Further Complex Methods, Examples sheet 3 (C8c) Lent Term 2024
Mathematical Tripos Part II - C Course Dr Daria Frank

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Starred questions or parts of questions are intended as extras: attempt them if you have time, but not at the expense of unstarred questions.

1  (a) Prove that for Re\(z > 1\),
\[
\frac{1}{\Gamma(z)} \int_{0}^{\infty} \frac{t^{z-1}}{e^t - 1} \, dt = \frac{\Gamma(1 - z)}{2i\pi} \int_{\gamma} \frac{t^{z-1}}{e^{-t} - 1} \, dt,
\]
where \(\gamma\) denotes the Hankel contour. Hence, deduce that the RHS of the above equation provides the analytic continuation of Riemann’s zeta function.

(b) The Bernoulli numbers \(B_n\) are defined by
\[
\frac{1}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^{m-1}}{m!},
\]
and \(B_0 = 1, B_1 = -\frac{1}{2}, B_{2m+1} = 0\) for \(m = 1, 2, \ldots\).

Use (a) and the residue theorem to compute \(\zeta(-n), n = 0, 1, 2, \ldots\) in terms of \(B_n\). Hence, deduce that the negative even integers are zeros of \(\zeta(z)\).

2  Show that for Re\(z > 1\)
\[
(1 - 2^{1-z})\zeta(z) = (1^{-z} - 2^{-z} + 3^{-z} - 4^{-z} \ldots) = \frac{1}{\Gamma(z)} \int_{0}^{\infty} \frac{t^{z-1}}{e^t + 1} \, dt.
\]
[Note: This result is actually valid for Re\(z > 0\).]

3  Show that
\[
\int_{-\infty}^{(0+)} \frac{\ln t}{e^{-t} - 1} \, dt = 0.
\]
Hence show that
\[
\lim_{z \to 1} (\zeta(z) - (z - 1)^{-1}) = \gamma,
\]
and
\[
\zeta'(0) = -\ln \sqrt{2\pi}.
\]
4  The psi-function is defined to be
\[ \psi(z) = \frac{d}{dz} \ln \Gamma(z). \]
Show that
\[ \psi'(z) = \sum_{s=0}^{\infty} \frac{1}{(s + z)^2}, \quad (z \neq 0, -1, -2, \ldots). \]
Then show that when \( z \) is real and positive, that \( \Gamma(z) \) has a single minimum which lies between \( z = 1 \) and \( z = 2 \).

Show that, for \( |z - 1| \) sufficiently small,
\[ \ln \Gamma(z) = -\gamma(z - 1) + \sum_{s=2}^{\infty} (-1)^s \frac{\zeta(s)}{s} (z - 1)^s. \]
What is the radius of convergence of this series?

5  Find two independent solutions of the Airy equation \( w'' - zw = 0 \) in the form
\[ w(z) = \int_{\gamma} e^{zt} f(t) \, dt, \]
where \( \gamma \) is to be specified in each case. Show that there is a solution for which \( \gamma \) can be chosen to consist of two straight line segments in the left half \( t \)-plane (Re \( t \leq 0 \)).

For this solution show that, if \( w(z) \) is normalised so that \( w(0) = iA 3^{-1/6} \Gamma(1/3) \), where \( A \) is a constant, then \( w'(0) = -iA 3^{1/6} \Gamma(2/3) \).
[Note: \( \Gamma(z) = \int_{0}^{\infty} e^{-t} t^{z-1} \, dt \) for Re \( z > 0 \).]

