

Comments and corrections: e-mail to bg268@cam.ac.uk.

**1** By using a contour consisting of the boundary of a quadrant, indented at the origin, show that (for a range of  $z$  to be stated)

$$\int_0^\infty t^{z-1} e^{-it} dt = e^{-\frac{1}{2}\pi iz} \Gamma(z).$$

Hence evaluate (again, for ranges of  $z$  to be stated)

$$\int_0^\infty t^{z-1} \cos t dt \quad \text{and} \quad \int_0^\infty t^{z-1} \sin t dt.$$

Use your results to evaluate  $\int_0^\infty \frac{\cos t}{t^{1/2}} dt$ ,  $\int_0^\infty \frac{\sin t}{t} dt$  and  $\int_0^\infty \frac{\sin t}{t^{3/2}} dt$ .

**2** Derive the formula  $B(p, q) = 2 \int_0^{\frac{\pi}{2}} \sin^{2p-1} \theta \cos^{2q-1} \theta d\theta$  and prove that

$$B(z, z) = 2^{1-2z} B(z, \frac{1}{2}).$$

For which values of  $z$  does this result hold?

**3** Show, using properties of the B function, that

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{\sqrt{32\pi}} \left(\Gamma\left(\frac{1}{4}\right)\right)^2$$

Using the change of variable  $x = t(2-t^2)^{-\frac{1}{2}}$ , deduce that

$$K\left(\frac{1}{\sqrt{2}}\right) = \frac{4}{\sqrt{\pi}} \left(\Gamma\left(\frac{5}{4}\right)\right)^2,$$

where  $K(k)$  is the complete elliptic integral  $\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$ .

**4** By using the infinite product representation of the Gamma function (Weierstrass canonical product), prove that

$$\frac{2^{2z} \Gamma(z) \Gamma(z + \frac{1}{2})}{\Gamma(2z)}$$

is a constant independent of  $z$ . Then, by letting  $z \rightarrow 0$  evaluate the relevant constant and thus establish the following identity:

$$2^{2z-1} \Gamma(z) \Gamma(z + \frac{1}{2}) = \sqrt{\pi} \Gamma(2z).$$

Furthermore, show that for  $m = 1, 2, 3, \dots$ , the following identity is valid:

$$\Gamma(z) \Gamma(z + \frac{1}{m}) \Gamma(z + \frac{2}{m}) \dots \Gamma(z + \frac{m-1}{m}) = m^{\frac{1}{2}-mz} (2\pi)^{\frac{m-1}{2}} \Gamma(mz).$$

5 Using  $t = s\tau$ ,  $s > 0$ , it follows that

$$\frac{\Gamma(z)}{s^z} = \int_0^\infty e^{-s\tau} \tau^{z-1} d\tau.$$

Letting  $z = 1$  and integrating the resulting formula with respect to  $s$  from 1 to  $t$ , show that

$$\ln t = \int_0^\infty \left( e^{-\tau} - e^{-t\tau} \right) \frac{d\tau}{\tau}.$$

Using this formula in the expression for  $\Gamma'(z)$ , prove that

$$\frac{\Gamma'(z)}{\Gamma(z)} = \int_0^\infty \left( e^{-\tau} - \frac{1}{(1+\tau)^z} \right) \frac{d\tau}{\tau}.$$

Hence, deduce that

$$\gamma = - \int_0^\infty \left( e^{-\tau} - \frac{1}{1+\tau} \right) \frac{d\tau}{\tau}.$$

6 Starting with the infinite product representation of the Gamma function and using the definition of  $\gamma$ , derive Euler's product representation, i.e.

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{z(1+z)(2+z)\dots(n+z)}.$$

7 (a) Prove that for  $\operatorname{Re} z > 1$ ,

$$\frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1} dt = \frac{\Gamma(1-z)}{2i\pi} \int_\gamma \frac{t^{z-1}}{e^{-t} - 1} dt,$$

where  $\gamma$  denotes the Hankel contour. Hence, deduce that the RHS of the above equation provides the analytic continuation of Riemann's zeta function.

(b) The Bernoulli numbers  $B_n$  are defined by

$$\frac{1}{e^t - 1} = \sum_{m=0}^\infty B_m \frac{t^{m-1}}{m!},$$

and  $B_0 = 1$ ,  $B_1 = -\frac{1}{2}$ ,  $B_{2m+1} = 0$  for  $m = 1, 2, \dots$ .

Use (a) and the residue theorem to compute  $\zeta(-n)$ ,  $n = 0, 1, 2, \dots$  in terms of  $B_n$ . Hence, deduce that the negative even integers are zeros of  $\zeta(z)$ .

8 Show that

$$E_1(k) = \int_k^\infty \frac{e^{-t}}{t} dt = -\gamma - \ln k + k - \frac{k^2}{4} + O(k^3), \quad k \rightarrow 0^+.$$

*Hint:*

$$E_1(k) = \int_k^\infty \frac{dt}{t(t+1)} + \int_0^\infty \left( e^{-t} - \frac{1}{t+1} \right) \frac{dt}{t} - \int_0^k \left( e^{-t} - \frac{1}{t+1} \right) \frac{dt}{t}.$$

**9**

Show that for  $\operatorname{Re} z > 1$

$$(1 - 2^{1-z})\zeta(z) = (1^{-z} - 2^{-z} + 3^{-z} - 4^{-z} \dots) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t + 1} dt.$$

[Note: This result is actually valid for  $\operatorname{Re} z > 0$ .]

**10**

Show that

$$\int_{-\infty}^{(0+)} \frac{\ln t}{e^{-t} - 1} dt = 0.$$

Hence show that

$$\lim_{z \rightarrow 1} (\zeta(z) - (z-1)^{-1}) = \gamma,$$

and

$$\zeta'(0) = -\ln \sqrt{2\pi}.$$

**11**

The psi-function is defined to be

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z).$$

Show that

$$\psi'(z) = \sum_{s=0}^{\infty} \frac{1}{(s+z)^2}, \quad (z \neq 0, -1, -2, \dots).$$

Then show that when  $z$  is real and positive, that  $\Gamma(z)$  has a single minimum which lies somewhere between  $z = 1$  and  $z = 2$ .

Hence show that

$$\ln \Gamma(z) = -\gamma(z-1) + \sum_{s=2}^{\infty} (-)^s \frac{\zeta(s)}{s} (z-1)^s.$$