Mathematical Tripos Part II — Lent term 2009

Further Complex Methods — Prof. A.S. Fokas

COURSE OUTLINE

I. Analytic Functions and Departure from Analyticity

1. The d-bar derivative

- (a) The Cauchy-Riemann equations
- (b) A Generalisation of Cauchy's Theorem
- (c) An explicit formula for computing an analytic function in terms of either its real or imaginary part

2. Lack of Analyticity at a point

- (a) Isolated singular points
- (b) Branch points

3. Lack of Analyticity on a curve and in a two dimensional domain

4. Analytic functions defined in terms of infinite series and products

5. Analytic functions defined in terms of integrals

6. The Gamma, Beta and Zeta functions

II. Integral representations of evolution PDEs

1. The Fourier transform and its variants

2. The solution of evolution PDEs via appropriate transforms

3. A new transform method for evolution PDEs

III. Second order ODEs

1. Classification of singular points and series solutions

2. Equations with three regular singular points (Papperitz)

3. The hypergeometric function

4. Integral representations

Chapter 1

Analytic Functions and Departure from Analyticity

1.1 The d-Bar Derivative

Let $u(x,y)$ and $v(x,y)$, $x,y \in \mathbb{R}$, be real differentiable functions. Let f be defined in terms of u and v by

$$f = u(x,y) + iv(x,y), \quad x,y \in \mathbb{R}. \tag{2.1}$$

The function f depends on x and y, or equivalently on z and $\bar{z}$, where

$$z = x + iy, \quad \bar{z} = x - iy. \tag{2.2}$$

However, if u and v satisfy the Cauchy Riemann equations in the neighborhood of the point (x_0, y_0), then in the neighborhood of z_0, f depends only on z and *not* on $\bar{z}$, hence $\partial f/\partial \bar{z} = 0$. This motivates the following result:

(a) *The Cauchy-Riemann Equations*

The single equation $\partial f/\partial \bar{z} = 0$ is equivalent to the Cauchy-Riemann equations:

$$\frac{\partial f}{\partial \bar{z}} = 0 \leftrightarrow u_x - v_y = 0, \quad u_y + v_x = 0. \tag{2.3}$$

Indeed, equations (2.2) imply the formulae

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right), \quad \partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right). \tag{2.4}$$

Hence,

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right)(u + iv) = \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x) \tag{2.5}$$

and (2.3) follows.

The d-bar derivative is also important for the following generalization of Cauchy's theorem:

(b) *Generalization of Cauchy's Theorem*

Theorem 2.1. Let $D \subset \mathbb{R}^2$ be a compact region bounded by the regular closed curve ∂D. Suppose that the complex-valued function $f(z, \bar{z})$ is continuously differentiable in $D \cup \partial D$, i.e. the first derivatives of the real and imaginary parts of f exist and are continuous. Then,

$$\int_{\partial D} f(z, \bar{z}) dz = \int\int_D \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz, \tag{2.6a}$$

where

$$d\bar{z} \wedge dz = (dx - idy) \wedge (dx + idy) = 2idxdy. \tag{2.6b}$$

In particular, if $f(z)$ is analytic in D, then

$$\int_{\partial D} f(z) dz = 0, \tag{2.7}$$

which is Cauchy's theorem.

Proof.

Equation (2.6a) is the direct consequence of Poincaré's Lemma,

$$\int_{\partial D} W = \int\int_D dW, \tag{2.8}$$

in the particular case that the differential form W is given by fdz; in this case

$$dW = f_z dz \wedge dz + f_{\bar{z}} d\bar{z} \wedge dz$$

and since

$$dz \wedge dz = 0, \quad dW = f_{\bar{z}} d\bar{z} \wedge dz.$$

If f is analytic, then $f_{\bar{z}} = 0$ and equation (2.6a) becomes equation (2.7). **QED**

Remark 2.1 Equation (2.6) can also be obtained from Green's theorem. Indeed, letting $f = u + iv$, equation (2.6a) can be rewritten in the following form:

$$\int_{\partial D} [(udx - vdy) + i(vdx + udy)] = \int\int_{\partial D} [-(v_x + y_y) + i(u_x - v_y)] \, dxdy.$$

Remark 2.2 It should be noted that the above proof of Cauchy's theorem requires $f'(z)$ to be continuous in D; there exists a proof due to Goursat that avoids this assumption [].

(c) *An explicit formula for computing an analytic formula in terms of either its real or its imaginary parts.*

Using the fact that an analytic function is independent of $\bar{z}$ yields the following important result []:

Theorem 2.2. Let the complex-valued function $f(z)$ be analytic in the neighborhood of the point z_0 and let $u(x, y)$ be the real part of $f(z)$. Then

$$f(z) = 2u\left(\frac{z + \bar{z}_0}{2}, \frac{z - \bar{z}_0}{2i}\right) - \overline{f(z_0)}. \tag{2.9a}$$

Similarly, let $v(x, y)$ be the imaginary part of $f(z)$. Then

$$f(z) = 2iv\left(\frac{z + \bar{z}_0}{2}, \frac{z - \bar{z}_0}{2i}\right) + \overline{f(z_0)}. \tag{2.9b}$$

Proof. Expressing x and y in terms of z and $\bar{z}$ via equations (2.2), it follows that

$$f(z) = u\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) + iv\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right). \tag{2.10}$$

Evaluating this equation at $z = z_0$ and then computing the complex conjugate of the resulting equation we find

$$\overline{f(z_0)} = u\left(\frac{\bar{z}_0 + z}{2}, \frac{z - \bar{z}_0}{2i}\right) - iv\left(\frac{\bar{z}_0 + z}{2}, \frac{z - \bar{z}_0}{2i}\right). \tag{2.11}$$

The function $f(z)$ is independent of $\bar{z}$, thus replacing in the RHS of (2.10) $\bar{z}$ by $\bar{z}_0$, equation (2.10) becomes

$$f(z) = u\left(\frac{z + \bar{z}_0}{2}, \frac{z - \bar{z}_0}{2i}\right) + iv\left(\frac{z + \bar{z}_0}{2}, \frac{z - \bar{z}_0}{2i}\right).$$

Adding or subtracting this equation to (2.11), we find equations (2.9). **QED**

Example 2.1. Find an analytic function whose real part is given by

$$u(x, y) = \frac{x}{x^2 + y^2}. \tag{2.12}$$

Equation (2.9a) implies

$$f(z) = \frac{z + \bar{z}_0}{z\bar{z}_0} - \overline{f(z_0)} = \frac{1}{z} + a + ib,$$

where the real constants a and b are defined by

$$a + ib = \frac{1}{\bar{z}_0} - \overline{f(z_0)}.$$

The real part of $1/z$ equals $x/(x^2 + y^2)$, hence $a = 0$. Thus,

$$f(z) = \frac{1}{z} + ib, \quad b \ \text{ real}. \tag{2.13}$$

1.2 Lack of Analyticity at a Point

If a function $f(z)$ is analytic in the entire complex z-plane including infinity, then according to Liouville's theorem it is constant, thus, in order to obtain interesting functions it is necessary to "break analyticity". The simplest situation occurs when a function loses analyticity at an *isolated point.* This situation is described by Laurent's theorem.

Theorem 2.3. *(Laurent's theorem)* Let D denote the following annulus in the complex z-plane,

$$D : \alpha < |z - z_0| < \beta, \tag{2.14a}$$

where z_0 is a complex number and $0 \le \alpha < \beta \le \infty$. Let $f(z)$ be analytic in the annulus D. Define the numbers a_n by

$$a_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{n+1}}, \quad n = 0, \pm 1, \pm 2, \cdots, \tag{2.14b}$$

where Γ denotes the circle $|\zeta - z_0| = \gamma$, with $\alpha < \gamma < \beta$. Then the following series converges uniformly to $f(z)$ on every compact subset of A:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n. \tag{2.14c}$$

Proof. Let z be a fixed point in D. The function $F(\zeta)$ defined by

$$F(\zeta) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z}, & \zeta \neq z \\ f'(z), & \zeta = z, \end{cases}$$

is analytic in D, thus according to Cauchy's formula, see equation (2.7),

$$\int_\Gamma F(\zeta)d\zeta = 0.$$

Hence,

$$\int_{\Gamma_1} \frac{f(\zeta) - f(z)}{\zeta - z}d\zeta = \int_{\Gamma_2} \frac{f(\zeta) - f(z)}{\zeta - z}d\zeta, \tag{2.15}$$

where Γ_1 and Γ_2 denote the circles with center at z_0 and radii γ_1 and γ_2, where

$$\alpha < \gamma_1, \quad \gamma < \gamma_2 < \beta.$$

Using

$$\int_{\Gamma_1} \frac{d\zeta}{\zeta - z} = 0, \quad \int_{\Gamma_2} \frac{d\zeta}{\zeta - z} = 2\pi i,$$

equation (2.15) becomes

$$f(z)=\frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\zeta)}{\zeta-z}d\zeta-\frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\zeta)}{\zeta-z}d\zeta, \tag{2.16}$$

where z is such that $\gamma_1<|z-z_0|<\gamma_2$.

Expanding the Cauchy kernel $(\zeta-z)^{-1}$ in a geometric series we find the following formulae, where $\hat{z}=z-z_0$:

$$\frac{1}{\zeta-z}=\frac{1}{\zeta-z_0-\hat{z}}=\frac{1}{\zeta-z_0}\frac{1}{1-\frac{\hat{z}}{(z-z_0)}}=\sum_{n=0}^{\infty}\frac{\hat{z}^n}{(\zeta-z_0)^{n+1}},\quad \zeta\in\Gamma_1$$

and

$$\frac{1}{\zeta-z}=-\frac{1}{\hat{z}-(\zeta-z_0)}=-\frac{1}{\hat{z}}\frac{1}{1-\frac{(z-z_0)}{\hat{z}}}=\sum_{n=0}^{\infty}\frac{(\zeta-z_0)^n}{\hat{z}^{n+1}},\quad \zeta\in\Gamma_2.$$

These series converge uniformly for $\zeta\in\Gamma_1$ and $\zeta\in\Gamma_2$ respectively, thus we may substitute them in (2.16) and integrate term by term. This yields (2.14c) with a_n defined as follows:

$$a_n=\frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}},\quad n=0,1,2,\cdots$$

and

$$a_{-n}=\frac{1}{2\pi i}\int_{\Gamma_2}f(\zeta)(\zeta-z_0)^{n+1},\quad n=1,2,\cdots.$$

The functions $f(\zeta)(\zeta-z_0)^m$ are analytic for all integers m, thus the integrals defining a_n and a_{-n} may be computed along any circle $|\zeta-z_0|=\gamma$, $\alpha<\gamma<\beta$. **QED**

Remark 2.3. Laurent's theorem contains Taylor expansion as a particular case. Indeed, if f is analytic in the disk $|z-z_0|<\beta$, then

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n,\quad |z-z_0|<\beta, \tag{2.17a}$$

where

$$a_n=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}},\quad n=0,1,2,\cdots. \tag{2.17b}$$

It will be shown in section 2.3 that

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\int_{\Gamma}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}},\quad n=0,1,2,\cdots, \tag{2.17c}$$

where $f^{(n)}(z)$ denotes the nth derivative of $f(z)$ and $f^{(0)}(z)=f(z)$. Hence, $a_n=f^{(n)}(z_0)/n!$.

