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Example Sheet 3

1. A particle starts at the origin $(0,0)$ at time $t=0$. In each of the cases below, derive the corresponding Fokker-Planck equation (in x, y). The particle moves in a random walk, where it takes steps with the step sizes below, each with probability rate λ:

   (a) A square grid: $(-1,0)$, $(+1,0)$, $(0,-1)$ and $(0,+1)$

   (b) A triangular grid: $(+1,0)$, $(-1,0)$, $(\frac{1}{2},\frac{\sqrt{3}}{2})$, $(\frac{1}{2},-\frac{\sqrt{3}}{2})$,$(-\frac{1}{2},+\frac{\sqrt{3}}{2})$ and $(-\frac{1}{2},-\frac{\sqrt{3}}{2})$

   (c) Square grid with bonus diagonal steps (breaking isotropy): $(-1,0)$, $(+1,0)$, $(0,-1)$, $(0,+1)$, $(+1,+1)$ and $(-1,-1)$.

   (d) For case (c), write down the master equation for $P_{m,n}$ (where m,n are the discretised x, y) and use it to find $\langle m^2+n^2\rangle$. Also calculate $\langle x^2+y^2\rangle$ from the Fokker-Planck equation.

2. A two-population dynamic model has the following transition rates:

$$\begin{aligned}(m,n)&\to(m+1,n) &:&\quad \mu+\lambda_1 n\\ (m,n)&\to(m-1,n) &:&\quad \beta_1 m\\ (m,n)&\to(m,n+1) &:&\quad \lambda_2 m\\ (m,n)&\to(m,n-1) &:&\quad \beta_2 n.\end{aligned}$$

   (a) Construct a master equation for $P_{m,n}$ and use it to derive equations for the time evolution of $\langle M\rangle,\langle N\rangle$. Find conditions on the parameters (μ, λ_1, λ_2, β_1 and β_2) for there to be *stable* fixed point with $\langle M\rangle,\langle N\rangle>0$.

   (b) Write down the Fokker-Planck equation using the method in lectures. Now consider small fluctuations near the fixed point found above, and approximate A as linear in x and B as constant. Show that the covariance matrix C satisfies
$$\dot{C}=a\,C+C\,a^T+b$$
   where a and b are matrices which should be given.

   (c) For the special case when $\lambda_1=\lambda_2=\lambda$ and $\beta_1=\beta_2=\beta$ consider the equation for $\dot{C}$ in components. Show that there is a fixed point for C (which need not be explicitly found) and that it is stable. Explain what this means for this model.

   (d) Same as (iii) but with general λ_1, λ_2, β_1 and β_2.

3. As in the example in lectures, the concentration of bacteria $C(x,t)$ in a thin channel of length L obeys the diffusion equation in one dimension with constant diffusivity D. Initially, the concentration of bacteria is $C(x,0)=C_0+(C_1-C_0)\,x/L$ (this was achieved by keeping the concentration at the ends $x=0$ and $x=L$ at $C=C_0$ and $C=C_1$ respectively and waiting for the system settle to its steady state). At $t=0$ the ends of the channel are both suddenly sealed. There is no flux ($\frac{\partial C}{\partial x}=0$) through$x=0$ or $x=L$ for $t>0$. Use separation of variables (in x and t) to find the bacterial concentration in$0\le x\le L$ for$t>0$, and hence show that

$$C(x,t)=C_0+(C_1-C_0)\left[\frac{1}{2}-\sum_{n\ \text{odd}\ge 1}\frac{4}{n^2\pi^2}\cos\left(\frac{n\pi}{L}x\right)\exp\left(-\frac{n^2\pi^2}{L^2}Dt\right)\right]$$

   Sketch the bacterial distribution for a range of times, paying particular attention to the shape at very early and late times. Explain what is happening.

4. In an axisymmetric *cylindrical* geometry, find and sketch the similarity solution of the diffusion equation $C_t = (D/r)(rC_r)_r$ (where $D > 0$ is a constant), which satisfies $C \to 0$ as $r \to \infty$ and $\int_0^\infty 2\pi r\, C(r,t)\, dr = M$, where $M > 0$ is a constant, by assuming that the solution is of the form

$$C(r,t) = \eta F(\xi), \qquad \eta = \frac{M}{Dt}, \qquad \xi = \frac{r}{(Dt)^{1/2}}.$$

Hence show that C is Gaussian. Find the analogous similarity solution in a *spherically symmetric* geometry, where $C(r,t)$ satisfies $C_t = (D/r^2)(r^2 C_r)_r$, with $\int_0^\infty 4\pi r^2\, C(r,t)\, dr = M$ and $C \to 0$ as $r \to \infty$.

5. The concentration of a chemical $C(x,t)$ satisfies the nonlinear diffusion equation

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D(C)\,\frac{\partial C}{\partial x}\right) \quad \text{and} \quad \int_{-\infty}^{\infty} C(x,t)\, dx = M$$

with $D(C) = k\, C^p$ for positive constants M, k and p. Use dimensional analysis to find a suitable space-like ξ and space-independent η for the similarity solution of the form $C(x,t) = \eta\, F(\xi)$. Use this to seek the solution initially localised to the origin, and show that F is of the form

$$F(\xi) = \begin{cases} \left(A - \frac{p}{2(2+p)}\xi^2\right)^{1/p} & \text{for} \quad |\xi| < \xi_0 \\ 0 & \text{otherwise.} \end{cases}$$

for some A and ξ_0. For the case when $p = 2$, find A and ξ_0.

6. A simple model of the spreading of an animal population $N(x,t)$ in a spatial domain is given by the nonlinear reaction-diffusion equation

$$N_t = D\,(N\,N_x)_x + \alpha N, \qquad N(x,0) = N_0\,\delta(x), \qquad N \to 0 \quad \text{as} \quad |x| \to \infty,$$

where D andN_0 are positive constants and α is a constant which may be positive or negative.

(a) By setting $N(x,t) = R(x,\tau)\, e^{\alpha t}$, where $\tau(t)$ is some time-like variable satisfying $\tau(0) = 0$, show that a suitable choice ofτ yields $R_\tau = (RR_x)_x$, $R(x,0) = N_0\,\delta(x)$.

(b) By setting $R(x,\tau) = \tau^{-1/3} F(\xi)$, $\xi = x/\tau^{1/3}$, show that the population is confined to a region $|x| < x_0$ where

$$x_0^3 = \frac{9N_0\, D}{2}\left(\frac{e^{\alpha t} - 1}{\alpha}\right).$$

Describe the evolution of the population in the cases $\alpha = 0$, $\alpha > 0$ and $\alpha < 0$.

7. A bistable system with diffusion is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u(u-r)(u-1)$$

where $0 < r < 1$. Seek a traveling wave solution by setting $\xi = x - ct$ and $u(x,t) = f(\xi)$, and find the differential equation satisfied by f.

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