master equation and $\langle x^2 + y^2 \rangle$ from the Fokker-Planck equation.