Partial Differential Equations Example sheet 1

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Books

In addition to the sets of lecture notes written by previous lecturers ([1, 2]) which are still useful, the books [4, 3] are very good for the PDE topics in the course, and go well beyond the course also. If you want to read more on distributions [6] is most relevant. Also [7, 8] are useful; the books [5, 9] are more advanced, but may be helpful.

References

[1] T.W. Körner, Cambridge Lecture notes on PDE, available at https://www.dpmms.cam.ac.uk/ twk

[2] M. Joshi and A. Wassermann, Cambridge Lecture notes on PDE,available at http://www.damtp.cam.ac.uk/user/dmas2

[3] G.B. Folland, Introduction to Partial Differential Equations, *Princeton 1995, QA 374 F6*

[4] L.C. Evans, Partial Differential Equations,*AMS Graduate Studies in Mathematics Vol 19, QA377.E93 1990*

[5] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, *Springer, New York 2011 QA320 .B74 2011*

[6] F.G. Friedlander, Introduction to the Theory of Distributions, *CUP 1982, QA324*

[7] F. John, Partial Differential Equations, *Springer-Verlag 1982, QA1.A647*

[8] Rafael Jos Iorio and Valria de Magalhes Iorio, Fourier analysis and partial differential equations *CUP 2001, QA403.5 .I57 2001*

[9] M.E. Taylor, Partial Differential Equations,Vols I-III*Springer 96, QA1.A647*

1 Introduction

1.1 Notation

We write partial derivatives as $\partial_t = \frac{\partial}{\partial t}, \partial_j = \frac{\partial}{\partial x_j}$ etc and also use suffix on a function to indicate partial differentiation: $u_t = \partial_t u$ etc. A general k^{th} order linear partial differential operator (pdo) acting on functions $u = u(x_1, \dots x_n)$ is written:

$$P = \sum_{|\alpha \leq k} a_\alpha \partial^\alpha u \,. \tag{1.1}$$

Here $\alpha = (\alpha_1, \dots \alpha_n) \in \mathbb{Z}_+^n$ is a multi-index of order $|\alpha| = \sum \alpha_j$ and

$$\partial^\alpha = \prod \partial_j^{\alpha_j} \,, \quad x^\alpha = \prod x_j^{\alpha_j} \,. \tag{1.2}$$

For a multi-index we define the factorial $\alpha! = \prod \alpha_j!$. For (real or complex) constants a_α the formula (1.1) defines a constant coefficient linear pdo of order k. (Of course assume always that at least one of the a_α with $|\alpha| = k$ is non-zero so that it is genuinely of order k.) If the coefficients depend on x it is a variable coefficient linear pdo. The word linear means that

$$P(c_1 u_1 + c_2 u_2) = c_1 P u_1 + c_2 P u_2 \tag{1.3}$$

holds for P applied to C^k functions u_1, u_2 and arbitrary constants c_1, c_2.

1.2 Basic definitions

If the coefficients depend on the partial derivatives of a function of order strictly less than k the operator

$$u \mapsto Pu = \sum_{|\alpha \leq k} a_\alpha(x, \{\partial^\beta u\}_{|\beta|<k})\, \partial^\alpha u \tag{1.4}$$

is called quasi-linear and (1.3) no longer holds. The corresponding equation $Pu = f$ for $f = f(x)$ is a quasi-linear partial differential equation (pde). In such equations the partial derivatives of highest order - which are often most important - occur linearly. If the coefficients of the partial derivatives of highest order in a quasi-linear operator P depend only on x (not on u or its derivatives) the equation is called semi-linear. If the partial derivatives of highest order appear nonlinearly the equation is called fully nonlinear; such a general pde of order k may be written

$$F(x, \{\partial^\alpha u\}_{|\alpha| \leq k}) = 0 \,. \tag{1.5}$$

Definition 1.2.1 *A classical solution of the pde* (1.5) *on an open set* $\Omega \subset \mathbb{R}^n$ *is a function* $u \in C^k(\Omega)$ *which is such that* $F(x, \{\partial^\alpha u(x)\}_{|\alpha| \leq k}) = 0$ *for all* $x \in \Omega$ *.*

