Classical Dynamics
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Mathematical Tripos, Part II
MT 2024

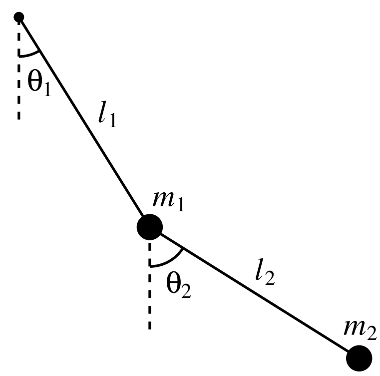
Example Sheet 1

1. A circular hoop of radius a lies in a vertical plane. The hoop rotates with constant angular velocity ω around a fixed vertical axis that goes through its centre, O. A bead of mass m is threaded on the hoop and moves without friction. Its location is denoted by A. The angle between the line OA and the downward vertical is $\psi(t)$.

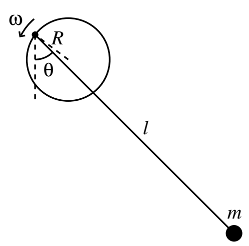
   (a) Using the Lagrangian formalism, derive a second-order differential equation for $\psi(t)$.

   (b) Assume now that the hoop rotates freely about the vertical axis without friction. Write down the Lagrangian of the system, neglecting the mass of the hoop. Find the additional conserved quantity.

2. A double pendulum is drawn below. Two light rods, of lengths l_1 and l_2, oscillate in the same plane. Attached to them are masses m_1 and m_2. How many degrees of freedom does the system have? Write down the Lagrangian describing the dynamics. Derive the equations of motion.



3. The pivot of a simple pendulum is attached to the rim of a disc of radius R, which rotates about its centre in the plane of the pendulum with constant angular velocity ω. (See the diagram below.) Write down the Lagrangian and derive the equation of motion for the dynamical variable θ.



4. A particle of mass m_1 is restricted to move on a circle of radius R_1 in the plane $z = 0$, with centre at $(x, y) = (0, 0)$. A second particle, of mass m_2, is restricted to move on a circle of radius R_2 in the plane $z = c$, with centre at $(x, y) = (0, a)$. The two particles are connected by a spring; the resulting potential energy is

$$V = \frac{1}{2}\omega^2 d^2 \,,$$

where d is the distance between the particles.

(a) Identify the two generalized coordinates and write down the Lagrangian of the system.

(b) Write down the Lagrangian in the case that one circle lies directly beneath the other, $a = 0$, and identify a conserved quantity that appears in this case.

5. Two particles, each of mass m, are connected by a light rope of length l. One particle moves on a smooth horizontal table at a variable distance r from a hole, through which the rope is threaded. The second particle hangs beneath the table.

(a) Assume initially that the second particle hangs directly beneath the hole. Write down the Lagrangian of the system in terms of r and a variable ψ, describing the angle that the first particle makes with respect to a fixed axis. Identify an ignorable coordinate. Write down the equation of motion for the remaining coordinate, assuming that the rope remains taut.

(b) Assume now that the second particle oscillates beneath the table, as a spherical pendulum. How many degrees of freedom does the system now have? Write down the Lagrangian describing this motion, assuming that the rope remains taut at all times. How many ignorable coordinates are there?

6. An electron, of mass m and charge $-e$, moves in a magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r})$. The Lagrangian for the motion is

$$L = \frac{1}{2}m|\dot{\mathbf{r}}|^2 - e\,\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}) \,.$$

Show that Lagrange's equations reproduce the Lorentz force law for the electron.

(a) With respect to cylindrical polar coordinates (r, θ, z), consider the vector potential

$$\mathbf{A} = \frac{f(r)}{r}\,\mathbf{e}_\theta \,,$$

where $\mathbf{e}_\theta$ is the unit vector in the θ direction. At some initial time, the electron is at a distance r_0 from the z axis; its velocity is then in the (r, z) plane. Show that the electron's angular velocity about the z axis is given by

$$\dot{\theta} = \frac{e}{mr^2}\left[f(r) - f(r_0)\right] \,.$$

(b) (Again, with respect to cylindrical polar coordinates.) Consider the (different) vector potential,

$$\mathbf{A} = rg(z)\,\mathbf{e}_\theta\,,$$

where $g(z) > 0$. Find two constants of the motion. The electron is projected from a point (r_0, θ_0, z_0) with velocity $(2er_0 g(z_0)/m)\,\mathbf{e}_\theta$. Show that the electron will then describe a circular orbit, provided that $g'(z_0) = 0$. Show that this orbit is stable against small translations in the z direction, provided that $g''(z_0) > 0$.

7. The Lagrangian for a relativistic point particle of mass m is

$$L = -mc^2\sqrt{1 - \frac{|\dot{\mathbf{r}}|^2}{c^2}} - V(\mathbf{r})\,,$$

where c is the speed of light. Derive the equation of motion, and show that it reduces to Newton's equation of motion in the limit $|\dot{\mathbf{r}}| \ll c$.

8. Consider a system with n dynamical degrees of freedom, and generalized coordinates denoted by q^a, with $a = 1, \ldots, n$. The most general form for a purely kinetic Lagrangian is

$$L = \frac{1}{2} g_{ab}(q^1, ..., q^n)\dot{q}^a\dot{q}^b\,, \qquad (*)$$

where the summation convention is being used. The functions $g_{ab} = g_{ba}$ depend on the generalized coordinates. Assume that $\det(g_{ab}) \neq 0$ so that the inverse matrix g^{ab} exists (obeying $g^{ab}g_{bc} = \delta^a{}_c$). Show that Lagrange's equations for this system are given by

$$\ddot{q}^a + \Gamma^a_{bc}\dot{q}^b\dot{q}^c = 0\,, \qquad (\dagger)$$

where you should define the objects Γ^a_{bc}, which depend on the coordinates q^d through g and its first derivatives.

Write down a conserved quantity for the equations ($\dagger$), and for the case $n = 4$ with $g_{11} = g_{22} = g_{33} = +1$ and $g_{44} = -1$ and compare with the concept of *proper time* from the *Dynamics and Relativity* notes. *Optional*: In the case $g_{11} = g_{22} = g_{33} = +1$ and $g_{44} = -1 + 2V$ with V and with $\{\dot{q}_i\}_{i=1}^3$ very small, work out ($\dagger$) to first order of approximation and interpret. (This calculation is behind the Newtonian approximation of general relativity, see pp. 77-79 in Weinberg's book *Gravitation and Cosmology*.)

Please send any comments and corrections to `dmas2@cam.ac.uk`