C6c Lent Term 2011
Mathematical Tripos Part II P.H. Haynes

Mathematical Biology – Examples Sheet 3

[Please communicate any errors in this sheet to `phh@damtp.cam.ac.uk`. Some further material for this course is available at `http://www.damtp.cam.ac.uk/user/phh/mathbio.html`.]

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1. Use separation of variables to solve Laplace's equation $\nabla^2 T = 0$ describing the steady conduction of temperature $T(x, y)$ in the strip $x \geq 0$, $0 \leq y \leq 1$, with $T(x, 0) = 0$, $T(x, 1) = 0$, $T \to 0$ as $x \to \infty$ and $T(0, y) = 1$ for $0 \leq y \leq 1$. Hence show that the cross-sectionally averaged temperature $\bar{T}(x) \equiv \int_0^1 T(x, y)\, \mathrm{d}y$ is given by
$$\bar{T}(x) = \sum_{n \text{ odd},\, n \geq 1} \frac{8}{n^2\pi^2}\, e^{-n\pi x}.$$

2. A pipe of radius a is maintained at a uniform temperature $T_0 + \Delta T$. It is insulated with a material of uniform thermal diffusivity, such that the insulation occupies $a \leq R \leq \lambda a$, for $\lambda > 1$, where R is the distance from the centre of the pipe. The rate at which heat is lost from the outer surface of the pipe is given by $T_r = -(\gamma/a)(T - T_0)$ on $R = \lambda a$, where $\gamma > 0$ is a constant. Find $T(R)$, the temperature field in the insulation, and compute the total temperature flux out of the pipe (per unit length of pipe). Show that if $\gamma < 1$ and $1 < \lambda < 1/\gamma$, it is possible for the rate at which heat is lost from the pipe to *increase* if the insulation is thickened. Explain this apparently paradoxical observation.

   Note: $\nabla^2 T(R) = (1/R)(RT_R)_R$ *in this cylindrical geometry.*

3. In an axisymmetric cylindrical geometry, find *and sketch* the similarity solution of the diffusion equation $C_t = (D/R)(RC_R)_R$ (where $D > 0$ is a constant), which satisfies $C \to 0$ as $R \to \infty$ and $\int_0^\infty 2\pi R C(R, t)\, \mathrm{d}R = M$, where $M > 0$ is a constant, by assuming that the solution is of the form
$$C(R, t) = \frac{M}{Dt} F(\xi), \qquad \xi = \frac{R}{(Dt)^{1/2}}.$$

   Hence show that C is Gaussian. If you feel keen, find the analogous similarity solution in a spherically symmetric geometry, where $C(r, t)$ satisfies $C_t = (D/r^2)(r^2 C_r)_r$, with $\int_0^\infty 4\pi r^2 C(r, t)\, \mathrm{d}r = M$ and $C \to 0$ as $r \to \infty$. [Hint: $C \propto (Dt)^{-3/2}$ — why?]

4. A bar of material in $0 < x < 1$ has an initial temperature distribution $T(x, 0) = x$. For $t > 0$ the temperature $T(x, t)$ adjusts, according to the diffusion equation $T_t = T_{xx}$, to no-flux boundary conditions $T_x(0, t) = T_x(1, t) = 0$. Use separation of variables to find the unsteady temperature distribution for $t > 0$, and hence show that
$$T(x, t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} e^{-(n\pi)^2 t} \cos n\pi x.$$

   Sketch the temperature distribution at late times, when the series can be approximated by just its first term. What do you think the solution looks like at earlier times?

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5. Find the separable solution in polar co-ordinates of the diffusion equation

$$C_t = \frac{1}{r}\left(rC_r\right)_r + \frac{1}{r^2}C_{\theta\theta}, \qquad r \geq 0, \quad 0 \leq \theta < 2\pi,$$

where $C = C(r, \theta, t)$. Consider only solutions that are bounded at the origin and that decay with time. Show that

$$C(r, \theta, t) = \int_0^\infty \sum_{n=0}^{\infty} e^{-\lambda^2 t}\left[B_n(\lambda)\sin n\theta + C_n(\lambda)\cos n\theta\right] J_n(\lambda r)\ \mathrm{d}\lambda$$

for some functions B_n and C_n of λ. If it is then assumed that $C = 0$ on $r = a$, where $a > 0$, and that C is axisymmetric, modify this expression accordingly.