Classical solutions do not always exist and we will define generalized solutions later in the course. The most general existence theorem for classical solutions is the Cauchy-Kovalevskaya theorem, to state which we need the following definitions:

Definition 1.2.2 *Given an operator* (1.1) *we define*

- $P_{principal} = \sum_{|\alpha=k} a_\alpha \partial^\alpha u$, *(principal part)*
- $p = \sum_{|\alpha \leq k} a_\alpha (i\xi)^\alpha$, $\quad \xi \in \mathbb{R}^n$, *(total symbol)*
- $\sigma = \sum_{|\alpha=k} a_\alpha (i\xi)^\alpha$, $\quad \xi \in \mathbb{R}^n$, *(principal symbol)*
- $Char_x(P) = \{\xi \in \mathbb{R}^n : \sigma(x,\xi) = 0\}$, *(the set of characteristic vectors at x)*
- $Char(P) = \{(x,\xi) : \sigma(x,\xi) = 0\} = \cup_x Char_x(P)$, *(characteristic variety)*.

Clearly σ, p depend on $(x,\xi) \in \mathbb{R}^{2n}$ for variable coefficient linear operators, but are independent of x in the constant coefficient case. For quasi-linear operators we make these definitions by substituting in $u(x)$ into the coefficients, so that p, σ and (also the definition of characteristic vector) depend on this $u(x)$.

Definition 1.2.3 *The operator* (1.1) *is elliptic at x (resp. everywhere) if the principal symbol is non-zero for non-zero ξ at x (resp. everywhere). (Again the definition of ellipticity in the quasi-linear case depends upon the function $u(x)$ in the coefficients.)*

The elliptic operators are an important class of operators, and there is a well-developed theory for elliptic equations $Pu = f$. Other important classes of operators are the parabolic and hyperbolic operators: see below for defintions of classes of parabolic and hyperbolic operators of second order.

1.3 The Cauchy-Kovalevskaya theorem

The *Cauchy problem* is the problem of showing that for a given pde and given data on a hypersurface $\mathcal{S} \subset \mathbb{R}^n$ there is a unique solution of the pde which agrees with the data on $\mathcal{S}$. This is a generalization of the initial value problem for ordinary differential equations, and by analogy the appropriate data to be given on $\mathcal{S}$ consists of u and its normal derivatives up to order $k-1$. A crucial condition is the following:

Definition 1.3.1 *A hypersurface $\mathcal{S}$ is non-characteristic at a point x if its normal vector $n(x)$ is non-characteristic, i.e. $\sigma(x, n(x)) \neq 0$. We say that $\mathcal{S}$ is non-characteristic if it is non-characteristic for all $x \in \mathcal{S}$.)*

Again for quasi-linear operators it is necessary to substitute $u(x)$ to make sense of this definition, so that whether or not a hypersurface is non-characteristic depends on $u(x)$, which amounts to saying it depends on the data which are given on $\mathcal{S}$.

Theorem 1.3.2 (Cauchy-Kovalevskaya theorem) *In the real analytic case there is a local solution to the Cauchy problem for a quasi-linear pde in a neighbourhood of a point as long as the hypersurface is non-characteristic at that point.*

This becomes clearer with a suitable choice of coordinates which emphasizes the analogy with ordinary differential equations: let the hypersurface be the level set $x_n = t = 0$ and let $x = (x_1, \dots x_{n-1})$ be the remaining $n-1$ coordinates. Then a quasi-linear P takes the form

$$Pu = a_{0k}\partial_t^k + \sum_{|\alpha|+j\le k, j<k} a_{j\alpha}\partial_t^j\partial^\alpha u \tag{1.6}$$

with the coefficients depending on derivatives of order $< k$, as well as on (x,t). Since the normal vector to $t = 0$ is $n = (0, 0, \dots 0, 1) \in \mathbb{R}^n$ the non-characteristic condition is just $a_{0k} \ne 0$, and ensures that the quasi-linear equation $Pu = f$ can be solved for $\partial_t^k u$ in terms of $\{\partial_t^j\partial^\alpha u\}_{|\alpha|+j\le k, j<k}$ to yield an equation of the form:

$$\partial_t^k u = G(x, t, \{\partial_t^j\partial^\alpha u\}_{|\alpha|+j\le k, j<k}) \tag{1.7}$$

to be solved with data

$$u(x,0) = \phi_0(x), \partial_t u(x,0) = \phi_1(x) \dots \partial_t^{k-1}u(x,0) = \phi_{k-1}(x)\,. \tag{1.8}$$