Remark 2.4. Theorem 2.3 immediately implies the following estimate for the coefficients a_n: Let f be analytic in the annulus D defined in (2.14a) and let

$$\mu(\gamma) = \max_{|z-z_0|=\gamma} |f(z)|, \quad \alpha < \gamma < \beta. \tag{2.18}$$

Then, the coefficients a_n defined in (2.14b) satisfy

$$|a_n| \le \mu(\gamma)\gamma^{-n}, \quad n = 0, \pm 1, \pm 2, \cdots.$$

Indeed,

$$|a_n| = \frac{1}{2\pi}\left|\int_\Gamma f(\zeta)(\zeta - z_0)^{-n-1} dt\right| \le \frac{1}{2\pi} 2\pi\mu(\gamma)\gamma^{-n-1}.$$

(a) *Poles and Essential Singularities*

In what follows we concentrate on the particular case of Laurent's theorem when $\alpha = 0$, i.e. we consider a function $f(z)$ which is analytic in the punctured disk $0 < |z - z_0| < \beta$. If $\lim_{z\to z_0} f(z)$ exists then $f(z)$ is analytic at z_0. If this limit does not exist, then z_0 is an *isolated singularity* of $f(z)$; actually we can distinguish two cases: If there exists a positive integer m such that $\lim_{z\to z_0}(z - z_0)^m f(z)$ exists and is non-zero, then z_0 is a *pole of order* m (this means that the Laurent expansion contains only a finite number of terms with negative powers of $(z - z_0)$). If on the other hand, there does not exist such an integer, then z_0 is an *essential singularity* (this means that the Laurent expansion contains an infinite number of terms with negative powers of $(z - z_0)$).

(b) *Branch Points*

In contrast to poles and essential singularities, branch points are *non-isolated singularities*. Such points are characteristic singularities of *multivalued* functions, which will be considered in detail in Chapter 3. Here, we will only discuss simple multivalued functions which can be treated in a straightforward manner by the introduction of the so called *branch cuts*. The branch points of a given multivalued function are fixed, whereas branch cuts are somewhat arbitrary curves, whose only requirement is that they connect branch points.

The point z_0 is a branch point of the multivalued function $w(z)$, if $w(z)$ is discontinuous upon traversing a small circuit around z_0.

For example, the multivalued function $w(z) = z^{\frac{1}{2}}$ has two branch points, $z_0 = 0$ and $z_0 = \infty$. Indeed, at the point $z = \rho\exp(0i)$, $w(z)$ equals $\sqrt{\rho}$; however, after transversing a circle around $z_0 = 0$, z is now given by $z = \rho\exp(2i\pi)$ and $w(z)$ equals $\sqrt{\rho}\exp(i\pi) = -\sqrt{\rho}$. The point $z_0 = \infty$ can be analyzed by using the transformation $\zeta = 1/z$ to map $z_0 = \infty$ to $\zeta_0 = 0$.

By using branch cuts it is possible to choose a *single* value at every point of the complex z-plane in such a way that the resulting function is single-valued and continuous. A single-valued function obtained in this way is called a *branch* of the multivalued function.

Let us return to the simple example of $z^{\frac{1}{2}}$; this multivalued function gives rise to a single-valued function by employing a curve connecting 0 and ∞. The particular choice of such a curve is often dictated by applications; two such natural choices are shown in Fig. 2.1. In the first case

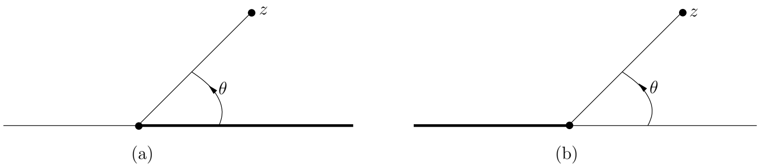

Figure 1.1:

$0 \le \theta < 2\pi$, whereas in the second case $-\pi \le \theta < \pi$. It is straightforward to verify that for each point in the "cut complex z-plane", $z^{\frac{1}{2}}$ has a unique value. For example, for a point at a distance ρ from the origin the values of $z^{\frac{1}{2}}$ above and below the branch cut of Figure 2.1a are given by $\sqrt{\rho}$ and $-\sqrt{\rho}$ respectively. Similarly, for the case of Figure 2.1b, the corresponding values are $\sqrt{\rho}\exp(i\pi/2) = i\sqrt{\rho}$ and $\sqrt{\rho}(\exp(-i\pi/2) = -i\sqrt{\rho}$, respectively.

Example 2.2. Show that

$$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2} dx = \pi(\sqrt{2}-1). \tag{2.19}$$

We will use this example to illustrate the point made earlier that the choice of the appropriate branch cut is often dictated by applications.

In order to establish (2.19) we consider the following integral:

$$I = \int_C \frac{(z^2-1)^{\frac{1}{2}}}{z^2+1} dz,$$

where the contour C is the union of the curves depicted in Figure 2.2. It is clear that in this case we must apply Cauchy's theorem in the shaded region of Figure 2.2, thus the function $(z^2-1)^{\frac{1}{2}}$ must be analytic in this region, here we must choose a branch cut along the real axis connecting the points -1 and $+1$.
The function $(z^2-1)^{\frac{1}{2}}$, is the product of the functions, $(z+1)^{\frac{1}{2}}$ and $(z-1)^{\frac{1}{2}}$. Thus, if we choose the branch cuts $(-1,\infty)$ and $(1,\infty)$ for these two functions, these branch cuts cancel from $(1,\infty)$ and the function $(z^2-1)^{\frac{1}{2}}$ has a branch cut in $(-1,1)$. This can be checked explicitly: Let

$$(z+1) = \rho_1 e^{i\theta_1},\ (z-1) = \rho_2 e^{i\theta_2}, \qquad 0 \le \theta_1, \theta_2 \le \pi.$$

Then

$$(z^2-1)^{\frac{1}{2}} = \sqrt{\rho_1\rho_2}e^{\frac{i(\theta_1+\theta_2)}{2}} = \sqrt{|z^2-1|}e^{\frac{i(\theta_1+\theta_2)}{2}}$$

and the values of θ_1, θ_2, $\theta = (\theta_1+\theta_2)/2$ are shown in Figure 2.3.

For $|x| < 1$, $|z^2-1| = 1-x^2$, hence for z above and below the branch cut, $(z^2-1)^{\frac{1}{2}}$ equals $i\sqrt{1-x^2}$ and $-i\sqrt{1-x^2}$ respectively.

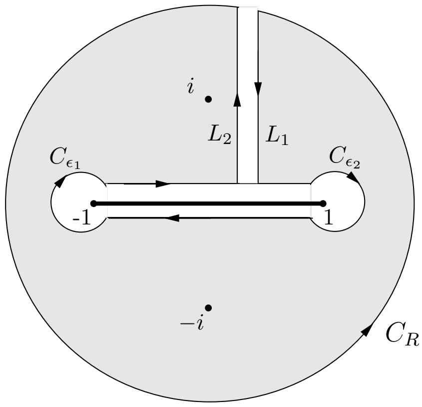

Figure 1.2:

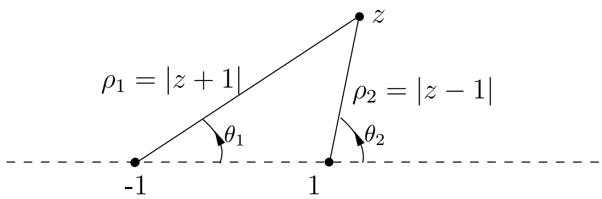

Figure 1.3:

The contributions around C_{ε_j}, $j = 1, 2$, vanish:

$$\left| \int_{C_{\varepsilon_j}} \frac{(z^2-1)^{\frac{1}{2}}}{z^2+1} dz \right| \leq \int_0^{2\pi} \frac{\sqrt{|z-1|}\sqrt{|z+1|}}{\left|2+\varepsilon_j^2 e^{2i\theta}+2\varepsilon_j e^{i\theta}\right|} \varepsilon_j d\theta \leq \int_0^{2\pi} \frac{\sqrt{\varepsilon_j^2+2\varepsilon_j}}{2-\varepsilon_j^2-2\varepsilon_j} \varepsilon_j d\theta \to 0,$$

as $\varepsilon_j \to 0$, $j = 1, 2$.

