Further Complex Methods, Examples sheet 4 (C8d) Lent Term 2025

Mathematical Tripos Part II - C Course Professor Peter Haynes

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1 Show that the most general linear second order ordinary differential equation whose only singularities are regular singular points at $z = a$ and $z = b$ can be written in the form

$$w'' + \left[\frac{1-A}{z-a} + \frac{1+A}{z-b}\right] w' + \frac{B(a-b)^2}{(z-a)^2(z-b)^2} w \ = \ 0\,, \qquad (\dagger)$$

where A and B are arbitrary constants.

Write down and solve the equation when the two singular points are at 0 and ∞, in the two cases $A^2 \neq 4B$ and $A^2 = 4B$. Use a Möbius transformation to deduce the general solution of ($\dagger$). What is the significance of the two constants α and α' which satisfy $\alpha + \alpha' = A$ and $\alpha\alpha' = B$?

2 Consider the second order linear differential equation with three, and only three, regular singular points in the finite complex plane, all other points being ordinary points,

$$\frac{d^2y}{dz^2} + \Big(\frac{A}{z-z_1} + \frac{B}{z-z_2} + \frac{C}{z-z_3}\Big)\frac{dy}{dz} + \frac{1}{(z-z_1)(z-z_2)(z-z_3)}\Big(\frac{D}{z-z_1} + \frac{E}{z-z_2} + \frac{F}{z-z_3}\Big)y = 0.$$

Show that by making a fractional linear transformation

$$z \to z' = \frac{(z_3-z_2)(z-z_1)}{(z_3-z_1)(z-z_2)}$$

that the new differential equation is of the same form but now with regular singular points at $0, 1, \infty$ with the same roots of the indicial equation as in the original equation.

Can a similar transformation be made to map $0, 1, \infty$ into any three points in the finite complex plane?

3 Consider the equation for $y(z)$ with P-symbol

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ \alpha_1 & \beta_1 & \gamma_1 & z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{matrix}\right\}.$$

By transforming $y \to \tilde{y}(z) = z^{-\alpha_1}(1-z)^{-\beta_1}y(z)$, show that the resulting equation for $\tilde{y}(z)$ is the hypergeometric equation.

4 By expanding $(1-tz)^{-a}$, show that

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt = \frac{\Gamma(c-b)}{\Gamma(a)}\sum_0^\infty \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!},$$

where $(1-tz)^{-a}$ takes its principal value, provided $\operatorname{Re} c > \operatorname{Re} b > 0$, and $|z|<1$. You should explain the reason for these conditions.

State the regions of the complex z-plane in which (i) the integral defines an analytic function and (ii) the sum defines an analytic function. Explain how the integral provides an analytic continuation in z of the function defined by the sum.

Given that the above integral is a multiple of $F(a,b;c;z)$, show that

$$F(a,b;c;1) = \frac{\Gamma(c-b-a)\Gamma(c)}{\Gamma(c-b)\Gamma(c-a)}.$$

when a, b and c satisfy a condition that you should give.

5 What can be said about the nature of the solutions of a second order linear ordinary differential equation in the neighbourhood of a regular singular point?

Explain carefully why the hypergeometric equation with the usual parameters a, b, and c (so that the exponents at $z=0$ are 0 and $1-c$), has, for any given value of the parameter c, a solution $w(z)$ that satisfies $w(0)=1$. Is $w(z)$ uniquely determined?

Is it the case that, for any given value of the parameter c, there is a solution $w(z)$ that is analytic at $z=0$, and satisfies $w(0)=1$? If such a solution exists, is it unique?

6 Is $(1-z)^{c-a-b}F(c-a,c-b;c;z)$ analytic at $z=0$? The branch of $(1-z)^{c-a-b}$ is defined by $|\arg(1-z)|<\pi$ (which implies a branch cut from 1 to ∞ along the positive real axis). [Assume $c \neq 0,-1,-2,\ldots$]

Show, by considering transformations of P-functions, that

$$(1-z)^{c-a-b}F(c-a,c-b;c;z) = F(a,b;c;z).$$

Show also that

$$(1-z)^{-a}F(a,c-b;c;\frac{z}{z-1}) = F(a,b;c;z).$$

7 The hypergeometric function, $F(a, b; c; z)$ may be defined for $|z| < 1$ (and as usual $c \neq 0, -1, -2, \ldots$) by

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}.$$

Let

$$g(z) = \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+t)\Gamma(b+t)}{\Gamma(c+t)} \Gamma(-t)(-z)^t dt$$

where the contour runs to the left of all poles of $\Gamma(-t)$ and to the right of all poles of $\Gamma(a+t)$ and $\Gamma(b+t)$, $|\arg(-z)| < \pi$ and $a \neq 0, -1, -2, \ldots$, $b \neq 0, -1, -2 \ldots$.

(i) Give a sketch of the complex t-plane showing the positions of the singularities of the integrand and the curve.

By closing the contour with a large semi-circle in the right half complex plane (which may be assumed to make a negligible contribution to the integral), show that $g(z) = F(a, b; c; z)$.

(ii) By closing the contour instead with a large semi-circle in the left half complex plane (which may be assumed to make a negligible contribution to the integral), show that the analytic continuation to $|z| > 1$ of the series for $F(a, b; c; z)$ in the case when a and b do not differ by an integer is provided by

$$\frac{\Gamma(b-a)\Gamma(c)}{\Gamma(c-a)\Gamma(b)}(-z)^{-a} F\big(a, 1-c+a; 1-b+a; z^{-1}\big) + \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(a)}(-z)^{-b} F\big(b, 1-c+b; 1-a+b; z^{-1}\big)$$

8 By changing variable in the hypergeometric equation, deduce that

$$F(a, b; 1+a+b-c; 1-z) \equiv y_1(z)$$

is a solution of the hypergeometric equation near $z = 1$ and deduce that another solution is

$$(1-z)^{c-a-b} F(c-a, c-b; 1+c-a-b; 1-z) \equiv y_2(z).$$

[Assume $c \neq 0, -1, -2, \ldots$]

Recall the result from Q4 that if $\operatorname{Re} c > \operatorname{Re} b > 0$ and $\operatorname{Re}(c-a-b) > 0$, then

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}.$$

Find constants A and B such that (under conditions to be specified)

$$F(a, b; c; z) = Ay_1(z) + By_2(z).$$