Notice that these data determine, for all $j < k$, the derivatives

$$\partial_t^j\partial^\alpha u(x,0) = \partial^\alpha\phi_j(x)\,, \tag{1.9}$$

(i.e. those involving fewer than k normal derivatives ∂_t) on the initial hypersurface.

Theorem 1.3.3 *Assume that $\phi_0, \dots \phi_{k-1}$ are all real analytic functions in some neighbourhood of a point x_0 and that G is a real analytic function of its arguments in a neighbourhood of $(x_0, 0, \{\partial^\alpha\phi_j(x_0)\}_{|\alpha|+j\le k, j<k})$. Then there exists a unique real analytic function which satisfies* (1.8)-(1.7) *in some neighbourhood of the point x_0.*

Notice that the non-characteristic condition ensures that the k^{th} normal derivative $\partial_t^k u(x,0)$ is determined by the data through the equation. Differentiation of (1.7) gives further relations which can be shown to determine all derivatives of the solution at $t = 0$, and the theorem can be proved by showing that the resulting Taylor series defines a real-analytic solution of the equation. Read section 1C of the book of Folland for the full proof.

In the case of first order equations with real coefficients the method of characteristics gives an alternative method of attack which does not require real analyticity. In this case we consider a pde of the form

$$\sum_{j=1}^n a_j(x,u)\partial_j u = b(x,u) \tag{1.10}$$

with data

$$u(x) = \phi(x)\,, \quad x \in \mathcal{S} \tag{1.11}$$

where $\mathcal{S} \subset \mathbb{R}^n$ is a hypersurface, given in paramteric form as $x_j = g_j(\sigma)\,, \sigma = (\sigma_1, \dots \sigma_{n-1}) \in \mathbb{R}^{n-1}$. (Think of $\mathcal{S} = \{x_n = 0\}$ parametrized by $g(\sigma_1, \dots \sigma_{n-1}) = (\sigma_1, \dots \sigma_{n-1}, 0)$.)

Theorem 1.3.4 *Let $\mathcal{S}$ be a C^1 hypersurface, and assume that the a_j, b, ϕ are all C^1 functions. Assume the non-characteristic condition:*

$$\Sigma_{j=1}^n a_j(x_0, \phi(x_0)) n_j(x_0) \neq 0$$

holds at a point $x_0 \in \mathcal{S}$. Then there is an open set $\mathcal{O}$ containing x_0 in which there exists a unique C^1 solution of (1.10) *which also satisfies* (1.11) *at all $x \in \mathcal{O} \cap \mathcal{S}$. If the non-characteristic condition holds at all points of $\mathcal{S}$, then there is a unique solution of* (1.10)-(1.11) *in an open neighbourhood of $\mathcal{S}$.*

This is proved by considering the characteristic curves which are obtained by integrating the system of $n+1$ characteristic ordinary differential equations (ode):

$$\frac{dx_j}{ds} = a_j(x,z)\,, \quad \frac{dz}{ds} = b(x,z) \tag{1.12}$$

with data $x_j(\sigma, 0) = g_j(\sigma), z(\sigma, 0) = \phi(g(\sigma))$; let $(X(\sigma, s), Z(\sigma, s)) \in \mathbb{R}^n \times \mathbb{R}$ be this solution. Now compute the Jacobian matrix of the mapping $(\sigma, s) \mapsto X(\sigma, s)$ at the point $(\sigma, 0)$: it is the $n \times n$ matrix whose columns are $\{\partial_{\sigma_j} g\}_{j=1}^{n-1}$ and the vector $a = (a_1, \ldots a_n)$, evaluated at $x = g(\sigma), z = \phi(g(\sigma))$. The non-characteristic condition implies that this matrix is invertible (a linear bijection) and hence, via the inverse function theorem, that the "restricted flow map" which takes $(\sigma, s) \mapsto X(\sigma, s) = x$ is locally invertible, with inverse $\sigma_j = \Sigma_j(x), s = S(x)$ and this allows one to recover the solution as $u(x) = Z(\Sigma(x), S(x))$. This just means we have found a locally unique characteristic curve passing through x, and have then found $u(x)$ by tracing its value back along the curve to a point $g(\Sigma(x))$ on the initial hypersurface.