The contribution around C_R can be computed explicitly:

$$\frac{(z^2-1)^{\frac{1}{2}}}{z^2+1} = \frac{\left[z^2\left(1-\frac{1}{z^2}\right)\right]^{\frac{1}{2}}}{z^2\left(1+\frac{1}{z^2}\right)} = \frac{1}{z}\left(1-\frac{1}{2z^2}+O\left(\frac{1}{z^4}\right)\right)\left(1-\frac{1}{z^2}+O\left(\frac{1}{z^4}\right)\right)$$

$$= \frac{1}{z} + O\left(\frac{1}{z^2}\right).$$

Thus

$$\lim_{R\to\infty} \int_{C_R} \frac{(z^2-1)^{\frac{1}{2}}}{z^2+1} dz = \int_0^{2\pi} i e^{i\theta} d\theta = 2\pi i.$$

Hence

$$2i \int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2} dx + 2\pi i = 2\pi i \left\{ \left(\frac{(z^2-1)^{\frac{1}{2}}}{2z} \right)_{z=i} + \left(\frac{(z^2-1)^{\frac{1}{2}}}{2z} \right)_{z=-i} \right\}.$$

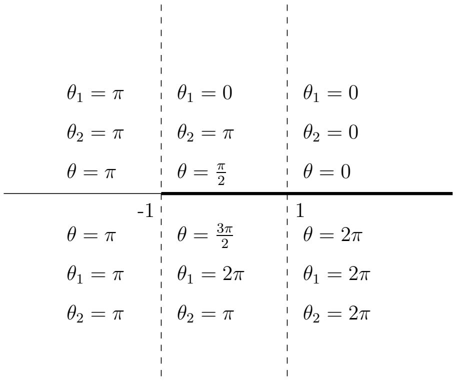

Figure 1.4:

At $z = i$, $\theta = [\pi/4 + (3\pi/4)]/2$, whereas at $z = -i$, $\theta = [7\pi/4 + 5\pi/4]/2$, and the above equation yields (2.19).

1.3 Lack of Analyticity on a Curve and in a Two-Dimensional Domain

Let L be a smooth curve (L may be an arc or a closed contour) and let $\varphi(\tau)$, $\tau \in L$, be a function satisfying the so-called *Hölder condition* on L, namely

$$|\varphi(\tau_1) - \varphi(\tau_2)| \leq \Lambda|\tau_1 - \tau_2|^\lambda, \quad \Lambda > 0, \quad 0 < \lambda \leq 1. \tag{2.20}$$

A Hölder function is certainly continuous, but it may not be differentiable.

Given a contour L and a Hölder function $\varphi(z)$, $z \in L$, we define the function $\Phi(z)$ by

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - z}d\tau, \quad z \in \mathbb{C}\backslash L. \tag{2.21}$$

It turns out that this function is analytic everywhere in the complex z-plane, including infinity, except for $z \in L$; its value at infinity is given by

$$\Phi(z) = \frac{\alpha_1}{z} + O\left(\frac{1}{z^2}\right), \quad z \to \infty, \quad \alpha_1 = -\frac{1}{2\pi i}\int_L \varphi(\tau)d\tau. \tag{2.22}$$

The function $\Phi(z)$ has a limit, denoted by $\Phi^+(t)$, as z approaches L along a curve in the $\oplus$ region, which is the region to the left of the increasing direction of L, see Figure 2.5. Similarly, $\Phi(z)$ has a limit, $\Phi^-(t)$, as z approaches L along a curve in the $\oplus$ region. These limits are given by the following formulae:

Theorem 2.5 (Plemelj Formulae) Let L be a smooth curve (L may be an arc or a closed contour) and let $\varphi(\tau)$ satisfy the Hölder condition (2.20) for $\tau \in L$. Then, the Cauchy integral

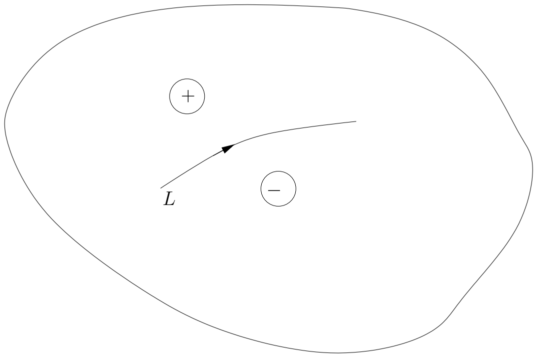

Figure 1.5:

$\Phi(z)$ defined in (2.21) has a limiting value $\Phi^+(t)$ as z approaches L along a nontangential curve which is in the $\oplus$ region, i.e. in the region to the left of the increasing direction of L. Similarly, $\Phi(z)$ has a limit $\Phi^-(t)$ as z approaches L along a nontangential curve in the $\oplus$ region. These limits are given by

$$\Phi^{\pm}(t) = \pm\frac{1}{2}\varphi(t) + \frac{1}{2\pi i}\fint_L \frac{\varphi(\tau)}{\tau - t}d\tau, \quad t \in L, \tag{2.23}$$

where t is *not* an endpoint of L and $\fint$ denotes the principal value integral, i.e.

$$\fint_L \frac{\varphi(\tau)d\tau}{\tau - t} = \lim_{\varepsilon\to 0}\int_{L-L_\varepsilon} \frac{\varphi(\tau)d\tau}{\tau - t}, \tag{2.24}$$

where L_ε is the part of L which has length 2ε and is centered around t.

Proof The proof is straightforward if $\varphi(\tau)$ is analytic at t, since in this case we can deform the contour L to $(L - L_\varepsilon) \cup C_\varepsilon$, where C_ε is a semi-circle of radius ε centered at t, see []. However, if $\varphi(t)$ is not analytic at t, the proof is quite elaborate, see [].

Remark 2.5 Let L be a closed curve dividing the complex z-plane into two regions D^+ and D^-
Let $\varphi(t)$, $t \in L$ be a given Hölder function. Given L and $\varphi(t)$, one can define a *scalar additive Riemann-Hilbert* problem, which involves finding two functions $\Phi^+(t)$ and $\Phi^-(t)$ such that:

(i) $\Phi^{\pm}(t)$ are the limits of the functions $\Phi^{\pm}(z)$ which are analytic in $D^{\pm}$, as z approaches L.

(ii) $\Phi^+(t) - \Phi^-(t) = \varphi(t), \quad t \in L.$

(iii) $\Phi^-(z) = O(\frac{1}{z}), \quad z \to \infty, \quad z \in D^-.$

The unique solution of this problem is given by

$$\Phi^{\pm}(z) = \Phi(z), \quad z \in D^{\pm}.$$

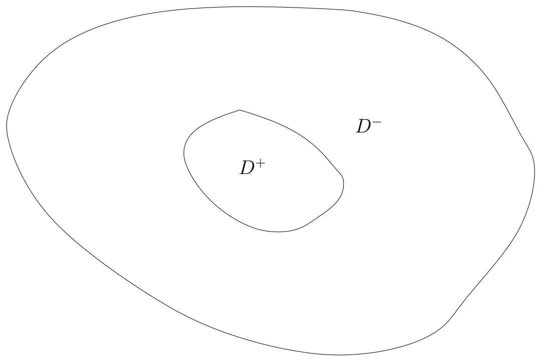

Figure 1.6:

Indeed, $\Phi^{\pm}(z)$ are analytic functions for $z \in D^{\pm}$ and the condition (ii) is satisfied in lieu of Plemelj's formulae; furthermore condition (iii) is also valid, see (2.22). This solution is unique, since if there did exist another solution, their difference denoted by Ψ would satisfy conditions (i), (iii), as well as $\Psi^+(t) = \Psi^-(t)$, $t \in L$; thus $\Psi(z)$ would be analytic in the entire complex z-plane, including infinity, hence Liouville's theorem implies that $\Psi = 0$.

Example 2.3 Solve the scalar additive Riemann-Hilbert problem specified in Remark 2.5, where L is the real axis and

$$\varphi(x) = \frac{\sin x}{x}, \quad x \in \mathbb{R}.$$

It suffices to compute the following principal value integral:

$$I(x) = \fint_{-\infty}^{\infty} \frac{\sin \xi}{\xi} \frac{d\xi}{\xi - x}, \quad x \in \mathbb{R}. \tag{2.25}$$

For this purpose, we consider the function

$$\frac{1 - e^{i\zeta}}{\zeta} \frac{1}{\zeta - x}, \quad x \in \mathbb{R},$$

and use Cauchy's theorem in the upper half of the complex ζ-plane. Clearly, the contribution along the semi-circle $\{|\zeta| = R, 0 \le \theta < \pi\}$ vanishes as $R \to \infty$. Thus,

$$\fint_{-\infty}^{\infty} \frac{1 - e^{i\xi}}{\xi} \frac{d\xi}{\xi - x} = \pi i \left[\frac{1 - e^{ix}}{x} \right], \quad x \in \mathbb{R}, \tag{2.26}$$

where the factor πi (as opposed to $2\pi i$), is due to the fact that the pole $\zeta = x$ is *on* the contour (the point $\zeta = 0$ is a removable singularity). The real and imaginary parts of equation (2.26) yield

$$\fint_{-\infty}^{\infty} \frac{(1 - \cos \xi)}{\xi} \frac{d\xi}{\xi - x} = \pi \frac{\sin x}{x}, \quad \fint_{-\infty}^{\infty} \frac{\sin \xi}{\xi} \frac{d\xi}{\xi - x} = \pi \frac{(\cos x - 1)}{x}, \quad x \in \mathbb{R}.$$

Hence, Plemelj formulae yield,

$$\Phi^{\pm}(x) = \pm\frac{1}{2}\frac{\sin x}{x} + \frac{1}{2i}\frac{(\cos x - 1)}{x}, \quad x \in \mathbb{R}.$$

Thus,

$$\Phi^{+}(x) = -\frac{i}{2x}\left(e^{ix} - 1\right), \quad \Phi^{-}(x) = -\frac{i}{2x}\left(e^{-ix} - 1\right). \tag{2.27}$$

Clearly $\Phi^{+}(z)$ and $\Phi^{-}(z)$ are analytic in the upper and lower half of the complex z-plane respectively and

$$\Phi^{\pm}(z) \sim \frac{i}{2z}, \quad z \to \infty.$$

Example 2.4. Solve the following singular integral equation:

$$\varphi(x) + \frac{\alpha}{i\pi}\fint_{-\infty}^{\infty}\frac{\varphi(\xi)}{\xi - x}d\xi = \frac{\sin x}{x}, \quad x \in \mathbb{R}, \tag{2.28}$$

where α is a constant different than ± 1.

