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Example Sheet 4

1. The SIR epidemic model can be extended to be a spatio-temporal model for the spread of an infectious disease:

$$\begin{aligned}\frac{\partial S}{\partial t} &= -\beta IS \qquad + D\frac{\partial^2 S}{\partial x^2}\\ \frac{\partial I}{\partial t} &= +\beta IS - \nu I + D\frac{\partial^2 I}{\partial x^2}\,.\end{aligned}$$

Suppose that an epidemic wave arrives in a previously uninfected region (so $S \approx N$ and $I \approx 0$). Consider the dynamics near this wave front by taking

$$\begin{aligned}S &= N - u(\xi)\\ I &= v(\xi)\end{aligned}$$

with $\xi = x - ct$ and linearise in u and v. You may assume that the system will settle to the slowest possible wave speed. Find the wave speed of the epidemic, and show that it is proportional to $\sqrt{R_0 - 1}$.

2. Consider the chemotactic system

$$\begin{aligned}\frac{\partial n}{\partial t} &= b\,n\left(1-\frac{n}{n_0}\right) - \frac{\partial}{\partial x}\left(n\,\chi(a)\frac{\partial a}{\partial x}\right) + D\frac{\partial^2 n}{\partial x^2}\\ \frac{\partial a}{\partial t} &= h\,n - d\,a + D_A\frac{\partial^2 a}{\partial x^2},\end{aligned}$$

where

$$\chi(a) = \frac{\chi_0\,a_0}{(a_0+a)^2}\,.$$

(a) Find a rescaling such that this reduces to

$$\begin{aligned}\frac{\partial u}{\partial \tau} &= u(1-u) - \beta\frac{\partial}{\partial \xi}\left[\frac{u}{(\alpha+v)^2}\frac{\partial v}{\partial \xi}\right] + \frac{\partial^2 u}{\partial \xi^2}\\ \frac{\partial v}{\partial \tau} &= \gamma(u-v) + \delta\,\frac{\partial^2 v}{\partial \xi^2}\,.\end{aligned}$$

Hint: do the rescaling over a few steps, keeping an eye on the intended final form.

(b) Show that the uniform, steady solution $u = v = 1$ is unstable to a spatial perturbation if

$$\frac{\beta\gamma}{(1+\alpha)^2} > (\sqrt{\gamma} + \sqrt{\delta})^2.$$

Find the critical wavenumber in the case when $\alpha = \gamma = \delta = 1$.

3. Investigate the possibility of a Turing instability for the reaction-diffusion system.

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{u^2}{v} - b\,u + \nabla^2 u\\ \frac{\partial v}{\partial t} &= u^2 - v + d\,\nabla^2 v\,.\end{aligned}$$

In particular, find the region of the parameter space (b, d) in which Turing instability can occur, and give the value for the critical wavenumber at the onset of instability in terms of d.

4. A space-dependent phytoplankton and zooplankton model can be reduced to the following equations

$$\frac{\partial u}{\partial t} = u + u^2 - \gamma\, u\, v + \nabla^2 u$$

$$\frac{\partial v}{\partial t} = \beta\, u\, v - v^2 + d\,\nabla^2 v\,.$$

(a) Find the regions in the $\beta - \gamma$ plane (i) in which there is a stable, homogeneous state (u_0, v_0) in which neither u_0 nor v_0 is zero and (ii) in which that state may be unstable to a Turing instability.

(b) In case (ii), for what values of d will the instability occur?

(c) Find the critical wavenumber for the onset of the instability in terms of β and d.

5. A variant of the Hutchinson-Wright equation is given by

$$\frac{dx(t)}{dt} = \alpha\left[x(t-T) - x(t)^2\right]\,,$$

where $\alpha, T > 0$.

(a) Give a brief interpretation of what this might represent in terms of population dynamics.

(b) Show that the constant solution with $x(t) = 1$ is stable for all $\alpha, T > 0$. [Hint: show that any 's' must have negative real part (as in lectures, the growth exponent of a small perturbation).]

6. Circulating red blood cells (RBCs) are destroyed in the spleen and created in the bone marrow in proportion to the number destroyed on the previous day.

(a) Treat days as discrete time units, and let R_n be the number of RBCs in circulation on day n, let M_n be the number produced by marrow on day n, let f be the fraction of RBCs removed by the spleen every day, and let γ be the number produced on day n for each cell lost on day $n-1$.

i. Write down equations for R_{n+1} and M_{n+1} in terms of R_n and M_n, and show that

$$R_n = A\lambda_1^n + \beta\lambda_2^n,$$

where

$$\lambda_{1,2} = \frac{1 - f \pm \sqrt{(1-f)^2 + 4\gamma f}}{2}.$$

ii. Deduce that the RBC count may fluctuate, but that it will remain constant at large times if and only if $\gamma = 1$.

(b) Start again, but this time formulate the model for RBC count as a delay differential equation for $R(t)$, where the production of new cells is proportional to $R(t-T)$. Now f should be a rate and γ is still the ratio of cells made for each one lost. Show that there exists a solution of the form $R(t) = Ce^{\lambda f t}$ if

$$\frac{1}{\gamma}(\lambda + 1) = e^{-\lambda f T}.$$

Show graphically that this equation has a positive real root if and only if $\gamma > 1$, and interpret this result.

7. The population density $n(a, t)$ of individuals of age a at time t satisfies

$$\frac{\partial n(a,t)}{\partial t} + \frac{\partial n(a,t)}{\partial a} = -\mu(a)n(a,t), \quad \text{with} \quad n(0,t) = \int_0^\infty b(a)n(a,t)da,$$

where $\mu(a)$ is the age-dependent death rate and $b(a)$ is the birth rate per individual of age a. Using the standard similarity solution $n(a,t) = e^{\gamma t}\, r(a)$ for each of the examples below, give (i) the mean number of offspring (ii) the population growth rate γ (solve where possible otherwise give an implicit expression) (iii) the value of the birth rate parameter B for which there is neither growth nor decay and sketch the age-profile of the population in this case.

(a) The birth rate $b(a)$ is a constant B for $a_1 < a < a_2$ and zero otherwise. The death rate $\mu(a)$ is a constant d for $a > a_2$ and zero otherwise.

(b) Individuals give birth only at age a^*: $b(a) = B\,\delta(a - a^*)$. The death rate $\mu(a)$ is a constant d for all ages.

(c) The birth rate $b(a)$ is a constant B for all ages. All individuals die at age A. [Hint: in this extreme case, rather than using $\mu(a)$, just reformulate the standard approach slightly.]