Example Sheet 3

1. *Inertia tensor*

(a) Prove that the principal moments of inertia, (I_1, I_2, I_3), are real and non-negative.

(b) During the lectures we outlined the proof of the Parallel Axis Theorem, which is a statement that the inertia tensor about a point P, which is displaced by $\mathbf{c}$ from the centre of mass C, is related to the inertia tensor about C by

$$I^P_{ab} = I^C_{ab} + M(c^2\delta_{ab} - c_a c_b)\,,$$

where M is the total mass of the body. Complete the proof of the theorem. (It will be helpful to choose the origin to be at the centre of mass.)

2. *Euler angles*

The rotation matrix that relates the body axes $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ to the space axes $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ is given in terms of the Euler angles $(\theta(t), \phi(t), \psi(t))$ by

$$\begin{aligned}\mathsf{R} &= \mathsf{R}_3(\psi)\mathsf{R}_\mathrm{n}(\theta)\mathsf{R}_z(\phi)\\ &= \begin{pmatrix}\cos\psi & \sin\psi & 0\\ -\sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}.\end{aligned}$$

(a) Calculate the matrix $\mathsf{A} = \dot{\mathsf{R}}\mathsf{R}^\top$ and verify that it is antisymmetric. By identifying A as the antisymmetric matrix associated with the angular velocity vector in the body frame, deduce that

$$\begin{pmatrix}\omega_1\\ \omega_2\\ \omega_3\end{pmatrix} = \begin{pmatrix}\dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi\\ -\dot\theta\sin\psi + \dot\phi\sin\theta\cos\psi\\ \dot\psi + \dot\phi\cos\theta\end{pmatrix}.$$

(b) * [*optional extra*] Calculate the matrix $\mathsf{B} = \mathsf{R}^\top\dot{\mathsf{R}}$ and verify that it is antisymmetric. By identifying B as the antisymmetric matrix associated with the angular velocity vector in the space frame, deduce that

$$\begin{pmatrix}\omega_x\\ \omega_y\\ \omega_z\end{pmatrix} = \begin{pmatrix}\dot\theta\cos\phi + \dot\psi\sin\theta\sin\phi\\ \dot\theta\sin\phi - \dot\psi\sin\theta\cos\phi\\ \dot\phi + \dot\psi\cos\theta\end{pmatrix}.$$

3. *Free symmetric top*

(a) Consider the torque-free motion of a round plate. Show that, in the body frame, the angular velocity vector $\boldsymbol{\omega}$ precesses around the body axis $\mathbf{e}_3$ perpendicular to the plate with constant angular frequency equal to ω_3.

(b) The physicist Richard Feynman tells the following story:

> "*I was in the cafeteria and some guy, fooling around, throws a plate in the air. As the plate went up in the air I saw it wobble, and I noticed the red medallion of Cornell on the plate going around. It was pretty obvious to me that the medallion went around faster than the wobbling.*
> *I had nothing to do, so I start figuring out the motion of the rotating plate. I discover that when the angle is very slight, the medallion rotates twice as fast as the wobble rate – two to one. It came out of a complicated equation!*
> *I went on to work out equations for wobbles. Then I thought about how the electron orbits start to move in relativity. Then there's the Dirac equation in electrodynamics. And then quantum electrodynamics. And before I knew it....the whole business that I got the Nobel prize for came from that piddling around with the wobbling plate.*"

[Here the 'wobble' is associated with precession of the top about $\mathbf{e}_z$ (motion in ϕ), not nutation (motion in θ).]

Feynman was right about quantum electrodynamics. But what about the plate?

[You could try two alternative methods. First, by using the expression for ω_3 in terms of Euler's angles together with the result of part (a). Second, by writing down the Lagrangian of the top and deriving the equation of motion for θ.]

(c) Consider a uniform symmetric ellipsoid of mass M with semi-axes $a = b \neq c$ (see Example 2.5(e)). Find the ratio of the semi-axes for which $\dot{\phi}$, the angular frequency of precession of the top about the angular momentum $\mathbf{L}$, equals $\omega_3/(5\cos\theta)$. Deduce further that $\dot{\psi} = \frac{4}{5}\omega_3$. What is the relationship between $\dot{\phi}$ and $\dot{\psi}$ for small values of θ? Compare with the result obtained in part (b).