Let $\Phi(z)$ be defined in terms of $\varphi(x)$ by

$$\Phi(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\varphi(\xi)}{\xi - z}d\xi, \quad z \in \mathbb{C}, \quad \text{Im } z \neq 0.$$

Then,

$$\Phi^{+}(x) - \Phi^{-}(x) = \varphi(x) \tag{2.29a}$$

and

$$\Phi^{+}(x) + \Phi^{-}(x) = \frac{1}{i\pi}\fint_{-\infty}^{\infty}\frac{\varphi(\xi)}{\xi - x}d\xi, \quad x \in \mathbb{R}. \tag{2.29b}$$

Replacing in equation (2.28) $\varphi(x)$ and the principal value integral by the LHS of equations (2.29) we find

$$(1+\alpha)\Phi^{+}(x) - (1-\alpha)\Phi^{-}(x) = \frac{\sin x}{x}, \quad x \in \mathbb{R}.$$

The definition of $\Phi(z)$ implies that $\Phi^{\pm}(z) = O(1/z)$ as $z \to \infty$. Thus, the functions $\Psi^{+}(x) = (1+\alpha)\Phi^{+}(x)$ and $\Psi^{-}(x) = (1-\alpha)\Phi^{-}(x)$ satisfy the Riemann-Hilbert problem of example 2.3. Thus

$$(1+\alpha)\Phi^{+}(x) = -\frac{i}{2x}\left(e^{ix} - 1\right), \quad (1-\alpha)\Phi^{-}(x) = -\frac{i}{2x}\left(e^{-ix} - 1\right).$$

Hence, equation (2.29a) yields

$$\varphi(x) = -\frac{i}{2x}\left[\frac{e^{ix} - 1}{1+\alpha} - \frac{e^{-ix} - 1}{1-\alpha}\right].$$

Example 2.5. Let $f(x)$ be Hölder for $x \in [0, X]$, where $X > 0$. Define $\hat{f}(k)$ by

$$\hat{f}(k) = \int_0^X e^{ik\xi} f(\xi) d\xi, \quad k \in \mathbb{C}. \tag{2.30a}$$

Show that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k) dk, \quad x \in \mathbb{R}. \tag{2.30b}$$

We multiply (2.30a) by $\exp(ikx)$ and integrate the resulting expression with respect to dk from 0 to ∞:

$$I_1(x) = \int_0^{\infty} e^{ikx} \hat{f}(k) dk = \int_0^{\infty} e^{ikx} \left(\int_0^X e^{-ik\xi} f(\xi) d\xi \right) dk.$$

If we interchange the order of integration of the above integrals, we obtain an integral in k which does not converge. Thus, before we interchange the order of integration we first regulate the k-integral:

$$\begin{aligned} I_1(x) &= \lim_{\varepsilon \to 0} \int_0^{\infty} e^{ikx - \varepsilon k} \left(\int_0^X e^{-ik\xi} f(\xi) d\xi \right) dk \\ &= \lim_{\varepsilon \to 0} \int_0^X f(\xi) \left(\int_0^{\infty} e^{ik(x - \xi + i\varepsilon)} dk \right) d\xi. \end{aligned}$$

Hence, computing the k-integral we find

$$\int_0^{\infty} e^{ikx} \hat{f}(k) dk = \lim_{\varepsilon \to 0} (-i) \int_0^X \frac{f(\xi) d\xi}{\xi - (x + i\varepsilon)}.$$

Similarly,

$$\int_{-\infty}^0 e^{ikx} \hat{f}(k) dk = \lim_{\varepsilon \to 0} i \int_0^X \frac{f(\xi) d\xi}{\xi - (x - i\varepsilon)}.$$

Using Plemelj formulae and adding the above expressions we find (2.30b).

Theorem 2.6. (Pompieu's Formula) Let $D \subset \mathbb{R}^2$ be a compact region bounded by the regular closed curve ∂D. Suppose that the complex-valued function $f(z, \bar{z})$ is continuously differential in $D \cup \partial D$, i.e. the first derivatives of the real and imaginary parts of f exist and are continuous. Then,

$$f(z, \bar{z}) = \frac{1}{2i\pi} \int_{\partial D} \frac{f(\zeta, \bar{\zeta}) d\zeta}{\zeta - z} + \frac{1}{2i\pi} \int\int_D \frac{\partial f(\zeta, \bar{\zeta})}{\partial \bar{\zeta}} \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}, \quad z \in D. \tag{2.31}$$

Proof Let D_ε denote D/C_ε, where C_ε denotes the disc $|\zeta - z| \leq \varepsilon$. Employing equation (2.6a) in D_ε we find

$$\int_{\partial D_\varepsilon} \frac{f(\zeta, \bar{\zeta}) d\zeta}{\zeta - z} - \int_{\partial C_\varepsilon} \frac{f(\zeta, \bar{\zeta})}{d\zeta} \zeta - z = \int\int_{D_\varepsilon} \frac{\partial f(\zeta, \bar{\zeta})}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z}, \tag{2.32}$$

where we have used the fact that $(\zeta - z)^{-1}$ is analytic for $\zeta \in D_\varepsilon$. As $\varepsilon \to 0$, the difference between the RHS of (2.32) and the RHS of (2.32) vanishes. Indeed, this difference is smaller than the following expression (using $\zeta = z + r\exp(i\theta)$):

$$\int_0^\varepsilon \int_0^{2\pi} \left|\frac{\partial f}{\partial \bar{\zeta}}\right| \frac{rdrd\theta}{r} = 2\pi M\varepsilon,$$

where we have used the fact that the continuity of $\partial f/\partial\bar{\zeta}$ in a bounded region implies that there exists a constant M such that $|\partial f/\partial\bar{\zeta}| \leq M$. On the other hand, letting $\zeta = z + \varepsilon\exp(i\theta)$ in the second term of the LHS of (2.32), it follows that as $\varepsilon \to 0$ this term yields $2\pi i f(z, \bar{z})$, hence (2.32) becomes (2.31) (where we have used that $d\bar{\zeta} \wedge d\zeta = -d\zeta \wedge d\bar{\zeta}$). **QED**

Remark 2.6. If f is analytic then $f_{\bar{\zeta}} = 0$ and (2.31) yields Cauchy's integral formula:

$$f(z) = \frac{1}{2i\pi}\int_{\partial D} \frac{f(\zeta)d\zeta}{\zeta - z}, \quad z \in D. \tag{2.33}$$

Repeated differentiations of this equation (for rigorous considerations see []), yield equation (2.17c).

Example 2.6. Let C_ρ denote the disc $|z| \leq \rho$ and let m, n be non-negative integers. Show that

$$\frac{1}{2i\pi}\int\int_{C_\rho} \frac{\zeta^m\bar{\zeta}^n}{\zeta - z} d\zeta \wedge d\bar{\zeta} = \begin{cases} \frac{1}{n+1} z^m\bar{z}^{n+1}, & m \leq n \\ \frac{1}{n+1} z^m\bar{z}^{n+1} - \frac{1}{n+1}\rho^{2n+2}z^{m-n-1}, & m > n. \end{cases} \tag{2.34}$$

Pompieu's formula (2.31) with $f = z^m\bar{z}^{n+1}$ yields the following equation:

$$z^m\bar{z}^{n+1} = \frac{1}{2\pi 1}\int_{\partial C_\rho} \frac{\zeta^m\bar{\zeta}^{n+1}}{\zeta - z} d\zeta + (n+1)I_{mn}(z), \tag{2.35}$$

where $I_{mn}(z)$ denotes the LHS of (2.34). Using $\zeta\bar{\zeta} = \rho^2$, it follows that the integrand of the first integral of the RHS of (2.35), denoted by g, is given by

$$g = \rho^{2n+2}\frac{\zeta^{m-n-1}}{\zeta - z}.$$

If $m > n$, the residue from the pole $\zeta = z$ equals $\rho^{2n+2}z^{m-n-1}$ and hence we find (2.34) for $m > n$. if $m \leq n$, then

$$g = \frac{\rho^{2n+2}}{\zeta^l(\zeta - z)}, \quad l = 1, 2, \cdots.$$

In this case the residues from the poles $\zeta = 0$ and $\zeta = z$ equal $-\rho^{2n+2}z^{-l}$ and $\rho^{2n+2}z^{-l}$ respectively, thus the first term of the RHS of (2.35) vanishes and hence we find (2.34) with $m \leq n$.

Remark 2.7. Let D be the entire complex z-plane and let $f = 1/(z - z_0)$. Then Pompieu's formula yields

$$\frac{1}{z - z_0} = \frac{1}{\pi} \int\int_{\mathbb{R}^2} \frac{\partial}{\partial \bar{\zeta}} \left(\frac{1}{\zeta - z_0} \right) \frac{dxdy}{z - \zeta}.$$

This formula suggests that

$$\frac{\partial}{\partial \bar{z}} \frac{1}{z - z_0} = \pi \delta(z - z_0), \tag{2.36}$$

which is indeed the case (for a rigorous derivation see []).

1.4 Analytic Functions Defined in Terms of Infinite Sequences and Products

A function is called *entire* if it is analytic in the entire finite complex plane. A function is called *meromorphic* if its only singularities in the finite plane are poles. A polynomial $P_n(z)$, which is a particular case of an entire function, can be uniquely factorized in terms of its zeros. This motivates the following natural question: Is it possible to express a non-polynomial entire function in terms of its infinitely many zeros? The answer is affirmative and the associated formula is due to Weierstrass. In the case that the zeros are simple, this formula is a straightforward corollary of a formula due to Mittag-Leffler which expresses a meromorphic function in terms of its poles as well as the associated residues. Before deriving these fundamental results, we first present two theorems which provide sufficient conditions for the convergence of infinite series and infinite products; the proofs of these theorems can be found in [].

Theorem 2.7. Let $a_j(z)$ be analytic for $z \in D$ and all integers j. Suppose that $|a_j(z)| \leq M_j$, $z \in D$, with $\{M_j\}$ constant. If $\sum_{j=1}^{\infty} M_j$ converges, then the series $\sum_{j=1}^{\infty} a_j(z)$ converges uniformly for $z \in D$ to an analytic function $S(z)$.

Example 2.7. The Riemann zeta function $\zeta(z)$, defined by

$$\zeta(z) = \sum_{m=1}^{\infty} \frac{1}{m^z}, \tag{2.37}$$

is an analytic function for Re $z > 1$.

Indeed, $m^{-z} = \exp[-z \ln m]$ is analytic for all integers m. Furthermore, $\sum_{m=1}^{\infty} m^{-\text{Re } z}$ converges for Re $z > 1$.

