Further Complex Methods, Examples sheet 3 (C8c) Lent Term 2025

Mathematical Tripos Part II - C Course Professor Peter Haynes

Comments and corrections: e-mail to phh1@cam.ac.uk.

Starred questions or parts of questions are intended as extras: attempt them if you have time, but not at the expense of unstarred questions.

1 (a) Prove that for $\operatorname{Re} z > 1$,

$$\frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1}\, dt = \frac{\Gamma(1-z)}{2i\pi} \int_\gamma \frac{t^{z-1}}{e^{-t} - 1}\, dt,$$

where γ denotes the Hankel contour. Hence, deduce that the RHS of the above equation provides the analytic continuation of Riemann's zeta function.

(b) The Bernoulli numbers B_n are defined by

$$\frac{1}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^{m-1}}{m!},$$

and $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_{2m+1} = 0$ for $m = 1, 2, \dots$.

Use (a) and the residue theorem to compute $\zeta(-n)$, $n = 0, 1, 2, \dots$ in terms of B_n. Hence, deduce that the negative even integers are zeros of $\zeta(z)$.

2 Show that for $Re\, z > 1$

$$(1 - 2^{1-z})\zeta(z) = (1^{-z} - 2^{-z} + 3^{-z} - 4^{-z} \cdots) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t + 1} dt.$$

[Note: This result is actually valid for $Re\, z > 0$.]

3 Show that

$$\int_{-\infty}^{(0+)} \frac{\ln t}{e^{-t} - 1} dt = 0.$$

Hence show that

$$\lim_{z \to 1} (\zeta(z) - (z-1)^{-1}) = \gamma,$$

and

$$\zeta'(0) = -\ln \sqrt{2\pi}.$$

4 The psi-function is defined to be

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z).$$

Show that

$$\psi'(z) = \sum_{s=0}^{\infty} \frac{1}{(s+z)^2}, \qquad (z \neq 0, -1, -2 \cdots).$$

[Recall Q9 on Example Sheet 2.]

Then show that when z is real and positive, that $\Gamma(z)$ has a single minimum which lies between $z = 1$ and $z = 2$.

Show that, for $|z - 1|$ sufficiently small,

$$\ln \Gamma(z) = -\gamma(z-1) + \sum_{s=2}^{\infty} (-1)^s \frac{\zeta(s)}{s} (z-1)^s.$$

What is the radius of convergence of this series?

5 Find two independent solutions of the Airy equation $w'' - zw = 0$ in the form

$$w(z) = \int_{\gamma} e^{zt} f(t)\, dt,$$

where γ is to be specified in each case. Show that there is a solution for which γ can be chosen to consist of two straight line segments in the left half t-plane (Re $t \leq 0$).

For this solution show that, if $w(z)$ is normalised so that $w(0) = iA\, 3^{-\frac{1}{6}} \Gamma(1/3)$, where A is a constant, then $w'(0) = -iA\, 3^{\frac{1}{6}} \Gamma(2/3)$.

[Note: $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ for $Re\, z > 0$.]

6 By writing $w(z)$ in the form of an integral representation with the Laplace kernel show that the confluent hypergeometric equation $zw'' + (c - z)w' - aw = 0$ has solutions of the form

$$w(z) = \int_{\gamma} t^{a-1} (1-t)^{c-a-1} e^{tz}\, dt,$$

provided the path γ is chosen such that $[t^a (1-t)^{c-a} e^{tz}]_{\gamma} = 0$.

In the case $\mathrm{Re}\, z > 0$, find paths which provide two independent solutions in each of the following cases (where m is a positive integer):

(i) $a = -m$, $c = 0$;
(ii) $\mathrm{Re}\, a < 0$, $c = 0$, a is not an integer;
(iii) $a = 0$, $c = m$;
(iv) $\mathrm{Re}\, c > \mathrm{Re}\, a > 0$, a and $c - a$ are not integers.