4. *Free asymmetric top (1)*

(a) Throw a book in the air. (Secure it with an elastic band first!) If the principal moments of inertia are $I_3 > I_2 > I_1$, convince yourself that the book can rotate in a stable manner about the principal axes $\mathbf{e}_1$ and $\mathbf{e}_3$, but not about $\mathbf{e}_2$.

(b) Use Euler's equations to show that the energy E and the squared angular momentum $|\mathbf{L}|^2$ of a free asymmetric top are conserved. Suppose that the initial conditions are such that

$$|\mathbf{L}|^2 = 2EI_2\,,$$

with the initial angular velocity $\boldsymbol{\omega}$ perpendicular to the intermediate principal axis $\mathbf{e}_2$. Show that $\boldsymbol{\omega}$ will ultimately end up parallel to $\mathbf{e}_2$. What is the characteristic timescale required to reach this steady state?

5. *Free asymmetric top (2)*

A rigid lamina (i.e. a two-dimensional object) has principal moments of inertia about the centre of mass given by

$$I_1 = (\mu^2 - 1)\,, \qquad I_2 = (\mu^2 + 1)\,, \qquad I_3 = 2\mu^2\,,$$

where $\mu > 1$. Write down Euler's equations for the lamina moving freely in space. Show that the component of the angular velocity in the plane of the lamina (i.e. $\sqrt{\omega_1^2 + \omega_2^2}$) is constant in time.

Choose the initial angular velocity to be $\boldsymbol{\omega} = \mu N \mathbf{e}_1 + N \mathbf{e}_3$. Define $\tan\alpha = \omega_2/\omega_1$, which is the angle the component of $\boldsymbol{\omega}$ in the plane of the lamina makes with $\mathbf{e}_1$. Show that it satisfies

$$\ddot{\alpha} + N^2 \cos\alpha \sin\alpha = 0$$

and deduce that, at time t,

$$\boldsymbol{\omega} = \mu N \,\mathrm{sech}(Nt)\, \mathbf{e}_1 + \mu N \tanh(Nt)\, \mathbf{e}_2 + N \,\mathrm{sech}(Nt)\, \mathbf{e}_3\,.$$

6. *Lagrange top*

Consider a heavy symmetric top of mass M, fixed at the point P which is a distance l from the centre of mass. The principal moments of inertia about P are (I_1, I_1, I_3) and the Euler angles are defined as in the lectures. The top is spun with initial conditions $\dot{\phi} = 0$ and $\theta = \theta_0$. Show that θ obeys the equation of motion

$$I_1 \ddot{\theta} = -\frac{dV_{\text{eff}}}{d\theta}\,,$$

where

$$V_{\text{eff}}(\theta) = \frac{I_3^2 \omega_3^2}{2 I_1} \frac{(\cos\theta - \cos\theta_0)^2}{\sin^2\theta} + Mgl\cos\theta\,.$$

Suppose that the top is spinning very fast so that

$$I_3 \omega_3 \gg \sqrt{MglI_1}\,.$$

Show that the minimum of $V_{\text{eff}}(\theta)$ is close to θ_0. Use this fact to deduce that the top nutates with angular frequency

$$\Omega \approx \frac{I_3}{I_1}\omega_3\,,$$

and sketch the subsequent motion.

7. *Lagrange top in Hamiltonian formalism*

The Lagrangian for the heavy symmetric top is

$$L = \frac{1}{2} I_1 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi} \cos \theta)^2 - Mgl \cos \theta \,.$$

Find the conjugate momenta p_θ, p_ϕ and p_ψ and the Hamiltonian $H(\theta, \phi, \psi, p_\theta, p_\phi, p_\psi)$. Derive Hamilton's equations.

8. *Hamilton's equations*

A system with two degrees of freedom x and y has the Lagrangian

$$L = x\dot{y} + y\dot{x}^2 + \dot{x}\dot{y} \,.$$

Derive Lagrange's equations. Obtain the Hamiltonian $H(x, y, p_x, p_y)$. Derive Hamilton's equations and show that they are equivalent to Lagrange's equations.

Please send any comments and corrections to `dbs26@cam.ac.uk`