Further Complex Methods, Examples sheet 2 (C8b) Professor Peter Haynes

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Starred questions or parts of questions are intended as extras: attempt them if you have time, but not at the expense of unstarred questions.

1 Define the branch of $f(z) = (1-z^2)^{\frac{1}{2}}$ by the branch cut along the real axis from -1 to $-\infty$ and from 1 to ∞, with $f(0) = 1$. Use this branch and a suitably chosen semi-circular contour (with finite radius R greater than 1) in the upper half plane to evaluate

$$\int_{-1}^{1} (1-x^2)^{\frac{1}{2}}\, dx\ .$$

2 The function $\sin^{-1} z$ is defined, for $0 \le \arg z < 2\pi$, by

$$\sin^{-1} z = \int_0^z \frac{dt}{\sqrt{1-t^2}}\ ,$$

where the integrand has a branch cut along the real axis from -1 to $+1$ and takes the value $+1$ at the origin on the upper side of the cut. The path of integration is a straight line for $0 \le \arg(z) \le \pi$ and is curved in a positive sense round the branch cut for $\pi < \arg z < 2\pi$. Express $\sin^{-1}(e^{i\pi} z)$ $(0 < \arg z < \pi)$ in terms of $\sin^{-1} z$ and deduce that $\sin(\phi - \pi) = -\sin\phi$. *Hint:* $\sin^{-1}(e^{i\pi}z) = -\pi + \sin^{-1} z$, as can be derived by calculating the integral half way round the cut and remembering that the integrand is an odd function.

3 Let $\omega_{m,n} = m\omega_1 + n\omega_2$, where (m, n) are integers not both zero, and let

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n}^{\infty} \left[\frac{1}{(z-\omega_{m,n})^2} - \frac{1}{{\omega_{m,n}}^2} \right]$$

be the Weierstrass elliptic function with periods (ω_1, ω_2) such that ω_1/ω_2 is not real. Show that, in a neighbourhood of $z = 0$,

$$\wp(z) = \frac{1}{z^2} + \frac{1}{20} g_2 z^2 + \frac{1}{28} g_3 z^4 + O(z^6)$$

where

$$g_2 = 60 \sum_{m,n} (\omega_{m,n})^{-4}, \quad g_3 = 140 \sum_{m,n} (\omega_{m,n})^{-6}.$$

Deduce that $\wp$ satisfies a 1st order nonlinear ODE

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3.$$

4 (a) Show that

$$4\wp(2z) - \left(\frac{\wp''(z)}{\wp'(z)}\right)^2 + 8\wp(z) = 0.$$

(b)* Show that

$$\wp(w+z) = \frac{1}{4}\left[\frac{\wp'(z)-\wp'(w)}{\wp(z)-\wp(w)}\right]^2 - \wp(z) - \wp(w).$$

The result in (a) is a special case of this.

5 * Show, by considering contour integrals around a suitably defined cell, that a doubly-periodic function cannot have a single pole of order 1 within the cell. Show also that the number of poles and the number of zeros within the cell are equal (with appropriate counting for repeated roots, poles of order greater than 1, etc).

6 By using a contour consisting of the boundary of a quadrant, indented at the origin, show that (for a range of z to be stated)

$$\int_0^\infty t^{z-1}e^{-it}\,dt = e^{-\frac{1}{2}\pi i z}\Gamma(z).$$

Hence evaluate (again, for ranges of z to be stated)

$$\int_0^\infty t^{z-1}\cos t\,dt \quad \text{and} \quad \int_0^\infty t^{z-1}\sin t\,dt.$$

Use your results to evaluate $\displaystyle\int_0^\infty \frac{\cos t}{t^{1/2}}\,dt$, $\displaystyle\int_0^\infty \frac{\sin t}{t}\,dt$ and $\displaystyle\int_0^\infty \frac{\sin t}{t^{3/2}}\,dt$.

7 Starting with the infinite product representation of the Gamma function (Weierstrass canonical product) and using the definition of γ, derive the Euler's product formula, i.e.

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\; n^z}{z(1+z)(2+z)\ldots(n+z)} \; .$$

8

(a) Use Stirling's approximation $\sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}/n! \to 1$ as $n \to \infty$ and the Euler's product formula to show that

$$\Gamma_n(z) := \frac{\sqrt{2\pi}e^{-n}n^{z+n+\frac{1}{2}}}{z(z+1)\cdots(z+n)} \to \Gamma(z)$$

as $n \to \infty$.

Hence, prove that

$$\frac{2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})}{\Gamma(2z)}$$

is a constant independent of z. Then, by letting $z \to \frac{1}{2}$ evaluate the relevant constant and thus establish the following identity:

$$2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}\Gamma(2z).$$

(b) *Furthermore, by constructing $\Gamma_n(z)\Gamma_n(z+\frac{1}{m})\cdots\Gamma_n(z+\frac{m-1}{m})/\Gamma_{nm}(mz)$, prove the Gauss multiplication formula

$$\Gamma(z)\Gamma(z+\frac{1}{m})\Gamma(z+\frac{2}{m})\ldots\Gamma(z+\frac{m-1}{m}) = (2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}-mz}\Gamma(mz),$$

for $m = 1, 2, \ldots$ and $mz \neq 0, -1, -2, \ldots$.

9 Using $t = s\tau$, $s > 0$, it follows that

$$\frac{\Gamma(z)}{s^z} = \int_0^\infty e^{-s\tau}\tau^{z-1}d\tau.$$

Letting $z = 1$ and integrating the resulting formula with respect to s from 1 to t, show that

$$\ln t = \int_0^\infty \left(e^{-\tau} - e^{-t\tau}\right)\frac{d\tau}{\tau}.$$

Using this formula in the expression for $\Gamma'(z)$, prove that

$$\frac{\Gamma'(z)}{\Gamma(z)} = \int_0^\infty \left(e^{-\tau} - \frac{1}{(1+\tau)^z}\right)\frac{d\tau}{\tau}.$$

Hence, deduce that

$$\gamma = -\int_0^\infty \left(e^{-\tau} - \frac{1}{1+\tau}\right)\frac{d\tau}{\tau}\ .$$

10 Show that

$$E_1(k) = \int_k^\infty \frac{e^{-t}}{t}\,dt = -\gamma - \ln k + k - \frac{k^2}{4} + O(k^3), \quad k \to 0^+.$$

Hint:

$$E_1(k) = \int_k^\infty \frac{dt}{t(t+1)} + \int_0^\infty \left(e^{-t} - \frac{1}{t+1}\right)\frac{dt}{t} - \int_0^k \left(e^{-t} - \frac{1}{t+1}\right)\frac{dt}{t}.$$

11 Derive the formula $\mathrm{B}(p,q) = 2\int_0^{\frac{\pi}{2}} \sin^{2p-1}\theta\cos^{2q-1}\theta\,d\theta$ and prove that

$$\mathrm{B}(z,z) = 2^{1-2z}\mathrm{B}(z,\tfrac{1}{2}).$$

For which values of z does this result hold?

12 Show, using properties of the B function, that

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \tfrac{1}{\sqrt{32\pi}}\left(\Gamma(\tfrac{1}{4})\right)^2$$

Using the change of variable $x = t(2-t^2)^{-\frac{1}{2}}$, deduce that

$$\mathrm{K}(\tfrac{1}{\sqrt{2}}) = \tfrac{4}{\sqrt{\pi}}\left(\Gamma(\tfrac{5}{4})\right)^2 ,$$

where $\mathrm{K}(k)$ is the complete elliptic integral $\displaystyle\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$.