1. Show that:

(i) \[ \int_0^\infty \frac{dx}{(x^2 + 1)^2(x^2 + 4)} = \frac{\pi}{18} ; \]

(ii) \[ \int_{-\infty}^{\infty} \frac{\cos x \, dx}{x^2 + a^2} = \frac{\pi}{a} e^{-a} \quad \text{where} \ a > 0 ; \]

(iii) \[ \int_{-\infty}^{\infty} \frac{x - \sin x}{x^3} \, dx = \frac{\pi}{2} ; \]

2. Write down the range of values of \( \alpha \) (real) or \( \beta \) (complex) for which the following integrals converge.

(i) \[ \int_\gamma e^{sz} \, dz \quad \text{where} \quad \{ \gamma : z = se^{i\alpha}, -\infty < s < \infty \} \]

(ii) \[ \int_\gamma e^{1/z} \, dz \quad \text{where} \quad \{ \gamma : z = se^{i\alpha}, 0 \leq s \leq 1 \} \]

(iii) \[ \int_0^\infty \frac{x^\beta \, dx}{1 + x} \]

(iv) \[ \int_\gamma (1 + \tanh z) \, dz, \quad \text{where} \quad \{ \gamma : z = se^{i\alpha}, 0 \leq s < \infty \} \]

3. Let \( f(t) \) be analytic at \( t = 0 \) with \( f(0) = 0 \) and \( f'(0) \neq 0 \). Let \( C \) be a circle centred on the origin, with interior \( D \), such that \( f \) is analytic in \( D \) and the inverse of \( f \) exists on \( f(D) \).

For a fixed point \( z \) within \( C \), let \( w = f(z) \). Assuming that \( w \) is small, show (using the residue theorem) that

\[ z = \frac{1}{2\pi i} \int_C \frac{tf'(t)}{f(t) - w} \, dt, \]

and hence that \( z = \sum_{n=1}^\infty b_n w^n \), where

\[ b_n = \frac{1}{2\pi i} \int_C \frac{tf'(t)}{(f(t))^n} \, dt = \frac{1}{2\pi i} \lim_{t \to 0} \frac{d^{n-1}}{dt^{n-1}} \left( \frac{t}{f(t)} \right)^n. \]

Show that the equation \( w = ze^{-z} \) has a solution, for sufficiently small \( w \) (how small?),

\[ z = \sum_{n=1}^\infty \frac{n^{n-1}}{n!} w^n. \]

Find also one solution of the equation \( w = 2z - z^2 \).

4. Let \( \phi(x, y) \) be a harmonic function. Show that \( \phi \) is the real part of any analytic function \( f(z) \) of the form

\[ f(z) = 2\phi((z + 1)/2, (z - 1)/2i) - \phi(1, 0) + ic \]

where \( c \) is a real constant (provided \( \phi \) is such that the right hand side exists). Use this formula to find analytic functions whose real parts are (i) \( x/(x^2 + y^2) \) and (ii) \( \tan^{-1} y/x \).
5 Let $f_1(z)$ be the branch of $(z^2 - 1)^{\frac{1}{2}}$ defined by branch cuts in the $z$-plane along the real axis from $-1$ to $-\infty$ and from $1$ to $\infty$, with $f_1(z)$ real and positive just above the latter cut. Let $f_2(z)$ be the branch of $(z^2 - 1)^{\frac{1}{2}}$ defined by a cut along the real axis from $-1$ to $+1$, with $f_2(x)$ real and positive for $(x - 1)$ real and positive. Show that $f_1(z) = f_1(-z)$ but $f_2(z) = -f_2(-z)$.

6 Let $P(z)$ be a polynomial of degree $n$, with $n$ simple roots, none of which lie on a simple close contour $L$. Show that

$$\frac{1}{2\pi i} \int_L \frac{P'(z)}{P(z)} \, dz = \text{number of roots lying within } L.$$

7 By integrating the function

$$\frac{(\ln z)^2}{z^2 + 1}$$

around an appropriate contour, compute the following integrals:

$$\int_0^\infty \frac{(\ln x)^m}{x^2 + 1} \, dx, \quad m = 1, 2.$$

8 Evaluate

$$\int_0^\infty \frac{x^{m-1}}{x^2 + 1} \, dx, \quad 0 < m < 2.$$

Why is it necessary for $m$ to satisfy the above restrictions?