6  By writing \( w(z) \) in the form of an integral representation with the Laplace kernel show that the confluent hypergeometric equation \( zw'' + (c - z)w' - aw = 0 \) has solutions of the form
\[ w(z) = \int_{\gamma} t^{a-1} (1-t)^{c-a-1} e^{tz} \, dt, \]
provided the path \( \gamma \) is chosen such that \( [t^{a}(1-t)^{c-a} e^{tz}]_{\gamma} = 0. \)

In the case Re \( z > 0 \), find paths which provide two independent solutions in each of the following cases (where \( m \) is a positive integer):
(i) \( a = -m, c = 0 \);
(ii) Re \( a < 0, c = 0, a \) is not an integer;
(iii) \( a = 0, c = m \);
(iv) Re \( c > \text{Re} \, a > 0, a \) and \( c - a \) are not integers.
Use the Laplace transform to solve the ordinary differential equation
\[ \frac{d^2y}{dt^2} - k^2y = f(t), \quad k > 0, \quad y(0) = y_0, \quad y'(0) = y'_0. \]

Let \( f(t) = e^{-k_0 t}, k_0 \neq k, k_0 > 0 \), so that the Laplace transform of \( f(t) \) is
\[ \hat{f}(s) = \frac{1}{s + k_0}. \]

Show that
\[ y(t) = y_0 \cosh kt + \frac{y'_0}{k} \sinh kt + \frac{e^{-k_0 t}}{k_2 - k^2} \cosh kt - \frac{k_0}{k_0^2 - k^2} \sinh kt. \] (1)

Now suppose that \( f(t) \) is an arbitrary continuous function that possesses a Laplace transform. Use the convolution theorem for Laplace transforms, or otherwise, to show that
\[ y(t) = y_0 \cosh kt + \frac{y'_0}{k} \sinh kt + \int_0^t f(t') \frac{\sinh k(t - t')}{k} dt'. \]

Put \( f(t) = e^{-k_0 t} \) and re-obtain your answer to the first part of this question. Suppose now that \( k_0 = k \). What is \( y(t) \)? Could you have found this solution by taking the limit in (1) as \( k_0 \to k \)?

The Schrödinger equation is
\[ i \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0. \]

Suppose that \( u(x, 0) = f(x) \).

Fourier transform this equation with respect to \( x \) to find
\[ u(x, t) = \frac{e^{-i\pi/4}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{\frac{i(x-x')^2}{4t}} f(x') dx'. \]

(You may it useful to recall that \( \int_{-\infty}^{\infty} e^{ia^2} du = e^{i\pi/4} \sqrt{\pi} \).)

Now use Laplace transform methods to find the same solution to this problem.
A simple version of the Klein-Gordon equation is
\[ \psi_{tt} = \psi_{xx} - \psi. \] (2)

(This equation describes, amongst other things, the propagation of large-scale variations in the height of the sea surface in the presence of rotation.)

(a) Solve this equation subject to the initial conditions \( \psi(x, 0) = 0, \psi_t(x, 0) = \delta(x) \) using Laplace transform methods. Show that, for \( t < |x|, \psi(x, t) = 0 \), and, for \( t > |x|, \)
\[
\psi(x, t) = \frac{1}{2\pi i} \int_{\gamma} e^{st} \exp(- (1 + s^2)^{1/2}|x|) \frac{ds}{2(1 + s^2)^{1/2}}
\]
where \( \gamma \), followed anticlockwise, encloses a branch cut along the imaginary axis from \( s = -i \) to \( s = i \).

(b) Show that, defining the variable \( w \) by \( (t^2 - x^2)^{1/2}w = st - (1 + s^2)^{1/2}|x| \), the above integral may be transformed to give
\[
\psi(x, t) = \frac{1}{2\pi i} \int_{\gamma} \exp((t^2 - x^2)^{1/2}w) \frac{dw}{2(1 + w^2)^{1/2}}
\]
with \( \gamma \) defined in the \( w \)-plane as in the \( s \)-plane.

(c) Show using Laplace’s method that \( J_0(z) \), which is the solution of \( zy'' + y' + zy = 0 \) such that \( y(0) = 1 \) and \( y'(0) = 0 \) can be represented as
\[
J_0(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{e^{zs}}{(1 + s^2)^{1/2}} ds
\]
with \( \gamma \) again as defined above.

(d) Deduce that the solution of (2) specified above for \( t > |x| \) may be written as
\[
\psi(x, t) = \frac{1}{2} J_0((t^2 - x^2)^{1/2}).
\]

Draw a sketch of the solution.