1 Show that:

(i) \[ \int_0^\infty \frac{dx}{(x^2 + 1)^2(x^2 + 4)} = \frac{\pi}{18}; \]

(ii) \[ \int_{-\infty}^{\infty} \frac{\cos x \, dx}{x^2 + a^2} = \frac{\pi}{a} e^{-a} \quad \text{where} \ a > 0; \]

(iii) \[ \int_{-\infty}^{\infty} \frac{x - \sin x}{x^3} \, dx = \frac{\pi}{2}; \]

2 Show that

(i) \[ \int_\gamma e^{z^2} \, dz, \quad \text{where} \ \{ \gamma : z = se^{i\alpha}, -\infty < s < \infty \} \ \text{and} \ \alpha \in \mathbb{R}, \ \text{converges for} \ \alpha \in (-3\pi/4, -\pi/4) \cup (\pi/4, 3\pi/4). \ * \ \text{Does the integral converge at the end points of these intervals?} \]

(ii) \[ \int_\gamma e^{1/z} \, dz, \quad \text{where} \ \{ \gamma : z = se^{i\alpha}, 0 \leq s \leq 1 \} \ \text{and} \ \alpha \in \mathbb{R}, \ \text{converges for} \ \alpha \in (-\pi, -\pi/2) \cup (\pi/2, \pi). \ \text{Does the integral converge at} \ \alpha = -\pi/2, \pi/2, \pi? \]

(iii) \[ \int_0^\infty \frac{x^\beta \, dx}{1 + x}, \quad \text{where} \ \beta \in \mathbb{C}, \ \text{converges for} \ -1 < \text{Re} \beta < 0. \]

(iv) \[ \int_\gamma (1 + \tanh z) \, dz, \quad \text{where} \ \{ \gamma : z = se^{i\alpha}, 0 \leq s < \infty \} \ \text{and} \ \alpha \in \mathbb{R}, \ \text{converges for} \ \alpha \in (-\pi, -\pi/2) \cup (\pi/2, \pi). \ \text{Does the integral converge at} \ \alpha = \pi? \]

3 Let \( f(t) \) be analytic at \( t = 0 \) with \( f(0) = 0 \) and \( f'(0) \neq 0 \). Let \( C \) be a circle centred on the origin, with interior \( D \), such that \( f \) is analytic in \( D \) and the inverse of \( f \) exists on \( f(D) \).

For a fixed point \( z \) within \( C \), let \( w = f(z) \). Assuming that \( w \) is small, show (using the residue theorem) that

\[ z = \frac{1}{2\pi i} \int_C \frac{tf'(t)}{f(t) - w} \, dt, \]

and hence that \( z = \sum_{n=1}^{\infty} b_n w^n \), where

\[ b_n = \frac{1}{2\pi i} \int_C \frac{tf'(t)}{(f(t))^{n+1}} \, dt = \frac{1}{2\pi i} \int_C \frac{1}{(f(t))^n} \, dt = \frac{1}{n!} \lim_{t \to 0} \frac{d^{n-1}}{dt^{n-1}} \left( \frac{t}{f(t)} \right)^n. \]

Show that the equation \( w = ze^{-z} \) has a solution, for sufficiently small \( w \) (how small?),

\[ z = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} w^n. \]

Find also one solution of the equation \( w = 2z - z^2 \).
4 Let $\phi(x, y)$ be a harmonic function. Show that $\phi$ is the real part of any analytic function $f(z)$ of the form

$$f(z) = 2\phi((z + 1)/2, (z - 1)/2i) - \phi(1, 0) + ic$$

where $c$ is a real constant (provided $\phi$ is such that the right hand side exists). Use this formula to find analytic functions whose real parts are (i) $x/(x^2 + y^2)$ and (ii) $\tan^{-1}y/x$. [Note: You might like to start by considering a harmonic conjugate $\psi(x, y)$, where $\phi$ and $\psi$ obey the Cauchy-Riemann conditions, and then write $f(z) = \phi(x, y) + i\psi(x, y) = \sum_{n=0}^{\infty} a_n(z-1)^n$.]

5 Let $f_1(z)$ be the branch of $(z^2 - 1)^{1/2}$ defined by branch cuts in the $z$-plane along the real axis from $-1$ to $-\infty$ and from $1$ to $\infty$, with $f_1(z)$ real and positive just above the latter cut. Let $f_2(z)$ be the branch of $(z^2 - 1)^{1/2}$ defined by a cut along the real axis from $-1$ to $+1$, with $f_2(x)$ real and positive for $(x - 1)$ real and positive. Show that $f_1(z) = f_1(-z)$ but $f_2(z) = -f_2(-z)$.

6 Let $P(z)$ be a polynomial of degree $n$, with $n$ roots, none of which lie on a simple close contour $L$. Show that

$$\frac{1}{2\pi i} \int_L \frac{P'(z)}{P(z)} \, dz = \text{number of roots lying within } L.$$  

[Note: Try to do this question without assuming the fundamental theorem of algebra.]

7 By integrating the function

$$\frac{(\ln z)^2}{z^2 + 1}$$

around an appropriate contour, compute the following integrals:

$$\int_0^\infty \frac{(\ln x)^m}{x^2 + 1} \, dx, \quad m = 1, 2.$$  

8 Evaluate

$$\int_0^\infty \frac{x^{m-1}}{x^2 + 1} \, dx, \quad 0 < m < 2.$$  

Why is it necessary for $m$ to satisfy the above restrictions?

9 Consider a rectangular contour $C$, with corners at $(N + \frac{1}{2})(\pm 1 \pm i)$ to evaluate

$$\frac{1}{2\pi i} \int_C \frac{\pi \cot \pi z \coth \pi z}{z^3} \, dz.$$  

In the limit as $N \to \infty$, show that

$$\sum_{n=1}^{\infty} \frac{\coth n\pi}{n^3} = \frac{7}{180} \pi^3.$$  

10 Evaluate the following integrals, where $|f(z)/z| \to 0$ as $|z| \to \infty$ and $f(z)$ is analytic in the upper half plane (including the real axis):

$$\text{(i)} \quad \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{ix}}{x} \, dx \quad \text{(ii)} \quad \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x(x-i)} \, dx \quad \text{(iii)} \quad \mathcal{P}\int_{-\infty}^{\infty} \frac{dx}{x-i} \quad \text{(iv)} \quad \mathcal{P}\int_{-\infty}^{\infty} \frac{e^{-x^2}}{x} \, dx.$$  

[In part (iv) try to use the parity of the integrand, because (as you will see) closing the contour by a large semi-circle does not work.]

11 Use the principal value technique to evaluate

$$\int_{0}^{\infty} \frac{\sin x}{x(x^2 + 1)} \, dx.$$