Examples Sheet 3

1. Consider a stochastic model of a population where the death rate is β per capita (so total rate βn), and M individuals are added at the same time at rate λ (where M is a positive integer).

   (i) Give the master equation and find the mean and variance of the population size at steady state. (Note: you might have already done this as an exercise from lectures, in which case it is acceptable just to quote the results.)

   (ii) Write down the Fokker-Planck equation for this system. Use this to find the mean and variance of the population.

2. A particle starts at the origin $(0, 0)$ at time $t = 0$. In each of the cases below, derive the corresponding Fokker-Planck equation (in x, y). The particle moves in a random walk, where it takes steps with the step sizes below, each with probability rate λ:

   (i) A square grid:
   $(-1, 0)$, $(+1, 0)$, $(0, -1)$ and $(0, +1)$

   (ii) A triangular grid:
   $(+1, 0)$, $(-1, 0)$, $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$, $(-\frac{1}{2}, +\frac{\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$

   (iii) Square grid with bonus diagonal steps (breaking isotropy):
   $(-1, 0)$, $(+1, 0)$, $(0, -1)$, $(0, +1)$, $(+1, +1)$ and $(-1, -1)$

   For case (iii), write down the master equation for $P_{m,n}$ (where m, n are the discretised x, y) and use it to find $\langle m^2 + n^2 \rangle$. Also calculate $\langle x^2 + y^2 \rangle$ from the Fokker-Planck equation.

3. A two-population dynamic model has the following transition probability rates:

$$\begin{aligned}
(m, n) \to (m + 1, n) \quad &: \quad \mu + \lambda_1 n \\
(m, n) \to (m - 1, n) \quad &: \quad \beta_1 m \\
(m, n) \to (m, n + 1) \quad &: \quad \lambda_2 m \\
(m, n) \to (m, n - 1) \quad &: \quad \beta_2 n.
\end{aligned}$$

   (i) Construct a master equation for $P_{m,n}$ and use it to derive equations for the time evolution of $\langle M \rangle , \langle N \rangle$. Find conditions on the parameters (μ, λ_1, λ_2, β_1 and β_2) for there to be a *stable* fixed point with $\langle M \rangle , \langle N \rangle > 0$.

   (ii) Write down the Fokker-Planck equation using the method in lectures. Now consider small fluctuations near the fixed point found above, and approximate A as linear in x and B as constant. Show that the covariance matrix C satisfies

$$\dot{C} = a\, C + C\, a^T + b$$

   where a and b are matrices which should be given.

   (iii) For the special case when $\lambda_1 = \lambda_2 = \lambda$ and $\beta_1 = \beta_2 = \beta$ consider the equation for $\dot{C}$ in components. Show that there is a fixed point for C (which need not be explicitly found) and that it is stable. Explain what this means for this model.

   * (iv) Same as (iii) but with general λ_1, λ_2, β_1 and β_2.

4. As in the example in lectures, the concentration of bacteria $C(x,t)$ in a thin channel of length L obeys the diffusion equation in one dimension with constant diffusivity D. Initially, the concentration of bacteria is $C(x,0) = C_0 + (C_1 - C_0)\,x/L$ (this was achieved by keeping the concentration at the ends $x = 0$ and $x = L$ at $C = C_0$ and $C = C_1$ respectively and waiting for the system settle to its steady state).

   At $t = 0$ the ends of the channel are both suddenly sealed. There is no flux ($\frac{\partial C}{\partial x} = 0$) through $x = 0$ or $x = L$ for $t > 0$. Use separation of variables (in x and t) to find the bacterial concentration in $0 \le x \le L$ for $t > 0$, and hence show that

$$C(x,t) = C_0 + (C_1 - C_0)\left[\frac{1}{2} - \sum_{n \text{ odd}\ge 1} \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)\exp\left(-\frac{n^2\pi^2}{L^2}Dt\right)\right]$$

   Sketch the bacterial distribution for a range of times, paying particular attention to the shape at very early and late times. Explain what is happening.

5. (i) In an axisymmetric cylindrical geometry, find and sketch the similarity solution of the diffusion equation $C_t = (D/r)(rC_r)_r$ (where $D > 0$ is a constant), which satisfies $C \to 0$ as $r \to \infty$ and $\int_0^\infty 2\pi r\, C(r,t)\, dr = M$, where $M > 0$ is a constant, by assuming that the solution is of the form

$$C(r,t) = \eta F(\xi), \qquad \eta = \frac{M}{Dt}, \qquad \xi = \frac{r}{(Dt)^{1/2}}.$$

   Hence show that C is Gaussian.

   (ii) Find the analogous similarity solution in a spherically symmetric geometry, where $C(r,t)$ satisfies $C_t = (D/r^2)(r^2C_r)_r$, with $\int_0^\infty 4\pi r^2\, C(r,t)\, dr = M$ and $C \to 0$ as $r \to \infty$.
   [*Hint: start by finding appropriate η (no r) and ξ (proportional to r)*]

6. The concentration of a chemical $C(x,t)$ satisfies the nonlinear diffusion equation

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D(C)\,\frac{\partial C}{\partial x}\right) \quad \text{and} \quad \int_{-\infty}^{\infty} C(x,t)\, dx = M$$

   with $D(C) = k\,C^p$ for positive constants M, k and p. Use dimensional analysis to find a suitable space-like ξ and space-independent η for the similarity solution of the form $C(x,t) = \eta\, F(\xi)$. Use this to seek the solution initially localised to the origin, and show that F is of the form

$$F(\xi) = \begin{cases} \left(A - \frac{p}{2(2+p)}\xi^2\right)^{1/p} & \text{for} \quad |\xi| < \xi_0 \\ 0 & \text{otherwise.} \end{cases}$$

   for some A and ξ_0. For the case when $p = 2$, find A and ξ_0.

In addition to the examples sheets, students are encouraged to do the exercises given in lectures.