7 Use the Laplace transform to solve the ordinary differential equation

$$\frac{d^2y}{dt^2} - k^2 y = f(t), \quad k > 0, \quad y(0) = y_0, \quad y'(0) = y_0'.$$

Let $f(t) = e^{-k_0 t}, k_0 \neq k, k_0 > 0$, so that the Laplace transform of $f(t)$ is

$$\hat{f}(s) = \frac{1}{s + k_0}.$$

Show that

$$y(t) = y_0 \cosh kt + \frac{y_0'}{k} \sinh kt + \frac{e^{-k_0 t}}{k_0^2 - k^2} - \frac{\cosh kt}{k_0^2 - k^2} + \frac{\frac{k_0}{k}}{k_0^2 - k^2} \sinh kt. \qquad (1)$$

Now suppose that $f(t)$ is an arbitrary continuous function that possesses a Laplace transform. Use the convolution theorem for Laplace transforms, or otherwise, to show that

$$y(t) = y_0 \cosh kt + \frac{y_0'}{k} \sinh kt + \int_0^t f(t') \frac{\sinh k(t - t')}{k} dt'.$$

Put $f(t) = e^{-k_0 t}$ and re-obtain your answer to the first part of this question. Suppose now that $k_0 = k$. What is $y(t)$? Could you have found this solution by taking the limit in (1) as $k_0 \to k$?

8 The Schrödinger equation is

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0.$$

Suppose that $u(x, 0) = f(x)$.
Fourier transform this equation with respect to x to find

$$u(x, t) = \frac{e^{-i\pi/4}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{\frac{i(x - x')^2}{4t}} f(x') dx'.$$

(You may it useful to recall that $\int_{-\infty}^{\infty} e^{iu^2} du = e^{\frac{i\pi}{4}} \sqrt{\pi}$.)

Now use Laplace transform methods to find the same solution to this problem.

9 A linear system has transfer function $G(s)$, where s is the Laplace transform variable. The output of the system is passed through a linear device with transfer function $H(s)$ and then provided to the input as a negative feedback. Show that the transfer function of the system with feedback is $G(s)/[1+G(s)H(s)]$. Explain the Nyquist procedure for determining the stability of the system with feedback by considering the variation of $G(s)H(s)$ as s travels round a path in the complex plane.

Consider the case

$$G(s) = \frac{K}{(1+sT)Js^2}, \quad H(s) = 1 + bs$$

with the constants J, K, b and T all positive. Take the path for s to be composed of AB: $s = ix, \epsilon < x < R$; BC: $s = Re^{i\theta}, \frac{1}{2}\pi > \theta > -\frac{1}{2}\pi$; CD: $s = ix, -R < x < -\epsilon$; DA: $s = \epsilon e^{i\theta}, -\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. Describe the corresponding path in the ω plane, with $\omega = G(s)H(s)$. In particular: (a) What is the image of BC? (b) Does each of the images of AB and CD lie above or below the real axis; (c) What is the image of DA, paying particular attention to whether the each of the images of A and D lie above or below the real axis. Deduce that the system is stable if $b > T$ and unstable if $b < T$. In the latter case how many zeros of $1 + G(s)H(s)$ are there in the right-hand half s-plane?

10 * A simple version of the Klein-Gordon equation is

$$\psi_{tt} = \psi_{xx} - \psi. \tag{2}$$

(This equation describes, amongst other things, the propagation of large-scale variations in the height of the sea surface in the presence of rotation.)

(a) Solve this equation subject to the initial conditions $\psi(x,0) = 0$, $\psi_t(x,0) = \delta(x)$ using Laplace transform methods. Show that, for $t < |x|$, $\psi(x,t) = 0$, and, for $t > |x|$,

$$\psi(x,t) = \frac{1}{2\pi i}\int_\gamma e^{st}\,\exp(-(1+s^2)^{1/2}|x|)\,\frac{ds}{2(1+s^2)^{1/2}}$$

where γ, followed anticlockwise, encloses a branch cut along the imaginary axis from $s = -i$ to $s = i$.

(b) Show that, defining the variable w by $(t^2 - x^2)^{1/2}w = st - (1+s^2)^{1/2}|x|$, the above integral may be transformed to give

$$\psi(x,t) = \frac{1}{2\pi i}\int_\gamma \exp((t^2-x^2)^{1/2}w)\,\frac{dw}{2(1+w^2)^{1/2}}$$

with γ defined in the w-plane as in the s-plane.

(c) Show using Laplace's method that $J_0(z)$, which is the solution of $zy'' + y' + zy = 0$ such that $y(0) = 1$ and $y'(0) = 0$ can be represented as

$$J_0(z) = \frac{1}{2\pi i}\int_\gamma \frac{e^{zs}}{(1+s^2)^{1/2}}\,ds$$

with γ again as defined above. Deduce that the solution of (2) specified above for $t > |x|$ may be written as

$$\psi(x,t) = \frac{1}{2}J_0((t^2-x^2)^{1/2}).$$

Draw a sketch of the solution.