Mathematical Tripos Part II C7b

Dynamical Systems (D course) Michaelmas 2014

Example Sheet 2 P. H. Haynes

Please send comments/amendments etc. to `phh1@cam.ac.uk`. Examples sheets, plus lecture notes and any other material, will be made available at
`http://www.damtp.cam.ac.uk/user/phh/dynsys.html`.

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1. *Non-uniqueness of the centre manifold for non-hyperbolic fixed points*. Consider the dynamical system $\dot{x} = -x^3,\ \dot{y} = -y$. Note that the line $y = 0$ is invariant (this corresponds to the eigenvector of the linearised system with zero eigenvalue, and is therefore the *centre subspace* described in lectures). Show that there are an infinite number of trajectories that become tangent to the line $y = 0$ as $x \to 0$ (and which satisfy $y = h(x)$ where all possible forms of h have zero Taylor expansion about the origin). (Each of these trajectories is a *centre manifold* – see §5 of lectures.)

2. Use the Lyapunov function $V = x^2 + y^2$ to show that the origin is asymptotically stable for the system

$$\dot{x} = -2x - y^2, \quad \dot{y} = -y - x^2,$$

and that the region $x^2 + y^2 \leqslant 1$ is included in its domain of stability.

3. Determine the values of k for which $V = x^2 + ky^2$ is a Lyapunov function in a sufficiently small neighbourhood of the origin (how small need not be determined) for the system

$$\dot{x} = -x + y - x^2 - y^2 + xy^2, \quad \dot{y} = -y + xy - y^2 - x^2y.$$

What information about the domain of stability of the origin can be deduced when $k = 1$?
[*Hint*: $\dot{V}$ has a homogeneous quadratic factor when $k = 1$.]

4. Sketch the phase portrait of the ideal pendulum

$$\dot{\theta} = p, \quad \dot{p} = -\sin\theta.$$

(The cylindrical topology of phase space is represented by $[-\pi, \pi) \times \mathbf{R}$ with periodic edges.) By considering the function $H(\theta, p) = \frac{1}{2}p^2 + 1 - \cos\theta$ explain informally why all of the periodic orbits are Lyapunov stable. Show that the invariant set $\Sigma = \{(\theta, p) : H(\theta, p) = 2\}$ consists of one fixed point and two homoclinic orbits. Are these orbits Lyapunov stable? Is Σ Lyapunov stable?

Consider the damped pendulum $\ddot{\theta} + k\dot{\theta} + \sin\theta = 0$. Find the fixed points. Use La Salle's Invariance Principle to show that $(0, 0)$ is asymptotically stable and obtain an estimate of the domain of stability.

5. Consider the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ with $\mathbf{x} \in \mathbf{R}^n$, where all of the eigenvalues λ_i of A are distinct and have negative real parts. Show that the function $V = \sum_{i=1}^{n} v_i|\mathbf{e}_i^\dagger \cdot \mathbf{x}|^2$ satisfies $\dot{V} < 0$ for $\mathbf{x} \neq \mathbf{0}$, where the $\mathbf{e}_i^\dagger$s are the (possibly complex) left eigenvectors of A normalized so that $\mathbf{e}_i^\dagger \cdot \mathbf{e}_j = \delta_{ij}$, and the v_is are any set of strictly positive constants. Deduce that $\mathbf{x} = \mathbf{0}$ is asymptotically stable.
[*Hint*: First let $a_i(t) = \mathbf{e}_i^\dagger \cdot \mathbf{x}(t)$ and calculate $\dot{a}_i$ and $\dot{\overline{a_i}}$.]

6. Consider the system in $\mathbf{R}^2$

$$\dot{\mathbf{x}} = A\mathbf{x} - |\mathbf{x}|^2\mathbf{x},$$

where A is a constant real matrix with complex eigenvalues $p \pm iq$ ($q > 0$). Use the divergence test (Dulac with $\phi = 1$) and the Poincaré-Bendixson theorem to prove that there exists at least one limit cycle for $p > 0$ and there are none for $p < 0$.

7. Prove that

$$\dot{x} = x - y - (x^2 + \tfrac{3}{2}y^2)x, \quad \dot{y} = x + y - (x^2 + \tfrac{1}{2}y^2)y,$$

has a periodic solution. [*Hint*: polar coordinates.]

8. Show using Dulac's criterion with a suitable weighting function, that there are no periodic orbits in $x \geqslant 0$, $y \geqslant 0$ for the population model

$$\dot{x} = x(2 - y - x), \quad \dot{y} = y(4x - x^2 - 3).$$

Find and analyse the fixed points and sketch the phase portrait. If x and y are the population densities of two species what final outcome does your sketch suggest?

9. Suppose a differential equation in $\mathbf{R}^2$ has only three fixed points, two of which are sinks and the other a saddle. Sketch examples, where possible, of flows with a periodic orbit such that the set of fixed points enclosed by it: (a) is empty; (b) contains just one sink; (c) contains just the saddle; (d) contains one sink and the saddle; (e) contains both sinks; (f) contains all three fixed points. Where impossible, justify your answer.

10. Consider the system

$$\dot{x} = 2x + x^2 - y^2, \quad \dot{y} = -2y + x^2 - y^2.$$

Use the Poincaré index test to show that there are no periodic orbits.

11. Consider the system

$$\dot{x} = -x - y + \tfrac{3}{2}\alpha xy^2 + x^3, \quad \dot{y} = \alpha^{-1}x - y + \tfrac{1}{2}x^2y + \alpha y^3,$$

where α is a positive constant. Show that the origin is asymptotically stable by finding a Lyapunov function $V(\mathbf{x}) = x^2 + cy^2$ for an obvious choice of the constant c. Find the domain of asymptotic stability for the origin. What happens outside this region? *Use elliptical polars to show that the boundary of the region is a periodic orbit only when $\alpha < 16$.

12. Show that $\ddot{x} + ax + x^2 = 0$ conserves $V = \frac{1}{2}p^2 + \frac{1}{2}ax^2 + \frac{1}{3}x^3$ where $p = \dot{x}$. Sketch the phase plane for $a > 0$, and describe the different sorts of orbits in the system. Show that when $k > 0$ each solution of $\ddot{x} + k\dot{x} + ax + x^2 = 0$ converges to one of two fixed points or diverges to infinity. Compute the linear stability of each fixed point. Draw a sketch showing the sets of points whose orbits converge to each of the fixed points.

13. The Lorenz equations are

$$\dot{x} = \sigma(y - x), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz,$$

where r, σ and b are positive constants. For $r \in (0, 1)$ show that the origin is globally asymptotically stable by considering a function $V_1(x, y, z) = \alpha x^2 + \beta y^2 + \gamma z^2$ for a suitable choice of the constants α, β and γ. For $r \geqslant 1$ show, by considering the function $V_2(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2$, that all trajectories eventually enter and then remain within a bounded region of phase space.