Example Sheet 4

1. Verify the Jacobi identity for Poisson brackets,

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0\,.$$

2. A particle with mass m, position $\mathbf{r}$ and momentum $\mathbf{p}$ has angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$. Evaluate $\{x_i, L_j\}$, $\{p_i, L_j\}$, $\{L_i, L_j\}$ and $\{L_i, |\mathbf{L}|^2\}$.

   The Laplace–Runge–Lenz vector is defined as

$$\mathbf{A} = \mathbf{p}\times\mathbf{L} - mk\,\hat{\mathbf{r}}\,,$$

   where k is a constant and $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$. Show that $\{L_i, A_j\} = \epsilon_{ijk}A_k$. For a system described by the Hamiltonian

$$H = \frac{|\mathbf{p}|^2}{2m} - \frac{k}{|\mathbf{r}|}\,,$$

   show, using Poisson brackets, that $\mathbf{A}$ is conserved.

3. A particle of charge q moves in a time-independent background magnetic field $\mathbf{B}$. Show that $\{m\dot{x}_i, m\dot{x}_j\} = q\epsilon_{ijk}B_k$ and $\{x_i, m\dot{x}_j\} = \delta_{ij}$.

   A *magnetic monopole* is a particle that produces a radial magnetic field of the form

$$\mathbf{B} = g\,\frac{\hat{\mathbf{r}}}{r^2}\,,$$

   where g is a constant and $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$. Consider a charged particle moving in the background of the magnetic monopole. Define the generalized angular momentum,

$$\mathbf{J} = m\,\mathbf{r}\times\dot{\mathbf{r}} - qg\,\hat{\mathbf{r}}\,.$$

   Show that $\{\mathbf{J}, H\} = \mathbf{0}$. Why does this imply that $\mathbf{J}$ is conserved?

4. In the lectures we constructed canonical transformations using generating functions. Consider canonical transformations $\mathbf{q}\mapsto\mathbf{Q}(\mathbf{q},\mathbf{p})$, $\mathbf{p}\mapsto\mathbf{P}(\mathbf{q},\mathbf{p})$ from the following perspective. Define the $2n$-dimensional vector $\mathbf{x} = (q_1,\ldots,q_n,p_1,\ldots,p_n)^\top$ and the $2n\times 2n$ matrix

$$\Omega = \begin{pmatrix} 0 & \mathtt{I}_n \\ -\mathtt{I}_n & 0 \end{pmatrix},$$

   where each entry is itself an $n\times n$ matrix.

   (a) Write Hamilton's equations for $\dot{\mathbf{x}}$ in terms of Ω and the Hamiltonian H.

(b) Hence deduce the following equation for the vector $\mathbf{X} = (Q_1, ..., Q_n, P_1, ..., P_n)^\top$:

$$\dot{\mathbf{X}} = \left(\mathsf{J}\Omega\mathsf{J}^\top\right) \frac{\partial H}{\partial \mathbf{X}} ,$$

where $J^A_{\ B} = \partial X^A / \partial x^B$ $(A, B = 1, \ldots, 2n)$ is the Jacobian matrix of the transformation. This implies that, if the Jacobian of a transformation satisfies

$$\mathsf{J}\Omega\mathsf{J}^\top = \Omega ,$$

then Hamilton's equations are invariant under that transformation. The transformations with such a Jacobian (said to be *symplectic*) are canonical.

(c) Use the above conclusion to prove that, if the Poisson bracket structure is preserved, then the transformation is canonical.

5. Show that the following transformations are canonical:

(a) $P = \frac{1}{2}(p^2 + q^2) , \qquad Q = \arctan\left(\frac{q}{p}\right) ,$

(b) $P = \frac{1}{q} , \qquad Q = pq^2 ,$

(c) $P = 2\sqrt{q}\,(1 + \sqrt{q}\cos p)\sin p , \qquad Q = \log\left(1 + \sqrt{q}\cos p\right) .$

6. Show that the following transformation is canonical, for any constant λ:

$$\begin{aligned} q_1 &= Q_1 \cos\lambda + P_2 \sin\lambda , & q_2 &= Q_2 \cos\lambda + P_1 \sin\lambda , \\ p_1 &= -Q_2 \sin\lambda + P_1 \cos\lambda , & p_2 &= -Q_1 \sin\lambda + P_2 \cos\lambda . \end{aligned}$$

Given that the original Hamiltonian is

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\left(q_1^2 + q_2^2 + p_1^2 + p_2^2\right) ,$$

determine the new Hamiltonian $H(\mathbf{Q}, \mathbf{P})$. Hence solve for the dynamics, subject to the constraints $Q_2 = P_2 = 0$.

7. A group of particles, all of the same mass m, have initial heights z_0 and vertical momenta p_0 lying in the rectangle $-a \leqslant z_0 \leqslant a$, $-b \leqslant p_0 \leqslant b$ in phase space. The particles fall freely in a uniform gravitational field for a time t. Find the region of phase space in which they lie at time t, and show by direct calculation that its area is still $4ab$.

8. A *Poisson structure* on $\mathbb{R}^n$ is an antisymmetric matrix whose components $\pi^{AB}(x)$ may depend on the coordinates $x^A \in \mathbb{R}^n$, $A = 1, \ldots, n$, such that the Poisson bracket

$$\{f, g\} = \sum_{AB} \pi^{AB}(x) \frac{\partial f}{\partial x^A} \frac{\partial g}{\partial x^B}$$

satisfies the Jacobi identity.