Theorem 2.8. Let $a_j(z)$ be analytic for $z \in D$ and all integers j. Suppose that for $z \in D$ and $j \geq k$, either $|a_j(z)| \leq M_j$ or $|\ln(1 + a_j(z))| \leq M_j$, with $\{M_j\}$ constant. If $\sum_{j=1}^{\infty} M_j$ converges, then $\prod_{j=1}^{\infty}(1 + a_j(z))$ converges uniformly to a function $P(z)$ analytic for $z \in D$.

Example 2.8. The following products define two entire functions:

$$\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right), \quad \prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{\frac{z}{n}}. \tag{2.38}$$

Regarding the first product we note that for $|z|<R$, $|a_n|\leq \frac{R^2}{n^2}$ and $\sum_{j=1}^{\infty}\frac{R^2}{n^2}$ converges. The positive constant R is fixed but arbitrary, thus the first of the two products in (2.38) defines an entire function.

Regarding the second product in (2.38), we let $w=z/n$ and restrict w by $|w|<\frac{1}{2}$. Then, we find the following:

$$\ln(1-w)e^w = \ln(1-w)+w = w-\left(w+\frac{w^2}{2}+\frac{w^3}{3}+\cdots\right)$$

$$= -w^2\left(\frac{1}{2}+\frac{w}{3}+\frac{w^2}{4}+\cdots\right).$$

Thus

$$|\ln(1-w)e^w| \leq |w|^2\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right) = \frac{|w|^2}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right)$$

$$= \frac{w^2}{2}\frac{1}{1-\frac{1}{2}} = |w|^2.$$

Hence,

$$\left|\ln\left(1-\frac{z}{n}\right)e^{\frac{z}{n}}\right| \leq \frac{|z|^2}{|n|^2}, \quad |z|<R \ \text{ and } \ n>2R,$$

where the last inequality is a consequence of $R/n<\frac{1}{2}$. The positive constant R is fixed but arbitrary, thus again the second product in (2.38) defines an entire function.

Before deriving Mittag-Leffler theorem, we note that by definition, the poles of a meromorphic function *cannot* have an accumulation point in the finite plane. Thus, any closed disk $|z|\leq R$ contains only a finite number of poles. Hence, the infinite number of poles is denumerable and these poles can be numbered as $z_1, z_2, \cdots$, with $z_m\to\infty$ as $m\to\infty$.

Theorem 2.9 (Mittag-Leffler's Expansion). Let $\{z_j\}$ and $\{a_j\}$ denote the location of the simple poles and the associated residues of a meromorphic function. Assume that this function has only simple poles and that $z=0$ is not a pole. Let $\{C_n\}$ be the sequence of circles of radii R_n, enclosing $\{z_1,\cdots,z_n\}$ but no other singularities. Assume that $|f(z)|<M$ on C_N as $R_N\to\infty$. Then, $f(z)$ can be expressed as the following infinite series:

$$f(z) = f(0)+\sum_{j=1}^{\infty}a_j\left[\frac{1}{z-z_j}+\frac{1}{z_j}\right]. \tag{2.39}$$

Proof. Consider the integral I_n defined by

$$I_n = \frac{1}{2\pi i}\int_{C_n} \frac{f(\zeta)d\zeta}{\zeta(\zeta - z)}.$$

The above integral can be computed via residues, where the residues associated with $\zeta = 0$, $\zeta = z$ and $\zeta = z_j$ are given by $-f(0)/z$, $f(z)/z$ and $a_j/z_j(z_j - z)$ respectively. Hence,

$$I_n = -\frac{f(0)}{z} + \frac{f(z)}{z} + \sum_1^n \frac{a_j}{z}\left[\frac{1}{z - z_j} + \frac{1}{z_j}\right].$$

As $n \to \infty$ the integral around C_n vanishes, thus the above equation becomes (2.39). **QED**

Example 2.9. The function $\cot z$ admits the representation

$$\cot z = \frac{1}{z} + 2z\sum_1^\infty \frac{1}{z^2 - n^2\pi^2}, \quad z \in \mathbb{C}. \tag{2.40}$$

Indeed,

$$\cot z = \frac{\cos z}{\sin z} = \frac{1 - \frac{z^2}{2!} + \cdots}{z - \frac{z^3}{3!} + \cdots} = \frac{1}{z} + O(z).$$

Hence, if $f(z) = \cot z - \frac{1}{z}$, $f(0) = 0$ and $z_j = n\pi$, $a_j = 1$. Thus, (2.39) yields

$$\cot z - \frac{1}{z} = \sum_{n=-\infty}^{\infty}{}' \left[\frac{1}{z - n\pi} + \frac{1}{n\pi}\right],$$

where prime indicates that $n = 0$ is excluded. Decomposing the above sum into a sum involving negative integers and a sum involving positive integers and then replacing n by $-n$ in the former sum, we find (2.38).

Theorem 2.10. (Weierstrass canonical product formula.) Let $f(z)$ be an entire function which has simple zeros $\{z_j\}$, $z_j \neq 0$, $\lim_{n\to\infty} |z_n| = \infty$. Then, $f(z)$ can be expressed as the following infinite product:

$$f(z) = f(0)e^{\frac{f'(0)z}{f(0)}} \prod_{j=1}^{\infty} \left\{\left(1 - \frac{z}{z_j}\right) e^{\frac{z}{z_j}}\right\}. \tag{2.41}$$

Proof. The point z_j is a simple zero, thus $f(z) = (z - z_j)g_j(z)$, where $g_j(z_j) \neq 0$. The function f'/f, given by

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_j} + \frac{g_j'(z)}{g_j(z)},$$

is a meromorphic function with simple poles at z_j and corresponding residues $a_j = 1$. Thus, Mittag-Leffler's expansion yields

$$\frac{f'(z)}{f(z)} = \frac{f'(0)}{f(0)} + \sum_1^\infty \left(\frac{1}{z - z_j} + \frac{1}{z_j}\right).$$

Integrating this equation and fixing the constant of integration by evaluating the resulting expression at $z = 0$, we find

$$\ln f(z) - \ln f(0) = \frac{f'(0)}{f(0)} z + \sum_{1}^{\infty} \left\{ \ln\left(1 - \frac{z}{z_j}\right) + \frac{z}{z_j} \right\}$$

and then (2.39) follows. **QED**

Example 2.10. The function $\sin z/z$ admits the representative

$$\frac{\sin z}{z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right), \quad z \in \mathbb{C}. \tag{2.42}$$

Letting $f(z) = \sin z/z$, it follows that $f(0) = f'(0)$ and $z_j = n\pi$, thus

$$\frac{\sin z}{z} = \prod_{n=-\infty}^{\infty}{}' \left(1 - \frac{z}{n\pi}\right) e^{\frac{z}{n\pi}},$$

where prime indicates that $n = 0$ is excluded. Decomposing the above product into a product involving negative integers and a product involving positive integers, we find (2.42).

It was verified earlier in example 2.8 that the RHS of (2.42) converges to an entire function; we now see that this function is $\sin \pi z/\pi z$.

1.5 Analytic Functions Defined in Terms of Integrals

The Cauchy integral $\Phi(z)$ defined in (2.21), provides an example of an analytic function defined in terms of an integral. In general, if $f(z,t)$, $t \in \mathbb{R}$, is an analytic function for $z \in D$ and if the integral along $t \in [a,b]$ converges uniformly, then this integral defines a function $F(z)$ which is analytic for $z \in D$. In general, it is possible to extend the domain of definition of $F(z)$ by employing the powerful procedure of analytic continuation. This procedure involves the following: Let $F_1(z)$ be analytic in the region D_1 and on the boundary Γ, see Figure 2.7. Suppose that we can find a function $F_2(z)$ which is analytic on the boundary Γ and in the domain D_2, $\Gamma = D_1 \cap D_2$, such that $F_1 = F_2$, $z \in \Gamma$. Then the function

$$F(z) = \left[\begin{array}{ll} F_1(z), & z \in D_1 \\ F_2(z), & z \in D_2 \end{array}\right.$$

is analytic in $D_1 \cup D_2$ and we say that $F_2(z)$ provides the analytic continuation of $F_1(z)$. The proof that $F(z)$ is an analytic function is a direct consequence of Morera's theorem. Indeed, by decomposing an arbitrary contour L in $D_1 \cup D_2$ into two contours, one in D_1 and one in D_2, and employing the analyticity of F_1 and F_2, it follows that $\int_L F dz = 0$ and hence F is analytic.

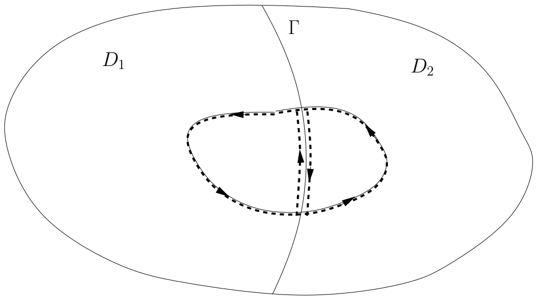

Figure 1.7:

If $F(z)$ is defined in terms of an integral, there exist several possible ways of implementing the procedure of analytic continuation. In what follows we will discuss three such ways: (a) Computing the relevant integral explicitly; (b) replacing the contour of integration of the real line, $t \in \mathbb{R}$, by a contour in the complex extension of t which will be denoted by ζ; (c) deriving a functional equation for $F(z)$.

