Mathematical Tripos Part II C7c

Dynamical Systems (D course) Michaelmas 2014

Example Sheet 3 P. H. Haynes

Please send comments/amendments etc. to `phh1@cam.ac.uk`. Examples sheets, plus lecture notes and any other material, will be made available at
`http://www.damtp.cam.ac.uk/user/phh/dynsys.html`.

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1. *[Nearly Hamiltonian flows]* Use the "energy balance" method to find the amplitude of the limit cycle in the nearly-Hamiltonian system

$$\ddot{x} - \epsilon(x-3)(x+1)\dot{x} + x = 0,$$

where $0 < \epsilon \ll 1$. Use the divergence of the flow to compute the non-trivial Floquet multiplier of the limit cycle, and hence determine its stability. (* Recall (4) on p34 of the lecture notes. How does your answer relate to $d\Delta H/dH$?)

2. *[Nearly Hamiltonian flows]* Describe the behaviour of the system

$$\dot{x} = 2y + \tfrac{1}{8}\epsilon\alpha x, \quad \dot{y} = -2x + \epsilon y^3(x^2 - \tfrac{1}{2}),$$

when $\epsilon = 0$ and α is a constant. Investigate whether there are any limit cycles for $0 < \epsilon \ll 1$. What are the possible amplitudes when $\alpha = 2$ and which are stable? What can you say about the system when $\alpha > \frac{9}{4}$ and when $\alpha = \frac{9}{4}$?

3. *[Stability of periodic orbits]* Use the divergence of the flow to find approximations to the non-trivial Floquet multiplier for the periodic orbit of the van der Pol oscillator

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0,$$

when (a) $0 < \mu \ll 1$, and (b) $\mu \gg 1$. [*Hint*: in case (b) $y = \dot{x}$ is not the optimal choice.Try the Liénard transformation.]

4. *[Stationary bifurcations]* (a) Sketch the position of the fixed points of

$$\dot{x} = f(x) = x(\mu + x - 2)(\mu + 2x - x^2)$$

in the (x, μ)-plane, indicating their stability. Hence draw the bifurcation diagram and classify the bifurcations.

(b) Repeat part (a) for the system

$$\dot{x} = x(9 - \mu x)(\mu + 2x - x^2)[(x+2)^2 + \frac{1}{25}(\mu - 3)^2 - 1]$$

State the type of bifurcation at each bifurcation point. Discuss the effect on the bifurcations of adding a small positive constant ϵ to the right-hand side.

5. *[Centre Manifold]* Find the critical value a_c at which there is a bifurcation at the origin in the system

$$\dot{x} = y - x - 2x^2, \quad \dot{y} = ax - y - 2y^2.$$

Change the variables to $u = x + y$, $v = x - y$, $\mu = a - a_c$ and explain why this is advantageous. Hence find the extended centre manifold and the evolution equation on it, each correct to third order. Deduce the nature of the bifurcation. What order was necessary for this deduction?

6. *[Centre Manifold]* Compute the extended centre manifold near $x = y = \mu = 0$ to sufficiently high order so as to identify the bifurcation type for the two different systems

$$(a) \qquad \dot{x} = \mu + y - 3x^2 + xy, \quad \dot{y} = -3y + y^2 - x^2,$$

and

$$(b) \qquad \dot{x} = -2x + y - x^2, \quad \dot{y} = \mu + x(y - x).$$

In each case sketch the bifurcation diagrams and sketch the (x, y)-phase portraits of the local dynamics close to the origin for $0 < \mu \ll 1$.

7. *[Centre Manifold]* Consider the system in $\mathbf{R}^2$

$$\dot{x} = x(1 - y - 4x^2), \quad \dot{y} = y(\mu - y - x^2).$$

Show that the fixed point $(0, \mu)$ has a bifurcation at $\mu = 1$, while the fixed points $(\pm\frac{1}{2}, 0)$ both have bifurcations at $\mu = \frac{1}{4}$. By finding the first approximation to the extended centre manifold in each case, construct the relevant evolution equation and determine the type of bifurcation. [*Hint*: Use an appropriate change of the origin.] Sketch the bifurcation diagrams.

8. *[Oscillatory/Hopf Bifurcations]* Investigate the linear stability at the origin for the system

$$\dot{x} = \mu x + \omega(x + y) - x(4x^2 + y^2), \quad \dot{y} = \mu y - \omega(2x + y) - y(4x^2 + y^2),$$

as the parameter μ is varied. What kind of bifurcation does this suggest? Find a transformation which reduces the *linear* problem to canonical form, and then rewrite the transformed system in polar coordinates. Deduce that the bifurcation is supercritical. Obtain the same result using the Poincaré-Bendixson theorem.

9. *[Oscillatory/Hopf Bifurcations]* Show that when u, $v = O(\epsilon)$ and $\mu = O(\epsilon^2)$, the system

$$\dot{u} = \mu u - v + u^2 - v^2 - u(u + v)^2, \quad \dot{v} = u + \mu v - uv - v(u^2 - uv + v^2)$$

can be transformed to the system

$$\dot{x} = \mu x - y - x(x^2 + y^2) + O(\epsilon^4), \quad \dot{y} = x + \mu y - y(x^2 + y^2) + O(\epsilon^4)$$

by the nonlinear transformation $u = x + xy$, $v = y$. Deduce the nature of the bifurcation at $\mu = 0$.

10.*. Investigate bifurcations as the parameter μ varies, in each of the systems

$$\dot{r} = \mu r + r^3 - r^5, \quad \dot{\theta} = 1,$$

and

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = \mu - \sin\theta,$$

where $\mu > 0$ in the second case. Draw phase portraits near the bifurcations. The first is a *saddle-node bifurcation of cycles* and the second is an *infinite-period bifurcation*.

11.*. By considering the behaviour of $\dot{r}$ and $\dot{\theta}$ in the system

$$\dot{r} = r(\mu + 2r^2 - r^4), \qquad \dot{\theta} = 1 - \nu r^2 \cos\theta\ ,$$

find the conditions on μ and ν under which there are zero, one and two periodic orbits. Deduce the stability of these orbits and show the results in the (μ, r) plane. Describe the types of bifurcation that occur as μ and ν are varied.