1 (a) Prove that for \( \text{Re} z > 1 \),
\[
\frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^t - 1} \, dt = \frac{\Gamma(1 - z)}{2i\pi} \int_{\gamma} \frac{t^{z-1}}{e^{-t} - 1} \, dt,
\]
where \( \gamma \) denotes the Hankel contour. Hence, deduce that the RHS of the above equation provides the analytic continuation of Riemann’s zeta function.

(b) The Bernoulli numbers \( B_n \) are defined by
\[
\frac{1}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!},
\]
and \( B_0 = 1, \, B_1 = -\frac{1}{2}, \, B_{2m+1} = 0 \) for \( m = 1, 2, \ldots \).

Use (a) and the residue theorem to compute \( \zeta(-n), \, n = 0, 1, 2, \ldots \) in terms of \( B_n \). Hence, deduce that the negative even integers are zeros of \( \zeta(z) \).

2 Show that for \( \text{Re} z > 1 \)
\[
(1 - 2^{1-z})\zeta(z) = (1^{-z} - 2^{-z} + 3^{-z} - 4^{-z} \cdots) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^t + 1} \, dt.
\]
[Note: This result is actually valid for \( \text{Re} z > 0 \).]

3 Show that
\[
\int_{-\infty}^{(0+)} \frac{\ln t}{e^{-t} - 1} \, dt = 0.
\]
Hence show that
\[
\lim_{z \to 1} (\zeta(z) - (z - 1)^{-1}) = \gamma,
\]
and
\[
\zeta'(0) = -\ln \sqrt{2\pi}.
\]
The psi-function is defined to be
\[ \psi(z) = \frac{d}{dz} \ln \Gamma(z). \]
Show that
\[ \psi'(z) = \sum_{s=0}^{\infty} \frac{1}{(s + z)^2}, \quad (z \neq 0, -1, -2 \cdots). \]
Then show that when \( z \) is real and positive, that \( \Gamma(z) \) has a single minimum which lies between \( z = 1 \) and \( z = 2 \).

Show that, for \( |z - 1| \) sufficiently small,
\[ \ln \Gamma(z) = -\gamma(z - 1) + \sum_{s=2}^{\infty} (-1)^s \frac{\zeta(s)}{s} (z - 1)^s. \]
What is the radius of convergence of this series?

Find two independent solutions of the Airy equation \( w'' - zw = 0 \) in the form
\[ w(z) = \int_{\gamma} e^{zt} f(t) \, dt, \]
where \( \gamma \) is to be specified in each case. Show that there is a solution for which \( \gamma \) can be chosen to consist of two straight line segments in the left half \( t \)-plane (Re \( t \leq 0 \)).

For this solution show that, if \( w(z) \) is normalised so that \( w(0) = iA 3^{-1/3} \Gamma(1/3) \), where \( A \) is a constant, then \( w'(0) = -iA 3^{2/3} \Gamma(2/3) \).

[Note: \( \Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} \, dt \) for Re \( z > 0 \).]

By writing \( w(z) \) in the form of an integral representation with the Laplace kernel show that the confluent hypergeometric equation \( zw'' + (c-z)w' - aw = 0 \) has solutions of the form
\[ w(z) = \int_{\gamma} t^{a-1} (1-t)^{c-a-1} e^{t z} \, dt, \]
provided the path \( \gamma \) is chosen such that \( [t^{a}(1-t)^{c-a} e^{t z}]_\gamma = 0 \).

In the case Re \( z > 0 \), find paths which provide two independent solutions in each of the following cases (where \( m \) is a positive integer):
(i) \( a = -m, \ c = 0; \)
(ii) Re \( a < 0, \ c = 0, \ a \) is not an integer;
(iii) \( a = 0, \ c = m; \)
(iv) Re \( c > \text{Re} \ a > 0, \ a \) and \( c - a \) are not integers.
Use the Laplace transform to solve the ordinary differential equation
\[
\frac{d^2 y}{dt^2} - k^2 y = f(t), \quad k > 0, \quad y(0) = y_0, \quad y'(0) = y'_0.
\]

Let \( f(t) = e^{-k_0 t}, k_0 \neq k, k_0 > 0 \), so that the Laplace transform of \( f(t) \) is
\[
\hat{f}(s) = \frac{1}{s + k_0}.
\]

Show that
\[
y(t) = y_0 \cosh kt + \frac{y'_0}{k} \sinh kt + \frac{e^{-k_0 t}}{k^2 - k^2} - \frac{\cosh kt}{k^2 - k^2} + \frac{k_0}{k^2 - k^2} \sinh kt. \tag{1}
\]

Now suppose that \( f(t) \) is an arbitrary continuous function that possesses a Laplace transform. Use the convolution theorem for Laplace transforms, or otherwise, to show that
\[
y(t) = y_0 \cosh kt + \frac{y'_0}{k} \sinh kt + \int_0^t f(t') \frac{\sinh k(t - t')}{k} dt'.
\]

Put \( f(t) = e^{-k_0 t} \) and re-obtain your answer to the first part of this question. Suppose now that \( k_0 = k \). What is \( y(t) \)? Could you have found this solution by taking the limit in (1) as \( k_0 \to k \)?

The Schrödinger equation is
\[
i \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0.
\]

Suppose that \( u(x, 0) = f(x) \).

Fourier transform this equation with respect to \( x \) to find
\[
u(x, t) = \frac{e^{-i\pi/4}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{i(x-x')^2/4t} f(x') dx'.
\]

(You may it useful to recall that \( \int_{-\infty}^{\infty} e^{iu^2} du = e^{i\pi/4} \sqrt{\pi} \).)

Now use Laplace transform methods to find the same solution to this problem.
A simple version of the Klein-Gordon equation is

$$\psi_{tt} = \psi_{xx} - \psi.$$  \hfill (2)

(This equation describes, amongst other things, the propagation of large-scale variations in the height of the sea surface in the presence of rotation.)

(a) Solve this equation subject to the initial conditions \(\psi(x, 0) = 0, \psi_t(x, 0) = \delta(x)\) using Laplace transform methods. Show that, for \(t < |x|\), \(\psi(x, t) = 0\), and, for \(t > |x|\),

$$\psi(x, t) = \frac{1}{2\pi i} \int_{\gamma} e^{st} \exp(-(1 + s^2)^{1/2}|x|) \frac{ds}{2(1 + s^2)^{1/2}}$$

where \(\gamma\), followed anticlockwise, encloses a branch cut along the imaginary axis from \(s = -i\) to \(s = i\).

(b) Show that, defining the variable \(w\) by \((t^2 - x^2)^{1/2} w = st - (1 + s^2)^{1/2}|x|\), the above integral may be transformed to give

$$\psi(x, t) = \frac{1}{2\pi i} \int_{\gamma} \exp(\{(t^2 - x^2)^{1/2} y\} \frac{dw}{2(1 + w^2)^{1/2}}$$

with \(\gamma\) defined in the \(w\)-plane as in the \(s\)-plane.

(c) Show using Laplace’s method that \(J_0(z)\), which is the solution of \(zy'' + y' + zy = 0\) such that \(y(0) = 1\) and \(y'(0) = 0\) can be represented as

$$J_0(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{e^{zs}}{(1 + s^2)^{1/2}} ds$$

with \(\gamma\) again as defined above.

(d) Deduce that the solution of (2) specified above for \(t > |x|\) may be written as

$$\psi(x, t) = \frac{1}{2} J_0((t^2 - x^2)^{1/2}).$$

Draw a sketch of the solution.