Example 2.11. Let $F(z)$ be defined by

$$F(z) = \int_0^\infty \frac{t^{z-1}}{t+1} dt, \quad 0 < \text{Re } z < 1. \tag{2.43}$$

As $t \to \infty$, $F(z)$, involves the integral of t^{z-2} which equals $t^{z-1}/z-1$, hence we require Re $(1-z) > 0$. As $t \to 0$, $F(z)$ involves the integral of t^{z-1} which equals t^z/z, hence we require Re $z > 0$. Actually for $0 <$ Re $z < 1$, the RHS of (2.43) defines a uniformly convergent integral and hence $F(z)$ is an analytic function for $0 <$ Re $z < 1$. By computing explicitly the above integral, it is possible to define $F(z)$ for all complex z. Indeed, let us consider the integral

$$I(z) = \int_C \frac{\zeta^{z-1} d\zeta}{\zeta + 1}, \quad 0 < \text{Re } z < 1,$$

where the contour C in the complex z-plane is depicted in Figure 2.8. Above and below the bracket cut, we have $\zeta = \rho \exp(i0)$ and
$\zeta = \rho \exp(2i\pi)$, respectively. Furthermore, the contributions along the contours $C_R = \{\zeta = R\exp(i\theta), 0 \le \theta < 2\pi\}$ and $C_\varepsilon = \{\zeta = \varepsilon \exp(i\theta), \pi/2 \le \theta \le 3\pi/2\}$ vanish. Thus,

$$\int_0^\infty \frac{\rho^{z-1} d\rho}{\rho+1} + \int_\infty^0 \frac{(\rho e^{2i\pi})^{z-1}}{\rho+1} d\rho = 2i\pi \left(e^{i\pi}\right)^{z-1}.$$

Hence,

$$F(z) = \frac{\pi}{\sin \pi z}, \quad 0 < \text{Re } z < 1. \tag{2.44}$$

The function $F(z)$ is analytic for $0 <$ Re $z < 1$, whereas $\pi/\sin \pi z$ is analytic for all $z \in \mathbb{C}$, $z \neq n\pi$, $n \in \mathbb{Z}$. Furthermore, these two functions are equal in the domain $0 <$ Re $z < 1$. Hence, $\pi/\sin \pi z$ provides the analytic continuation of $F(z)$ for all $z \neq n\pi$, $n \in \mathbb{Z}$. Furthermore, we can say that $F(z)$ is a meromorphic function with simple poles at $n\pi$, $n \in \mathbb{Z}$.

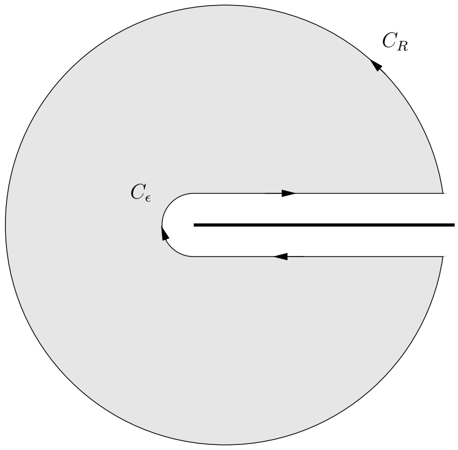

Figure 1.8:

Example 2.12. Let $F_1(z)$ be defined by

$$F_1(z) = \int_0^\infty \frac{e^{-zt}}{1+t^4} dt, \quad |\arg z| < \frac{\pi}{2}. \tag{2.45}$$

For $t > 0$ and for $|\arg z| < \frac{\pi}{2}$, the function $\exp(-zt)$ decays exponentially and the RHS of (2.45) defines an analytic function. It is clear that the restriction on z is a consequence of the fact that $t \in \mathbb{R}^+$, thus in order to extend the validity of $F_1(z)$ we replace the integral along the positive real axis with an integral along the negative imaginary axis of the complex extension of t, which we denote with ζ: Let $\tilde{F}_2(z)$ be defined by

$$\tilde{F}_2(z) = \int_0^{\infty e^{\frac{-i\pi}{2}}} \frac{e^{-z\zeta}}{1+\zeta^4} d\zeta, \quad 0 < \arg z < \pi. \tag{2.46}$$

In order for the exponential $\exp[-z\zeta]$ to decay, we require

$$-\frac{\pi}{2} < \arg z + \arg\zeta < \frac{\pi}{2},$$

and since $\arg\zeta = -\frac{\pi}{2}$, the function $\exp[-z\zeta]$ decays exponentially provided that $0 < \arg z < \pi$.

The functions $F_1(z)$ and $\tilde{F}_2(z)$ can be related: Indeed, let $I(z)$ denote the integral

$$I(z) = \int_C \frac{e^{-z\zeta}}{1+\zeta^4} d\zeta, \quad \frac{\pi}{2} < \arg z < \pi, \tag{2.47}$$

where C denotes the boundary of the fourth quadrant of the complex ζ-plane, see Figure 2.9. On the contour $C_R = \{\zeta = R\exp(i\theta), -\frac{\pi}{2} \le \theta \le 0\}$. Using the inequalities

$$-\frac{\pi}{2} \le \arg\zeta \le 0, \quad \frac{\pi}{2} < \arg z < \pi.$$

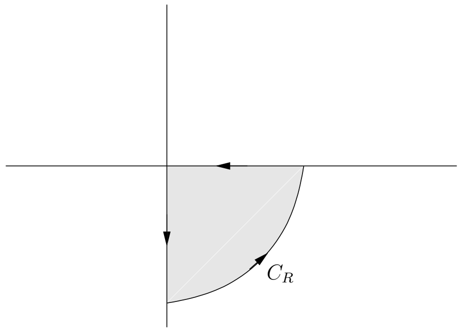

Figure 1.9:

it follows that $0 \leq \arg \zeta z \leq \pi$ and hence the contribution of the integral along C_R vanishes. Thus

$$\tilde{F}_2(z) - F_1(z) = 2i\pi \left(\frac{e^{-z\zeta}}{4\zeta^3} \right)_{\zeta = e^{-i\frac{\pi}{4}}} .$$

Hence

$$F_1(z) = \tilde{F}_2(z) - \frac{i\pi}{2}\, e^{i\frac{3\pi}{4}}\, e^{-ze^{-i\frac{\pi}{4}}}, \quad -\frac{\pi}{2} < \arg z < \pi. \tag{2.48}$$

Let us denote the RHS of (2.48) by $F_2(z)$. This function consists of the function $\tilde{F}_2(z)$ which is analytic for $0 < \arg z < \pi$, as well as of a constant multiple of the entire function $\exp[-ze^{-i\pi/4}]$; hence $F_2(z)$ is analytic for $0 < \arg z < \pi$. Furthermore, $F_2(z)$ coincides with $F_1(z)$ in the region $\pi/2 < \arg z < \pi$, thus $F_2(z)$ provides the analytic continuation of $F_1(z)$.

Example 2.13. Let $F_1(z)$ be defined by

$$F_1(z) = \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi} d\xi}{(\xi - i)(t - z)}, \quad \text{Im } z > 0. \tag{2.49}$$

Let $\tilde{F}_2(z)$ denote the RHS of (2.49) but for Im $z < 0$. Then, according to Plemelj formulae,

$$F_1(x) = \tilde{F}_2(x) + \frac{e^{ix}}{x - i}, \quad x \in \mathbb{R}. \tag{2.50}$$

The function $F_1(z)$ is analytic for Im $z > 0$ and is continuous for Im $z \geq 0$. Similarly, the function $F_2(z)$ defined by

$$F_2(z) = \tilde{F}_2(z) + \frac{e^{iz}}{z - i},$$

is analytic for Im $z < 0$ and continuous for Im $z \leq 0$. Furthermore, $F_1(x) = F_2(x)$, $x \in \mathbb{R}$. We claim that the function $F(z)$ defined by

$$F(z) = \begin{cases} F_1(z), & \text{Im } z > 0 \\ F_1(x) = F_2(x), & \text{Im } z = 0 \\ F_2(z), & \text{Im } z < 0 \end{cases}$$

in an entire function. This is direct consequence of the following fact: The statement about analytic continuation at the beginning of section 4 is also valid if $F_1(z)$ and $F_2(z)$ are continuous (instead of analytic) on Γ. Indeed, it is straightforward to verify that the relevant proof (see Figure 2.7) remains valid even if F_1 and F_2 are simply continuous on Γ, see [].

It should be emphasized that, in general, a Cauchy integral defined with respect to a curve L, defines a function which is *not* analytic on L. In the above example, we were able to define an entire function only because the associated *"jump"* function $\varphi(z) = \exp(iz)/(z - i)$, is analytic.

Example 2.14. Let $F(z)$ denote the following integral

$$F(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \text{Re } z > 0. \tag{2.51}$$

$F(z)$ is well defined as $t \to \infty$ because of the exponential term; as $t \to 0$, $F(z)$ involves the integral of t^{z-1} which equals t^z/z, hence we require Re $z > 0$. It is clear that this restriction is a consequence of the given power of t, hence we can extend the validity of $F(z)$ by changing the power of t: Let us integrate by parts the function $F(z+1)$; although this function is well defined for Re $z > -1$, we consider $F(z+1)$ with Re $z > 0$,

$$F(z+1) = \int_0^\infty e^{-t} t^z dt = -e^{-t} t^z \Big|_0^\infty + z \int_0^\infty e^{-t} t^{z-1} dt,$$

and the term $(e^{-t} t^z)_{t=0}$ vanishes because of the restriction Re $z > 0$. Hence,

$$F(z+1) = zF(z), \quad \text{Re } z > 0.$$

The function $F(z+1)$ is analytic for Re $z > -1$, whereas the function $zF(z)$ is analytic for Re $z > 0$, thus the above equation provides the analytic continuation of $zF(z)$. By repeating this procedure, we can define the following meromorphic function, called the gamma function:

$$\Gamma(z) = \begin{cases} F(z), & \text{Re } z > 0 \\ \frac{F(z+1)}{z}, & \text{Re } z > -1 \\ \vdots & \\ \frac{F(z+n+1)}{(z+n)(z+n-1)\cdots z}, & \text{Re } z > -(n+1) \\ \vdots & \end{cases} \tag{2.52}$$

Example 2.15 Let $w(z)$ denote the formal inverse of $\sin z$, i.e.

$$w = \arcsin z \leftrightarrow z = \sin w.$$

It is straightforward to obtain an explicit formula for $w(z)$, by solving the equation

$$z = \frac{1}{2i}\left(e^{iw} - e^{-iw}\right)$$

This yields

$$iw(z) = \ln\left[iz + (1-z^2)^{\frac{1}{2}}\right].$$

This expression shows that $w(z)$ has two sources of multivaluedness, namely the square root and the ln. Hence, after using an appropriate branch cut to fix $(1-z^2)^{\frac{1}{2}}$, we will find a multivalued function involving multiples of 2π; this of course reflects the periodicity of $\sin z$. A convenient way to analyze $w(z)$ is to use the fact that the derivative of $w(z)$ takes a particular simple form:

$$\frac{dw(z)}{dz} = \frac{1}{(1-z^2)^{\frac{1}{2}}}.$$

Hence, we define $w(z)$ by the following integral in the complex ζ plane:

$$w(z) = \int_0^z \frac{d\zeta}{(1-\zeta^2)^{\frac{1}{2}}}, \quad 0 \le \arg z < 2\pi. \tag{2.53}$$

We first discuss the relevant integral; we take a branch cut along the real ζ axis connecting the branch points ± 1 and we choose a branch such that $(1-\zeta^2)^{-\frac{1}{2}} = 1$ at $\zeta = 0^+$; hence

$$\frac{1}{(1-\zeta^2)^{\frac{1}{2}}} = \frac{e^{\frac{i\pi}{2}}}{\sqrt{|1-\zeta^2|}} e^{-\frac{i(\theta_1+\theta_2)}{2}}, \quad 0 \le \theta_1, \theta_2 < 2\pi \tag{2.54}$$

where the angles θ_1 and θ_2 are shown in Figure 2.10 (for $\zeta = 0^+$,
$\theta_1 = 0$, $\theta_2 = \pi$ and hence (2.54) implies that indeed $(1-\zeta^2)^{-\frac{1}{2}} = 1$).

