1. Show that the most general linear second order ordinary differential equation whose only singularities are regular singular points at \(z = a\) and \(z = b\) can be written in the form
\[
 w'' + \left[\frac{1 - A}{z - a} + \frac{1 + A}{z - b}\right] w' + \frac{B(a - b)^2}{(z - a)^2(z - b)^2} w = 0,
\]
where \(A\) and \(B\) are arbitrary constants.

Write down and solve the equation when the two singular points are at 0 and \(\infty\), in the two cases \(A^2 \neq 4B\) and \(A^2 = 4B\). Use a Möbius transformation to deduce the general solution of (†). What is the significance of the two constants \(\alpha\) and \(\alpha'\) which satisfy \(\alpha + \alpha' = A\) and \(\alpha \alpha' = B\)?

2. Consider the second order linear differential equation with three, and only three, regular singular points in the finite complex plane, all other points being ordinary points,
\[
 \frac{d^2y}{dz^2} + \left(\frac{A}{z - z_1} + \frac{B}{z - z_2} + \frac{C}{z - z_3}\right) \frac{dy}{dz} + \frac{1}{(z - z_1)(z - z_2)(z - z_3)} \left(\frac{D}{z - z_1} + \frac{E}{z - z_2} + \frac{F}{z - z_3}\right) y = 0.
\]

Show that by making a fractional linear transformation
\[
 z \to z' = \frac{(z_3 - z_2)(z - z_1)}{(z_3 - z_1)(z - z_2)}
\]
that the new differential equation is of the same form but now with regular singular points at 0, 1, \(\infty\) with the same roots of the indicial equation as in the original equation.

Can a similar transformation be made to map 0, 1, \(\infty\) into any three points in the finite complex plane?

3. Consider the equation for \(y(z)\) with \(P\)-symbol
\[
 P \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{array} \right\}.
\]

By transforming \(y \to \tilde{y}(z) = z^{-\alpha_1} (1 - z)^{-\beta_1} y(z)\), show that the resulting equation for \(\tilde{y}(z)\) is the hypergeometric equation.
4. By expanding \((1 - tz)^{-a}\), show that

\[
\int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a} \, dt = \frac{\Gamma(c - b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n)\Gamma(b + n) \, z^n}{\Gamma(c + n) \, n!},
\]

where \((1 - tz)^{-a}\) takes its principal value, provided \(\text{Re } c > \text{Re } b > 0\), and \(|z| < 1\). You should explain the reason for these conditions.

State the regions of the complex \(z\)-plane in which (i) the integral defines an analytic function and (ii) the sum defines an analytic function. Explain how the integral provides an analytic continuation in \(z\) of the function defined by the sum.

Given that the above integral is a multiple of \(F(a, b; c; z)\), show that

\[
F(a, b; c; 1) = \frac{\Gamma(c - b - a)\Gamma(c)}{\Gamma(c - b)\Gamma(c - a)}.
\]

when \(a\), \(b\) and \(c\) satisfy a condition that you should give.

5. What can be said about the nature of the solutions of a second order linear ordinary differential equation in the neighbourhood of a regular singular point?

Explain carefully why the hypergeometric equation with the usual parameters \(a\), \(b\), and \(c\) (so that the exponents at \(z = 0\) are 0 and \(1 - c\)), has, for any given value of the parameter \(c\), a solution \(w(z)\) that satisfies \(w(0) = 1\). Is \(w(z)\) uniquely determined?

Is it the case that, for any given value of the parameter \(c\), there is a solution \(w(z)\) that is analytic at \(z = 0\), and satisfies \(w(0) = 1\)? If such a solution exists, is it unique?

6. Is \((1 - z)^{c-a-b}F(c-a, c-b; c; z)\) analytic at \(z = 0\)? The branch of \((1 - z)^{c-a-b}\) is defined by \(|\arg(1 - z)| < \pi\) (which implies a branch cut from 1 to \(\infty\) along the positive real axis). [Assume \(c \neq 0, -1, -2, \ldots\)]

Show, by considering transformations of \(P\)-functions, that

\[
(1 - z)^{c-a-b}F(c-a, c-b; c; z) = F(a, b; c; z).
\]

Show also that

\[
(1 - z)^{-a}F(a, c-b; c; \frac{z}{z-1}) = F(a, b; c; z).
\]
7 The hypergeometric function, $F(a, b; c; z)$ may be defined for $|z| < 1$ (and as usual $c \neq 0, -1, -2, \ldots$) by

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}.$$

Let

$$g(z) = \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+t)\Gamma(b+t)}{\Gamma(c+t)} \Gamma(-t)(-z)^t dt$$

where the contour runs to the left of all poles of $\Gamma(-t)$ and to the right of all poles of $\Gamma(a+t)$ and $\Gamma(b+t)$, $|\arg(-z)| < \pi$ and $a \neq 0, -1, -2, \ldots, b \neq 0, -1, -2, \ldots$.

(i) Give a sketch of the complex $t$-plane showing the positions of the singularities of the integrand and the curve.

By closing the contour with a large semi-circle in the right half complex plane (which may be assumed to make a negligible contribution to the integral), show that $g(z) = F(a, b; c; z)$.

(ii) By closing the contour instead with a large semi-circle in the left half complex plane (which may be assumed to make a negligible contribution to the integral), show that the analytic continuation to $|z| > 1$ of the series for $F(a, b; c; z)$ in the case when $a$ and $b$ do not differ by an integer is provided by

$$\frac{\Gamma(b-a)\Gamma(c)}{\Gamma(c-a)\Gamma(b)} (-z)^{-a} F(a, 1-c+a; 1-b+a; z^{-1}) + \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(a)} (-z)^{-b} F(b, 1-c+b; 1-a+b; z^{-1}).$$

8 By changing variable in the hypergeometric equation, deduce that

$$F(a, b; 1+a+b-c; 1-z) \equiv y_1(z)$$

is a solution of the hypergeometric equation near $z = 1$ and deduce that another solution is

$$(1-z)^{-a-b} F(c-a, c-b; 1+c-a-b; 1-z) \equiv y_2(z).$$

[Assume $c \neq 0, -1, -2, \ldots$]

Recall the result from Q4 that if $0 < \arg(z-1) < 2\pi$, $\text{Re } c > \text{Re } b > 0$ and $\text{Re}(c-a-b) > 0$, then

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}.$$

Find constants $A$ and $B$ such that (under conditions to be specified)

$$F(a, b; c; z) = Ay_1(z) + B y_2(z).$$