Note: the solutions of $x^2F_{xx} + xF_x + (x^2 - \nu^2)F = 0$ *bounded at* $x = 0$ *are the Bessel functions* $J_\nu(x)$*; you may assume that* $J_0(x) = 0$ *has positive roots* $x = z_0, z_1, z_2, \ldots$

6. For the nonlinear diffusion equation $C_t = D(C^pC_x)_x$, where p and D are strictly positive constants, show that the self similar solution defined in $x \geq 0$ of the form

$$C(x, t) = \frac{M^{2/(2+p)}}{(Dt)^{1/(2+p)}}F(\xi), \qquad \xi = \frac{x}{(M^pDt)^{1/(2+p)}}$$

which satisfies $\int_0^\infty C(x, t)\ \mathrm{d}x = M$, $C_x(0, t) = 0$ and $C(x \to \infty, t) \to 0$ for some constant $M > 0$, is given by

$$F(\xi) = \left[A - \frac{p\xi^2}{2(2+p)}\right]^{1/p}, \qquad 0 < \xi < \left[\frac{2(2+p)A}{p}\right]^{1/2}$$

and $F = 0$ otherwise. For the case $p = 1$, prove that $A = (3/8)^{1/3}$.

7. A material with concentration $c(x)$ satisfies the nonlinear advection-diffusion equation

$$Uc_x = D(c^mc_x)_x$$

where U, D and m are positive constants. Assuming that $c(0) = 1$, and that $c \to \epsilon$ as $x \to -\infty$, find an implicit formula for $c(x)$ in $x < 0$ in the case $m = 1$, $\epsilon > 0$. Sketch the concentration distribution in the case $\epsilon \to 0$. If $\epsilon = 0$, $m > 0$, show that the solution in $x < 0$ is confined to a region of length D/mU.

8. A line source of solute is located at the origin $(x, y) = (0, 0)$ in a uniform stream of fluid of speed U_o. Show that the steady advection-diffusion equation can be scaled to the form

$$C_x = \nabla^2 C,$$

with $C \to 0$ as $x \to \infty$ and as $|y| \to \infty$. Set $C = e^{x/2}F(x, y)$ and find the PDE satisfied by F. Then, assuming $F = F(r)$ where $r = (x^2 + y^2)^{\frac{1}{2}}$, show that $F(r) = AK_0(r/2)$, where A is a constant and K_0 is the modified Bessel function of order zero.

The solute flux through a circle of radius r centred on the origin is kQ where $Q = -\int_0^{2\pi} rC_r\ \mathrm{d}\theta$. Given that $Q \to Q_0$ as $r \to 0$, where $Q_0 > 0$ is a constant, find A. Then show that when x is large and positive, and $y = O(x^{\frac{1}{2}})$,

$$C \sim \frac{Q_0}{(4\pi x)^{\frac{1}{2}}}\exp\left[-\frac{y^2}{4x}\right]. \qquad (*)$$

Explain how $(*)$ can be obtained by assuming from the start that C_{xx} is negligible relative to C_{yy} in the region in which C is significantly different from zero (the "boundary layer approximation").

Note: $K_0(z)$ *is the solution of* $z^2K_{0zz} + zK_{0z} = z^2K_0$ *which decays as* $z \to \infty$*;* $K_0(z) \sim -\log z$ *as* $z \to 0$*, and* $K_0(z) \sim (\pi/2z)^{\frac{1}{2}}e^{-z}$ *as* $z \to \infty$.

9. A simple model of the spreading of an animal population $P(x,t)$ in a spatial domain is given by the nonlinear reaction-diffusion equation

$$P_t = D\left(PP_x\right)_x + \alpha P, \qquad P(x,0) = P_0\delta(x), \qquad P \to 0 \quad \text{as} \quad |x| \to \infty,$$

where D and P_0 are positive constants; α is a constant which may be positive or negative. By setting $P(x,t) = R(x,\tau)e^{\alpha t}$, where $\tau(t)$ is some time-like variable satisfying $\tau(0) = 0$, show that a suitable choice of τ yields $R_\tau = (RR_x)_x$, $R(x,0) = P_0\delta(x)$. Using the constant-mass solution of the nonlinear diffusion equation (question 6), with $R(x,\tau) = \tau^{-1/3}F(\xi)$, $\xi = x/\tau^{1/3}$, show that the population is confined to a region $|x| < x_0$ where

$$x_0^3 = \frac{9P_0D}{2}\left(\frac{e^{\alpha t}-1}{\alpha}\right).$$

Describe the evolution of the population in the cases $\alpha = 0$, $\alpha > 0$ and $\alpha < 0$.

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