(a) Show that

$$\{fg, h\} = f\{g, h\} + \{f, h\}g\,.$$

(b) Assume that the matrix π is invertible, and suppose that its inverse is an anti-symmetric matrix ω whose components $\omega_{AB}(x)$ obey $\pi^{AB}(x)\,\omega_{BC}(x) = \delta^A_C$ for all x. Show that ω satisfies

$$\partial_A\omega_{BC} + \partial_C\omega_{AB} + \partial_B\omega_{CA} = 0$$

where $\partial_A = \dfrac{\partial}{\partial x^A}$. [*Hint*: Note that $\pi^{AB} = \{x^A, x^B\}$.]

(c) Set $x^A = (x, y, z)$. Show that

$$\{x, y\} = z\,, \qquad \{y, z\} = x\,, \qquad \{z, x\} = y$$

defines a Poisson structure on $\mathbb{R}^3$, and find Hamilton's equations corresponding to a Hamiltonian $H = \alpha x^2 + \beta y^2 + \gamma z^2$, where α, β and γ are non-zero constants. Is this Poisson structure invertible?

9. Explain what is meant by an *adiabatic invariant* for a mechanical system with one degree of freedom.

A light string passes through a small hole in the roof of a lift compartment of a very high skyscraper, and a small weight is attached to the lower end. Initially, the lift is at rest and the system behaves like a simple pendulum executing small oscillations. Construct a Hamiltonian for the system and use the theory of adiabatic invariants to discuss what happens to the frequency, linear and angular amplitudes of the motion if:

(a) the lift begins to move upwards with slowly increasing acceleration, with the string attached at the hole;

(b) the lift stays at rest, but the string is slowly withdrawn through the roof.

10. Consider a system with Hamiltonian

$$H = \frac{p^2}{2m} + \lambda\, q^{2n}\,,$$

where λ is a positive constant and n is a positive integer. Show that the action variable I and the energy E are related by

$$E = \lambda^{1/(n+1)} \left(\frac{n\pi I}{J_n}\right)^{2n/(n+1)} \left(\frac{1}{2m}\right)^{n/(n+1)}\,,$$

where $J_n = \int_0^1 (1-x)^{1/2} x^{(1-2n)/2n}\, dx$.

Consider a particle that moves in a potential $V(q) = \lambda\, q^4$. Assuming that λ varies slowly with time, show that the particle's total energy E is proportional to $\lambda^{1/3}$. Conversely, in the case that λ is fixed, show that the period of the motion is proportional to $(\lambda E)^{-1/4}$.

11. A pulsar of mass m moves in a planar orbit around a luminous supergiant star with mass $M \gg m$. You may regard the supergiant as being fixed at the origin of a plane-polar coordinate system (r, θ), and the neutron star as moving in a central potential $V(r) = -GMm/r$. Construct the Hamiltonian for the motion, and show that p_θ and the total energy E are constants of motion.

   The neutron star is in a non-circular orbit with $E < 0$. Give an expression for the adiabatic invariant $J(E, p_\theta, M)$ associated with the radial motion. The supergiant is steadily losing mass in a radiatively driven wind. Show that, over a long timescale, we have $E \propto M^2$.

   Eventually the supergiant becomes a supernova, throwing off its outer layers on a short timescale, and leaving behind a remnant black hole of mass $M/2$. Explain why the theory of adiabatic invariants cannot be used to calculate the new orbit.

   [You may find the following integral helpful:

$$\int_{r_1}^{r_2} \left[\left(1 - \frac{r_1}{r}\right)\left(\frac{r_2}{r} - 1\right)\right]^{1/2} dr = \frac{\pi}{2}(r_1 + r_2) - \pi\sqrt{r_1 r_2}\,,$$

   where $0 < r_1 < r_2$.]

12. [*Optional, based on 2010 Paper 4, Section II, Question 15D*]

   A system is described by the Hamiltonian $H(q, p, t)$. Define the *Poisson bracket* $\{f, g\}$ of two functions $f(q, p, t)$ and $g(q, p, t)$. Show from Hamilton's equations that

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}\,.$$

   Consider the Hamiltonian

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right)\,,$$

   where $\omega = \omega(t)$, and define

$$a = \frac{p - i\omega q}{\sqrt{2\omega}}\,, \qquad a^* = \frac{p + i\omega q}{\sqrt{2\omega}}\,,$$

   where $i^2 = -1$. Evaluate $\{a, a\}$ and $\{a, a^*\}$, and show that $\{a, H\} = -i\omega a$ and $\{a^*, H\} = i\omega a^*$. Show further that, when $f(q, p, t)$ is regarded as a function of the independent complex variables (a, a^*) and of t, one has

$$\frac{df}{dt} = i\omega\left(a^*\frac{\partial f}{\partial a^*} - a\frac{\partial f}{\partial a}\right) - \frac{1}{2}\frac{\dot\omega}{\omega}\left(a\frac{\partial f}{\partial a^*} + a^*\frac{\partial f}{\partial a}\right) + \frac{\partial f}{\partial t}\,.$$

   Deduce that, in the case $d\omega/dt = 0$, both $(\log a^* - i\omega t)$ and $(\log a + i\omega t)$ are constant during the motion.

   Consider now the case in which $\omega(t)$ varies slowly with time. Writing $f = (H/\omega)$, show that the time-average of (df/dt) over one period, $(2\pi/\omega)$, is approximately zero (that is, to order $(\dot\omega^2, \ddot\omega)$). [*Hint*: You might like to start by writing $a = A(t)e^{-i\omega t} = A(0)e^{-i\omega t} + O(\dot\omega)$.]