Dynamical Systems (D course) Michaelmas 2014

Example Sheet 4 P. H. Haynes

Please send comments/amendments etc. to `phh1@cam.ac.uk`. Examples sheets, plus lecture notes and any other material, will be made available at `http://www.damtp.cam.ac.uk/user/phh/dynsys.html`.

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1. Consider the graph for each of the maps $F : \mathbb{R} \to \mathbb{R}$,

$$(a)\, F(x) = -x, \qquad (b)\, F(x) = x - x^3, \qquad (c) F(x) = x + x^3, \qquad (d)\, F(x) = x + x^2.$$

State whether the non-hyperbolic fixed point at the origin is Liapunov stable, asymptotically stable or neither. In case (d) what set of points are attracted to the origin?

2. Find and analyse the successive bifurcations in the 1-dimensional map $F(x, \mu) = \mu - \frac{1}{4}x^2$ as μ increases from $-\infty$ to 5. [*Hint*: In part of this analysis you will find it useful to consider $F^2(x, \mu)$. Finding fixed points of F^2 requires solution of a quartic, but two of the roots are fixed points of F.]

3. Consider the logistic map F and the quadratic map G on $\mathbb{R}$,

$$F(x, \mu) = \mu x(1 - x), \qquad G(x, \nu) = \nu - x^2.$$

Show that for appropriate values of μ and ν, which should be determined, F and G are topologically conjugate, i.e., there exists a map $h : \mathbb{R} \to \mathbb{R}$ such that $h \circ F = G \circ h$. [*Hint*: look for a linear map.] In the light of this result comment on your answer to Q2.

4. Consider the following generalised sawtooth maps on $[0, 1]$: $S_n(x) = nx[\text{mod } 1]$. By considering the binary representation, determine explicitly the 3-cycles of S_2 and express them as fractions.

How many 3-cycles does S_3 have?

5.* Find an aperiodic orbit of the sawtooth map of the previous question that is dense in $[0, 1)$.

6. Let F be a continuous map on the real line and let $x_0 < x_1 < x_2 < x_3$ be the members of a 4-cycle with $F(x_n) = x_{n+1[\text{mod}\,4]}$. Prove (formally, using the IVT etc.) that F has a fixed point, a 2-cycle and a 3-cycle. Explain informally (using a directed graph) why F has at least one 4-cycle in addition to $(x_0, \ldots, x_3)$ and at least two 3-cycles.

7. Let F be a continuous map on the real line and let $x_4 < x_2 < x_0 < x_6 < x_1 < x_3 < x_5$ be the members of a 7-cycle with $F(x_n) = x_{n+1[\text{mod}\,7]}$. Show (informally, using a directed graph) that F has N-cycles for all $N > 8$ and for all even N.

8. Consider the skewed tent map

$$F(x) = \begin{cases} \mu x & x \in [0, a], \\ \mu a(1-x)/(1-a) & x \in [a, 1], \end{cases}$$

where $a \in (0, 1)$. When is F a map of $[0, 1]$ into itself? When is there a non-trivial (orientation reversing) fixed point? Show that this fixed point is at $x^* = \mu/(\mu + \mu_s(a))$ (with $\mu_s(\frac{1}{2}) = 1$). When is x^* stable?

Now consider F^2. Under what conditions does F^2 have an orientation reversing fixed point? Show that there is a value $\mu_c(a)$ (with $\mu_c(\frac{1}{2}) = \sqrt{2}$) such that F^2 has a horseshoe for $\mu > \mu_c$.

*When is the map F chaotic? (*Hint:* You may find it helpful to express F in terms of μ and μ_s, rather than μ and a, and to consider the sequence $G, G^2, G^4, \ldots$, where $G = F^4$).

9. (i) Show for the logistic map $F(x, \mu)$ that when $2 < \mu < 3.678\ldots$ there are exactly two points for each $n > 1$ such that

$$F^{n+1}(x, \mu) = 1 - \mu^{-1}, \qquad F^n(x, \mu) \neq 1 - \mu^{-1},$$

where $\mu_c = 3.678\ldots$ is a root of $\mu^4 - 4\mu^3 + 16 = 0$. Show further that the set of all such points as n varies has 0 and 1 as its only points of accumulation.

(ii) What can you say about the domain of attraction of the 2-cycle in $3 < \mu < 1 + \sqrt{6}$?

10.* The bifurcation diagram for the logistic map $F(x, \mu) = \mu x(1-x)$ shows a broad window around $\mu \approx 3.83$, because of the creation of a stable 3-cycle. Investigate the appearance of the 3-cycle as μ increases using the following approach.

A 3-cycle may be represented by

$$x_n = \alpha + \beta\omega^n + \beta^*\omega^{*n}$$

where $\omega = e^{2\pi i/3}$ and * denotes complex conjugate. Show from the logistic map that

$$\begin{aligned} \alpha &= \mu\alpha(1-\alpha) - 2\mu|\beta|^2 \\ \mu\beta^{*2} &= [\mu(1-2\alpha) - \omega]\beta \\ \mu\beta^2 &= [\mu(1-2\alpha) - \omega^*]\beta^*. \end{aligned}$$

Hence by eliminating β show that

$$9\mu^2\alpha^2 - (9\mu^2 + 3\mu)\alpha + 2(\mu^2 + \mu + 1) = 0.$$

Deduce that the 3-cycle appears at $\mu = 1 + \sqrt{8} \approx 3.8284$.

11.* In the bifurcation diagram for the logistic map there are several smooth dark tracks running through the chaotic part of the diagram. What are they? [*Hint* Think about $F(\frac{1}{2}, \mu)$.]

These tracks all intersect at the tip of the "white wedge" at $\mu \approx 3.67$. Can you obtain a more precise value of μ?