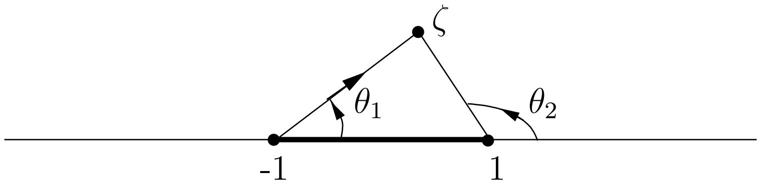

Figure 1.10:

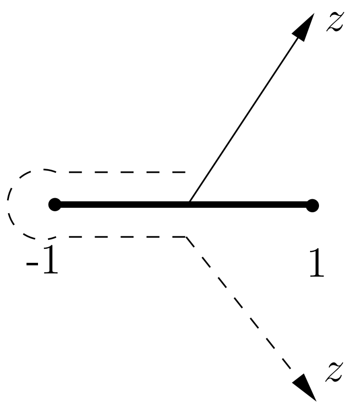

Figure 1.11:

We next discuss the contour of integration of $w(z)$: for $0 \leq \arg z \leq \pi$, the path of integration consists of the ray from $\zeta = 0^+$ to $\zeta = z$; for $\pi < \arg z < 2\pi$, the path of integration consists of a contour in the anticlockwise direction around the branch cut from $\zeta = 0^+$ to $\zeta = 0^-$, followed by the ray connecting 0^- and z, see Figure 2.11
For $\pi < \arg z < 3\pi$ we define the function $\tilde{w}(z)$ by (2.53), where now the contour of integration is chosen as follows: for $\pi < \arg z < 2\pi$ we use the same contour as in $w(z)$, whereas for $2\pi \leq \arg z < 3\pi$ we use a contour in the anticlockwise direction around the branch cut followed by the ray connecting 0^+ and z. Hence,

$$w\left(ze^{2\pi i}\right) = w(z) + \oint \frac{d\zeta}{(1-\zeta^2)^{\frac{1}{2}}}, \tag{2.55}$$

where $\oint$ denotes the contour around the branch cut. This integral can be evaluated explicitly by using Cauchy's theorem in the domain enclosed by the branch cut and the circle C_R, $|\zeta| = R$. As $R \to \infty$, $\theta_1 = \theta_2 = \theta$, thus the above integral equals

$$\lim_{R\to\infty} \int_0^{2\pi} e^{\frac{i\pi}{2}} \frac{e^{-i\theta}}{R} ie^{i\theta} R d\theta = -2\pi.$$

Equation (2.55), with the integral $\oint$ replaced by -2π, provides the analytic continuation of $w(z)$.

1.6 The Beta, Gamma and Zeta Functions

The so-called beta, gamma and zeta functions appear in a wide range of applications. In what follows we derive some basic facts about these functions.

(a) *The Gamma Function*

The Gamma function was introduced in (2.51) and (2.52). The *recursion formula*

$$\Gamma(z+1) = z\Gamma(z), \quad z \in \mathbb{C}, \tag{2.56a}$$

together with the identity $\Gamma(1) = 1$, immediately imply

$$\Gamma(n+1) = n! \tag{2.56b}$$

The representation (2.52) shows that $\Gamma(z)$ is a *meromorphic function* with

$$\text{simple poles at } z = 0, -1, -2, -n, \cdots, \quad \text{with residues } \frac{(-1)^n}{n!}. \tag{2.56c}$$

It will be shown later in this section that $\Gamma(z)$ satisfies the following *functional equation*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad z \in \mathbb{C}. \tag{2.56d}$$

Hence, letting $z = \frac{1}{2}$, we find

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}. \tag{2.56e}$$

Equation (2.56d) implies that

$$\frac{1}{\Gamma(z)} = \Gamma(1-z)\frac{\sin \pi z}{\pi}.$$

The function $\Gamma(1-z)$ has simple poles at $1 - z = 0, -1, -2, \cdots$, i.e. at $z = 1, 2, \cdots$, hence these points are removable singularities of the function $1/\Gamma(z)$. Thus, $1/\Gamma(z)$ is an *entire function* and have $\Gamma(z) \neq 0$. Thus, the zeros of $1/\Gamma(z)$ are only due to $\sin \pi z$. Therefore, the function $1/\Gamma(z+1)$ is an entire function with simple zeros at $z = -1, -2, \cdots$. Thus, letting $f(z) = 1/\Gamma(z+1)$ and noting that $f(0) = 1$, $f'(0) = -\Gamma'(1)$, Weiestrass canonical product expansion yields the following formula:

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}, \quad \gamma = -\Gamma'(1), \tag{2.56f}$$

where we have replaced $1/\Gamma(z+1)$ by $1/z\Gamma(z)$.

Evaluating (2.56f) at $z = 1$, we find

$$e^{-\gamma} = \lim_{N\to\infty} \prod_{n=1}^{\infty} \left(\frac{n+1}{n}\right) e^{-\frac{1}{n}} = \lim_{N\to\infty} (N+1)e^{-\sum_{n=1}^{N} \frac{1}{n}}.$$

Hence

$$\gamma = \lim_{N\to\infty} \sum_{n=1}^{N} \left[\frac{1}{n} - \ln(N+1)\right]. \tag{2.56g}$$

Equation (2.56f) can be written in the form

$$\frac{1}{\Gamma(z)} = \lim_{N\to\infty} z(1+z)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{N}\right)e^{\gamma z}e^{-z(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{N})}.$$

Replacing in this expression γ by the RHS of (2.56g) we obtain the reciprocal of Euler's product formula, i.e. we obtain

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)(z+2)\cdots(z+n)}. \tag{2.56h}$$

By using (2.56f) it is straightforward to show that for any positive integer m, the ratio

$$\frac{m^{nz}\Gamma(z)\Gamma\left(z+\frac{1}{m}\right)G\left(z+\frac{2}{m}\right)\cdots\Gamma\left(z+\frac{m-1}{m}\right)}{\Gamma(mz)}$$

is a constant independent of z. By letting $z \to 0$ this constant can be evaluated and this yields the following identity:

$$\Gamma(z)\Gamma\left(z+\frac{1}{m}\right)\Gamma\left(z+\frac{2}{m}\right)\cdots\Gamma\left(z+\frac{m-1}{m}\right)$$
$$= \sqrt{m}m^{-mz}(2\pi)^{\frac{m-1}{2}}\Gamma(mz), m=1,2,\cdots \tag{2.56i}$$

For $m=2$ this becomes the so-called *duplication formula*

$$2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}\Gamma(2z). \tag{2.56j}$$

Replacing in equation (2.51) t by $s\tau, s>0$, we obtain

$$\frac{\Gamma(z)}{s^z} = \int_0^\infty e^{-s\tau}\tau^{z-1}d\tau, \quad s>0, \quad \text{Re } z>0.$$

Letting $z=1$ and integrating the resulting expansion with respect to s from $s=0$ to $s=t$ we find

$$\ln t = \int_0^\infty \left(e^{-z}-e^{-tz}\right)\frac{d\tau}{\tau}, \quad t>0. \tag{2.57a}$$

Differentiating equation (2.51) with respect to z we find

$$\Gamma'(z) = \int_0^\infty e^{-t}t^{z-1}\ln t dt, \quad \text{Re } z>0. \tag{2.57b}$$

Replacing in this equation $\ln t$ by the RHS of (2.57a) we find

$$\frac{\Gamma'(z)}{\Gamma(z)} = \int_0^\infty \left(e^{-\tau} - \frac{1}{(1+\tau)^z}\right)\frac{d\tau}{\tau}, \quad \text{Re } z>0 \tag{2.57c}$$

It was noted in the discussion of example 2.11, that the reason for restricting Re z in the definition (2.43) was the requirement of convergence at $t = 0$. This motivates the definition of $\Gamma(z)$ in terms of a contour which *avoids* the origin. Such a contour is the so-called Hankel contour denoted by H and depicted in Figure 2.12. We will now show that the following integral provides the analytic continuation of the gamma function:

$$\Gamma(z) = \frac{1}{2i\sin\pi z}\int_H e^{\zeta}\zeta^{z-1}d\zeta, \quad z \in \mathbb{C}. \tag{2.58}$$

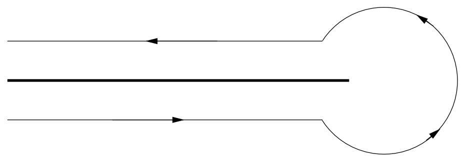

Figure 1.12:

Below the branch cut, on the straight line part of the contour H, $\zeta = \rho\exp(-i\pi) = -\rho$ and above the branch cut, $\zeta = \rho\exp(i\pi) = -\rho$, thus as $\rho \to \infty$, e^{ζ} decays exponentially and hence the integral appearing in (2.58) defines an analytic function. In order to prove that the RHS of (2.58) provides the analytic continuation of $\Gamma(z)$, it suffices to show that

$$I(z) = 2i\sin\pi z\int_0^{\infty} e^{-t}t^{z-1}dt, \quad \text{Re } z > 0, \tag{2.59}$$

where $I(z)$ denotes the integral appearing in the RHS of (2.58). This integral can be computed as follows:

$$I(z) = \lim_{\varepsilon\to 0}\left\{\int_{\infty}^{\varepsilon} e^{-\rho}\left(\rho e^{-i\pi}\right)^{z-1}e^{-i\pi}d\rho + \int_{-\pi}^{\pi} e^{\varepsilon e^{i\theta}}\left(\varepsilon e^{i\theta}\right)^{z-1} i\varepsilon e^{i\theta}d\theta \right.$$
$$\left. + \int_{\varepsilon}^{\infty} e^{-\rho}\left(\rho e^{i\pi}\right)^{z-1}e^{i\pi}d\rho\right\}. \tag{2.60}$$

The first and the third integrals converge as $\varepsilon \to 0$ because of the restriction Re $z > 1$, furthermore the second integral is proportional to ε^z and hence, as a consequence of the same restriction, this integral vanishes. Thus (2.60) becomes (2.59).

