# Further Complex Methods, Examples sheet 1

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**1** *Show that:*

*(i)* $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2(x^2+4)} = \frac{\pi}{18}$ ;

(ii) $\displaystyle\int_{-\infty}^\infty \frac{\cos x\, dx}{x^2+a^2} = \frac{\pi}{a}e^{-a}$ where $a>0$ ;

(iii) $\displaystyle\int_{-\infty}^\infty \frac{x-\sin x}{x^3}\,dx = \frac{\pi}{2}$ ;

**2** Show that

(i) $\displaystyle\int_\gamma e^{z^2}dz$, where $\{\gamma : z = se^{i\alpha}\ ,\ -\infty < s < \infty\}$ and $\alpha \in \mathbb{R}$, converges for $\alpha \in (-3\pi/4, -\pi/4) \cup (\pi/4, 3\pi/4)$. Does the integral converge at the end points of these intervals?

(ii) $\displaystyle\int_\gamma e^{1/z}dz$, where $\{\gamma : z = se^{i\alpha}, 0 \le s \le 1\}$ and $\alpha \in \mathbb{R}$, converges for $\alpha \in (-\pi, -\pi/2) \cup (\pi/2, \pi)$. Does the integral converge at $\alpha = -\pi/2, \pi/2, \pi$?

(iii) $\displaystyle\int_0^\infty \frac{x^\beta\, dx}{1+x}$, where $\beta \in \mathbb{C}$, converges for $-1 < \operatorname{Re}\beta < 0$.

(iv) $\displaystyle\int_\gamma (1+\tanh z)\ dz$, where $\{\gamma : z = se^{i\alpha}, 0 \le s < \infty\}$ and $\alpha \in \mathbb{R}$, converges for $\alpha \in (-\pi, -\pi/2) \cup (\pi/2, \pi)$. Does the integral converge at $\alpha = \pi$?

**3** Let $f(t)$ be analytic at $t=0$ with $f(0)=0$ and $f'(0) \neq 0$. Let $C$ be a circle centred on the origin, with interior $D$, such that $f$ is analytic in $D$ and the inverse of $f$ exists on $f(D)$.

For a fixed point $z$ within $C$, let $w = f(z)$. Assuming that $w$ is small, show (using the residue theorem) that

$$z = \frac{1}{2\pi i}\int_C \frac{tf'(t)}{f(t)-w}dt,$$

and hence that $z = \sum_{n=1}^\infty b_n w^n$, where

$$b_n = \frac{1}{2\pi i}\int_C \frac{tf'(t)}{(f(t))^{n+1}}dt = \frac{1}{2\pi i n}\int_C \frac{1}{(f(t))^n}dt = \frac{1}{n!}\lim_{t\to 0}\frac{d^{n-1}}{dt^{n-1}}\left(\frac{t}{f(t)}\right)^n.$$

Show that the equation $w = ze^{-z}$ has a solution, for sufficiently small $w$ (how small?),

$$z = \sum_{n=1}^\infty \frac{n^{n-1}}{n!}w^n.$$

Find also one solution of the equation $w = 2z - z^2$.

**4** Let $\phi(x, y)$ be a harmonic function. Show that $\phi$ is the real part of any analytic function $f(z)$ of the form

$$f(z) = 2\phi\big((z+1)/2, (z-1)/2i\big) - \phi(1,0) + ic$$

where $c$ is a real constant (provided $\phi$ is such that the right hand side exists). Use this formula to find analytic functions whose real parts are (i) $x/(x^2+y^2)$ and (ii) $\tan^{-1} y/x$.

[Note: You might like to start by considering a harmonic conjugate $\psi(x, y)$, where $\phi$ and $\psi$ obey the Cauchy-Riemann conditions, and then write $f(z) = \phi(x, y) + i\psi(x, y) = \sum_{n=0}^{\infty} a_n(z-1)^n$.]

**5** Let $f_1(z)$ be the branch of $(z^2-1)^{\frac{1}{2}}$ defined by branch cuts in the $z$-plane along the real axis from $-1$ to $-\infty$ and from $1$ to $\infty$, with $f_1(z)$ real and positive just above the latter cut. Let $f_2(z)$ be the branch of $(z^2-1)^{\frac{1}{2}}$ defined by a cut along the real axis from $-1$ to $+1$, with $f_2(x)$ real and positive for $(x-1)$ real and positive. Show that $f_1(z) = f_1(-z)$ but $f_2(z) = -f_2(-z)$.

**6** Let $P(z)$ be a polynomial of degree $n$, with $n$ roots, none of which lie on a simple close contour $L$. Show that

$$\frac{1}{2\pi i}\int_L \frac{P'(z)}{P(z)}dz = \text{number of roots lying within } L.$$

[Note: Try to do this question without assuming *the fundamental theorem of algebra.*]

**7** By integrating the function

$$\frac{(\ln z)^2}{z^2+1}$$

around an appropriate contour, compute the following integrals:

$$\int_0^{\infty} \frac{(\ln x)^m}{x^2+1}dx, \quad m = 1, 2.$$

**8** Evaluate

$$\int_0^{\infty} \frac{x^{m-1}}{x^2+1}dx, \quad 0 < m < 2.$$

Why is it necessary for $m$ to satisfy the above restrictions?

**9** Consider a rectangular contour $C$, with corners at $(N+\frac{1}{2})(\pm 1 \pm i)$ to evaluate

$$\frac{1}{2\pi i}\int_C \frac{\pi \cot \pi z \coth \pi z}{z^3}dz.$$

In the limit as $N \to \infty$, show that

$$\sum_{n=1}^{\infty} \frac{\coth n\pi}{n^3} = \frac{7}{180}\pi^3.$$

**10** Evaluate the following integrals, where $|f(z)/z| \to 0$ as $|z| \to \infty$ and $f(z)$ is analytic in the upper half plane (including the real axis):

$$\text{(i)}\ \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx \qquad \text{(ii)}\ \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x(x-i)} dx \qquad \text{(iii)}\ \mathcal{P}\int_{-\infty}^{\infty} \frac{dx}{x-i} \qquad \text{(iv)}\ \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{-x^2}}{x} dx.$$

**11** Use the principal value technique to evaluate

$$\int_0^{\infty} \frac{\sin x}{x(x^2+1)} dx.$$