Further Complex Methods, Examples sheet 1 (C8a) Lent Term 2025

Mathematical Tripos Part II - C Course Professor Peter Haynes

Comments and corrections: e-mail to phh1@cam.ac.uk.

Questions 1-7, 12 and 13 revise material presented in IB Complex Methods. Questions 8-11 contain material that is new in Further Complex Methods.

Starred questions or parts of questions are intended as extras: attempt them if you have time, but not at the expense of unstarred questions.

1 Show that:

(i) $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2(x^2+4)} = \frac{\pi}{18}$;

(ii) $\displaystyle\int_{-\infty}^\infty \frac{\cos x\,dx}{x^2+a^2} = \frac{\pi}{a}e^{-a}$ where $a > 0$;

(iii) $\displaystyle\int_{-\infty}^\infty \frac{x-\sin x}{x^3}\,dx = \frac{\pi}{2}$;

2 Show that

(i) $\displaystyle\int_0^\infty \frac{x^\beta\,dx}{1+x}$, where $\beta \in \mathbb{C}$, converges for $-1 < \operatorname{Re}\beta < 0$.

(ii) $\displaystyle\int_\gamma (1+\tanh z)\ dz$, where $\{\gamma\ :\ z = se^{i\alpha},\ 0 \le s < \infty\}$ and $\alpha \in \mathbb{R}$, converges for $\alpha \in (-\pi, -\pi/2) \cup (\pi/2, \pi)$. Does the integral converge at $\alpha = \pi$?

3 Let $f(t)$ be analytic at $t = 0$ with $f(0) = 0$ and $f'(0) \neq 0$. Let C be a circle centred on the origin, with interior D, such that f is analytic in D and the inverse of f exists on $f(D)$.

For a fixed point z within C, let $w = f(z)$. Assuming that w is small, show (using the residue theorem) that

$$z = \frac{1}{2\pi i}\int_C \frac{tf'(t)}{f(t)-w}dt,$$

and hence that $z = \sum_{n=1}^{\infty} b_n w^n$, where

$$b_n = \frac{1}{2\pi i}\int_C \frac{tf'(t)}{(f(t))^{n+1}}dt = \frac{1}{2\pi in}\int_C \frac{1}{(f(t))^n}dt = \frac{1}{n!}\lim_{t\to 0}\frac{d^{n-1}}{dt^{n-1}}\left(\frac{t}{f(t)}\right)^n.$$

Show that the equation $w = ze^{-z}$ has a solution, for sufficiently small w (how small?),

$$z = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}w^n.$$

Find also one solution of the equation $w = 2z - z^2$.

4 Let $\phi(x, y)$ be a harmonic function. Show that ϕ is the real part of any analytic function $f(z)$ of the form

$$f(z) = 2\phi\big((z+1)/2, (z-1)/2i\big) - \phi(1,0) + ic$$

where c is a real constant (provided ϕ is such that the right hand side exists). Use this formula to find analytic functions whose real parts are (i) $x/(x^2+y^2)$ and (ii) $\tan^{-1} y/x$.

[Note: You might like to start by considering a harmonic conjugate $\psi(x, y)$, where ϕ and ψ obey the Cauchy-Riemann conditions, and then write $f(z) = \phi(x, y) + i\psi(x, y) = \sum_{n=0}^{\infty} a_n (z-1)^n$.]

5 Let $P(z)$ be a polynomial of degree n, with n roots, none of which lie on a simple closed contour L. Show that

$$\frac{1}{2\pi i}\int_L \frac{P'(z)}{P(z)} dz = \text{number of roots lying within } L\,,$$

where the roots should be counted according to their multiplicity.
[Note: Try to do this question without assuming a factorization for $P(z)$.]

6 Consider a rectangular contour C, with corners at $(N+\frac{1}{2})(\pm 1 \pm i)$ to evaluate

$$\frac{1}{2\pi i}\int_C \frac{\pi \cot \pi z \coth \pi z}{z^3} dz.$$

In the limit as $N \to \infty$, show that

$$\sum_{n=1}^{\infty} \frac{\coth n\pi}{n^3} = \frac{7}{180}\pi^3.$$

7 Evaluate

$$\int_0^{\infty} \frac{x^{m-1}}{x^2+1} dx, \quad 0 < m < 2.$$

Why is it necessary for m to satisfy the above restrictions?

8 Let

$$F(z) = \int_{-\infty}^{\infty} \frac{e^{uz}}{1+e^u}\, du\,.$$

For what region of the z-plane is the integral defined?

Show by closing the contour (use a rectangle) in the upper half plane that, when the integral is defined,

$$F(z) = \pi \operatorname{cosec} \pi z\,.$$

Explain how this result provides the analytic continuation of $F(z)$.

9 Evaluate the following integrals, where, in (ii), $|f(z)/z| \to 0$ as $|z| \to \infty$ and $f(z)$ is analytic in the upper half plane (including the real axis):

(i) $\mathcal{P}\displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx$ (ii) $\mathcal{P}\displaystyle\int_{-\infty}^{\infty} \frac{f(x)}{x(x-i)} dx$ (iii) $\mathcal{P}\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x-i}$ (iv) $\mathcal{P}\displaystyle\int_{-\infty}^{\infty} \frac{e^{-x^2}}{x} dx.$

Which of the integrals (i), (iii) and (iv) is real?

10 Use the principal value technique to evaluate

$$\int_0^\infty \frac{\sin x}{x(x^2+1)}dx.$$

11 By considering the function

$$\frac{e^{iz}-1}{z},$$

compute the Hilbert transforms of

$$\frac{\cos x - 1}{x} \quad \text{and} \quad \frac{\sin x}{x}.$$

12 Let $f_1(z)$ be the branch of $(z^2-1)^{\frac{1}{2}}$ defined by branch cuts in the z-plane along the real axis from -1 to $-\infty$ and from 1 to ∞, with $f_1(z)$ real and positive just above the latter cut. Let $f_2(z)$ be the branch of $(z^2-1)^{\frac{1}{2}}$ defined by a cut along the real axis from -1 to $+1$, with $f_2(x)$ real and positive for $(x-1)$ real and positive. Show that $f_1(z) = f_1(-z)$ but $f_2(z) = -f_2(-z)$.

13 * By integrating the function

$$\frac{(\ln z)^2}{z^2+1}$$

around an appropriate contour, compute the following integrals:

$$\int_0^\infty \frac{(\ln x)^m}{x^2+1}dx, \quad m = 1, 2.$$