Remark 2.8 The function $\Gamma(z)$ has simples poles at $z = 0, -1, -2, \cdots$, thus the integral in (2.58) must have simple zeros at $z = 1, 2, \cdots$, in order to cancel the zeros of $\sin\pi z$ at $z = 1, 2, \cdots$. This is indeed the case as can be verified directly. Actually for $z = n \in \mathbb{Z}^+$ the branch cut disappears and hence the integral around the Hankel contour equals the integral around the circle, which we have already shown that vanishes provided that Re $z > 0$.

Equation (2.58) can also be used to verify that the residue of the pole at $z = -n$, $n = 0, 1, \cdots$ equals $(-1)^n/n!$ In this case we *cannot* take the limit $\varepsilon \to 0$ since the relevant integrals diverge. Nevertheless, the integral along H still equals the integral around the circle, which can be computed explicitly:

$$\int_0^{2\pi} e^{\varepsilon e^{i\theta}} \left(\varepsilon e^{i\theta}\right)^{-n-1} \varepsilon e^{i\theta} i d\theta = i \int_0^{2\pi} \frac{e^{\varepsilon e^{i\theta}}}{(\varepsilon e^{i\theta})^n} d\theta = \frac{2i\pi}{n!}.$$

Hence, the residue at $z = -n$, $n \in \mathbb{Z}$, equals

$$\frac{1}{2i\pi} \frac{1}{\cos \pi n} \frac{2i\pi}{n!} = \frac{(-1)^n}{n!}.$$

(b) *The Beta Function*

The beta function, denoted by $B(z_1, z_2)$ is defined by

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)}, \quad z_1, z_2 \in \mathbb{C}. \tag{2.61a}$$

We will now show that this definition implies that

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt, \quad \text{Re } \alpha > 0, \text{Re } \beta > 0. \tag{2.61b}$$

Indeed, with the above restriction on α and β, we have

$$\Gamma(\alpha)\Gamma(\beta) = \left(\int_0^\infty e^{-t} t^{\alpha-1} dt\right) \left(\int_0^\infty e^{-s} s^{\beta-1} ds\right).$$

Using the change of variables $t = u^2$, $s = v^2$, followed by the introduction of the polar coordinates $u = \rho \cos \theta$, $v = \rho \sin \theta$, we find

$$\Gamma(\alpha)\Gamma(\beta) = 4 \int_0^\infty \int_0^\infty e^{-(u^2+v^2)} u^{2\alpha-1} v^{2\beta-1} du dv$$

$$= \left(2 \int_0^\infty e^{-\rho^2} \rho^{2(\alpha+\beta)-1} d\rho\right) \left(2 \int_0^{\frac{\pi}{2}} (\cos \theta)^{2\alpha-1} (\sin \theta)^{2\beta-1} d\theta\right). \tag{2.62}$$

Making the change of variables $\rho^2 = r$ and $(\cos \theta)^2 = \varphi$, the first integral in the RHS of (2.62) becomes $\Gamma(\alpha + \beta)$, whereas the second integral yields the RHS of (2.61b). Hence, (2.62) and (2.61a) yield (2.61b)

$$2 \int_0^{\frac{\pi}{2}} (\cos \theta)^{2\alpha-1} (\sin \theta)^{2\beta-1} d\theta = \int_0^1 \varphi^{\alpha-1} (1-\varphi)^{\beta-1} d\varphi. \tag{2.63}$$

If we employ the change of variables $\tan \theta = v$ instead of $(\cos \theta)^2 = \varphi$, we find that the second integral of the RHS of (2.62) equals an integral involving $v^{\beta-1}/(1+v)^{\alpha+\beta}$. Hence,

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt = \int_0^\infty \frac{v^{\beta-1} dv}{(1+v)^{\alpha+\beta}} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{2.64}$$

Letting $\alpha + \beta = 1$, the last equation above implies

$$\int_0^\infty \frac{v^{-\alpha}}{1+v} dv = \Gamma(\alpha)\Gamma(1-\alpha), \quad 0 < \text{Re } \alpha < 1. \tag{2.65}$$

However, it was shown in example 2.11 that the above integral also equal $\pi/\sin \pi\alpha$ and hence (2.65) yields (2.56d).

(c) *The Riemann Zeta Function*

This function was defined as an infinite sum in (2.37). This sum can be rewritten in the form

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t - 1} d\tau, \quad \text{Re } z > 1. \tag{2.66}$$

Indeed, making the change of variables $\tau = nt$, $n \in \mathbb{Z}^+$, in the representation (2.51) of the gamma function, we find

$$\Gamma(z) = n^z \int_0^\infty e^{-nt} t^{z-1} dt.$$

Thus,

$$\frac{1}{n^z} = \frac{1}{\Gamma(z)} \int_0^\infty e^{-nt} t^{z-1} dt, \quad \text{Re } z > 1. \tag{2.67}$$

Summing the above equations from $n = 1$ to $n = \infty$ and using

$$\sum_1^\infty e^{-nt} = e^{-t} + e^{-2t} + \cdots = e^{-t}\left(1 + e^{-t} + (e^{-t})^2 + \cdots\right) = \frac{e^{-t}}{1 - e^{-t}} = \frac{1}{e^t - 1},$$

equation (2.67) becomes equation (2.66).

It is straightforward to show that the following integral provides the analytic continuation of $\zeta(z)$,

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi 1} \int_H \frac{\chi^{z-1}}{e^{-\chi} - 1} d\chi, \quad z \in \mathbb{C}, \tag{2.68}$$

where H is the Hankel contour depicted in Figure 2.12. Indeed, following precisely the same steps used for the analogous result for the gamma function, it can be shown that for Re $z > 1$, the RHS of (2.68) equals the RHS of (2.66).

Using (2.68), we will now shown that the Riemann zeta function satisfies the functional equation

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z) \zeta(1-z), \quad z \in \mathbb{C}. \tag{2.69}$$

It suffices to prove this equation for Re $z < 0$, since analytic continuation extends its validity to all $z \in \mathbb{C}$. In this respect we consider the integral

$$I(z) = \int_C \frac{\chi^{z-1}}{e^{-\chi} - 1} d\chi, \quad \text{Re } z < 0,$$

where the contour C consists of the union of the Hankel contour and of the rectangle with vertices at $\chi = \pm R \pm (2N+1)\pi i$, see Figure (2.13)

The integral along the vertical side $\chi = R + iy$ can be bounded by

$$\int_{-(2N+1)\pi}^{(2N+1)\pi} \frac{|\chi|^{z-1}}{|e^{-\chi} - 1|} dy \le \int_{-(2N+1)\pi}^{(2N+1)\pi} \frac{dy}{(R^2 + y^2)^{\frac{1-z}{2}} (1 - e^{-R})},$$

and the integral involving R vanishes as $R \to \infty$ (recalling that Re $z < 0$). The analysis of the integrals around the other three sides is similar and therefore the integral along the Hankel

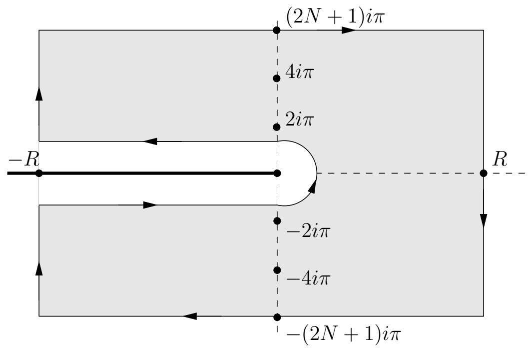

Figure 1.13:

contour equals the sum of the associated residues. The residue of the pole at $\chi = 2im\pi$ equals

$$\left(\frac{\chi^{z-1}}{-e^{-\chi}}\right)_{\chi=2m\pi i} = -(2m\pi i)^{z-1}.$$

Thus,

$$\frac{\zeta(z)}{\Gamma(1-z)}2\pi i = -2\pi i \sum_{m=1}^{\infty}\left[-\left(2m\pi e^{\frac{i\pi}{2}}\right)^{z-1} - \left(2m\pi e^{-\frac{i\pi}{2}}\right)^{z-1}\right]$$

$$= 2\pi i(2\pi)^{z-1}(-i)(2i)\sin\left(\frac{\pi z}{2}\right)\sum_{m=1}^{\infty} m^{z-1}.$$

The restriction Re $z < 0$ ensures that the above sum converges to $\zeta(1-z)$ and the above equation becomes (2.69).

Remark 2.9 Equation (2.69) suggests that $\zeta(z)$ has zeros at the negative even integers. This can be verified directly using the integral representation (2.68). Indeed, the so called Bernoulli numbers, B_n, are defined by the generating function

$$\frac{\chi}{e^{\chi}-1} = \sum_{m=0}^{\infty} B_m \frac{\chi^m}{m!}, \quad |\chi| < 2\pi \tag{2.70}$$

and $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_{2m+1} = 0$ for $m \in \mathbb{Z}^+$. For $z = -n$, $n \in N^+$ the branch cut disappears and $\zeta(-n)$ is expressed in terms of a small circle around the origin, $|\chi| < 2\pi$:

$$\zeta(-n) = \frac{n!}{2\pi i}\int_{|\chi|=\frac{1}{2}} \chi^{-n-z}\left(\frac{\chi}{e^{-\chi}-1}\right)d\chi.$$

Replacing in this equation $\chi/(e^{-\chi}-1)$ by the RHS of (2.70) and employing Cauchy's theorem we find

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}, \quad n = 0, 1, 2, \ldots. \tag{2.71}$$

Hence, $\zeta(-n)$ vanishes for $n = 2, 4, 6, \ldots$.