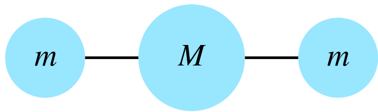
Example Sheet 2

1. The linear triatomic molecule drawn below consists of two identical outer atoms of mass m and a middle atom of mass M. It is a rough approximation to CO_2.



The interactions between neighbouring atoms are governed by a complicated potential $V(|\mathbf{r}_{i+1} - \mathbf{r}_i|)$. If we restrict attention to motion in the x direction parallel to the molecule, the Lagrangian is

$$L = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 - V(x_2 - x_1) - V(x_3 - x_2)\,,$$

where x_i is the position of the i^{th} particle. Let $r_0 = |x_{i+1} - x_i|$ be the separation of neighbouring atoms in equilibrium. Write down the equation describing small deviations from equilibrium in terms of the masses and the quantity $k = V''(r_0)$. Show that the system has three normal modes and calculate the frequencies of oscillation of the system. One of these frequencies vanishes; what is the interpretation of this?

2. A horizontal square wire frame with vertices $ABCD$ and side length $2a$ rotates with constant angular velocity ω about a vertical axis through A. A bead of mass m is threaded on BC and moves without friction. The bead is connected to B and C by two identical light springs of spring constant k and equilibrium length a.

   (a) Introducing the displacement $\eta(t)$ of the particle from the midpoint of BC, determine the Lagrangian $L(\eta, \dot{\eta})$.

   (b) Derive the equation of motion and identify the constant of the motion.

   (c) Describe the motion of the bead. Find the condition for there to be a stable equilibrium and find the frequency of small oscillations about it when it exists.

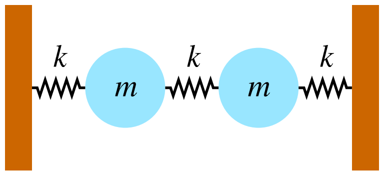
3. A pendulum consists of a mass m at the end of light rod of length l. The pivot of the pendulum is attached to a mass M, which is free to slide without friction along a horizontal rail. Take the generalized coordinates to be the position x of the pivot and the angle θ that the pendulum makes with the vertical.

   (a) Write down the Lagrangian and derive the equations of motion.

   (b) Find the non-zero frequency of small oscillations around the stable equilibrium.

   (c) Now suppose that a force acts on the the mass M, causing it to travel with constant acceleration a in the positive x direction. Find the equilibrium angle θ of the pendulum.

4. Two equal masses m are connected to each other and to fixed points by three identical springs of spring constant k as shown below. Write down the equations describing the motion of the system in the direction parallel to the springs. Find the normal modes and their frequencies.



5. Show that, for any solid body, the sum of any two principal moments of inertia is not less than the third. For what shapes is the sum of two equal to the third?

   Calculate the moments of inertia of:

   (a) A uniform solid sphere of mass M and radius R about a diameter.

   (b) A hollow sphere of mass M and radius R about a diameter.

   (c) A uniform solid circular cone of mass M, height h and base radius R with respect to the principal axes whose origin is at the vertex of the cone.

   (d) A uniform solid cylinder of radius R, height $2h$ and mass M about its centre of mass. For what height-to-radius ratio does the cylinder spin like a sphere?

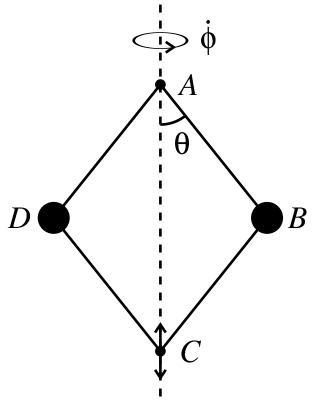
   (e) A uniform solid ellipsoid of mass M and semi-axes a, b and c, defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1$$

   with respect to the (x, y, z) axes with origin at the centre of mass. [*Hint:* With a change of coordinates, you can reduce this problem to that of the solid sphere.]

6. Four uniform rods of mass m and length $2a$ are hinged together to form a rhombus $ABCD$. The point A is fixed, while C lies directly beneath it and is free to slide up and down. Two uniform solid balls of mass M and radius R are rigidly attached to the upper rods with their centres at B and D as shown below. The whole system rotates freely around the vertical axis. Let θ be the angle that AB makes with the vertical, and let $\dot{\phi}$ be the angular velocity around the vertical axis.

   Find the Lagrangian for this system and show that there are two conserved constants of motion.

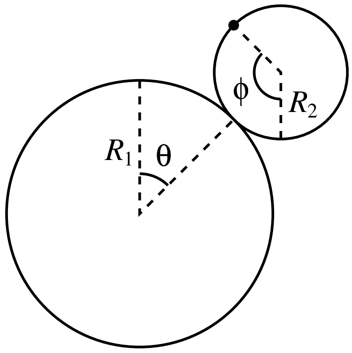


   [This is a model of a *centrifugal governor* – a device that in the past has been used to regulate the speed of engines, in particular steam engines.]

7. A cylindrical shell of radius R_2 rolls, without slipping, on a fixed cylinder of radius R_1 as shown below. Denote the angle through the centre of the first cylinder and the point of contact by θ. Denote the angle of a marked point on the upper cylinder with respect to a vertical axis by ϕ. Assume that the upper cylinder starts perched near the top at $\theta = 0$, and that it rolls without slipping, acted upon by gravity. Show that the constraint for small θ is

$$R_1\theta = R_2(\phi - \theta)\,. \qquad (*)$$

   Is this constraint holonomic? Can the system be described by holonomic constraints for all θ? Write down the Lagrangian for the system assuming that this constraint holds. (Remember that the cylinder has kinetic energy both from the translation of its centre of mass and also from its spin.) Work out the equation of motion for θ. If the upper cylinder starts from rest at $\theta = 0$, show that it falls off the lower cylinder at $\theta = \pi/3$.



[*Note*: The question of when the cylinder falls off is not obviously captured by the Lagrangian you wrote down, which assumes the constraint $(*)$ holds. To solve this you will have to revert to Newtonian thinking and consider the constraint forces at play.]

8*. An extension of question 4 [*optional, and certainly not at the expense of other questions*].

Suppose now that there are N equal masses joined by $N+1$ springs with fixed end-points. Write down the equations of motion in matrix form. Find the normal mode frequencies.

[*Hint:* To find the modes, you could try the following strategies:

(a) Solve the problem first with 'periodic boundary conditions': introduce an additional, $(N+1)^{\text{th}}$ mass and identify it with the 0^{th} mass. This is a mathematical device, but corresponds approximately (for large N) to the physical problem in which $N+1$ masses are joined in a circle. You can now solve the original problem by modifying the periodic boundary conditions by requiring zero displacement of the additional mass.

(b) Construct and solve a recurrence relation between determinants of suitably defined matrices of sizes N, $N-1$ and $N-2$.]

Please send any comments and corrections to `dbs26@cam.ac.uk`