1.4 Various types of equations

We have defined *elliptic* operators on an open set Ω as partial differential operators with the property that, for all $x \in \Omega$, the princpal symbol $\sigma(x, \xi)$ vanishes only for $\xi = 0$. Examples to keep in mind are the Laplacian and the Cauchy-Riemann operator.

In addition to elliptic operators, later on we will consider *parabolic* operators of the form

$$Lu = \partial_t u + Pu$$

where

$$Pu = -\sum_{j,k=1}^n a_{jk} \partial_j \partial_k u + \sum_{j=1}^n b_j \partial_j u + cu \tag{1.13}$$

is a second order elliptic operator - the quadratic form $\sum_{j,k=1}^n a_{jk} \xi_j \xi_k$ is positive definite. A useful slightly stronger condition, which we will use , is that of uniform ellipticity: there exist positive constants m, M such that

$$m\|\xi\|^2 \leq \sum_{j,k=1}^n a_{jk} \xi_j \xi_k \leq M\|\xi\|^2 \tag{1.14}$$

holds everywhere. The standard example of a parabolic operator is provided by the heat, or diffusion, equation $u_t - \Delta u = 0$, There are other, more general notions of parabolicity in the literature.

A second order equation of the form

$$u_{tt} + \sum_j \alpha_j \partial_t \partial_j u + Pu = 0$$

with P as in (1.13) (with coefficients potentially depending upon t and x), is called *strictly hyperbolic* if the principal symbol

$$\sigma(\tau, \xi; t, x) = -\tau^2 - (\alpha \cdot \xi)\tau + \sum_{jk} a_{jk} \xi_j \xi_k$$

considered as a polynomial in τ has two distinct real roots $\tau = \tau_{\pm}(\xi; t, x)$ for all nonzero ξ. We will mostly study the wave equation

$$u_{tt} - \Delta u = 0\,, \tag{1.15}$$

which is the basic hyperbolic operator. Again there are various alternative, more general notions of hyperbolicity in the literature, in particular that of a strictly hyperbolic system (see §11.1 in the book of Evans).

These three basic types of operators do not generally form a classification of all possible operators: for example, the operator $\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2$ is not elliptic, parabolic or hyperbolic.

However, the case of second order equations in two space dimensions is special. An equation of the form

$$au_{xx} + 2bu_{xy} + cu_{yy} = f\,, \tag{1.16}$$

where a, b, c, f are real-valued smooth functions of x, y, u, u_x, u_y, is classified as:

- elliptic in Ω if $b^2 - ac < 0$ throughout Ω ;
- hyperbolic in Ω if $b^2 - ac > 0$ throughout Ω .

The intermediate case, $b^2 = ac$ is degenerate - it can lead to an equation which is parabolic in the sense explained above, but there are other possibilities depending upon lower order terms: for example the case of an ordinary differential equation like $u_{xx} = 0$ satisfies $b^2 = ac$ everywhere.

The real significance of the conditions $b^2 \gtrless 0$ is for the existence of the characteristic curves for (1.16). These are defined to be real integral curves of the differential equation

$$a(y')^2 - 2by' + c \;=\; 0\,.$$

In the elliptic case there are no *real* characteristic curves - this is the case for the Laplacian. But for hyperbolic equations there are two distinct families of real characteristic curves (corresponding to the two distinct roots of the quadratic equation for y'). These curves determine a change of coordinates $(x, y) \mapsto (X, Y)$ under which (1.16) can be transformed into the form

$$U_{XY} = F(X, Y, U, U_X, U_Y)\,.$$

Thus this is a canonical form for second order hyperbolic equations in the plane. The wave equation $u_{yy} - u_{xx} = 0$ is a well-known example. The characteristic curves are $x+y = X$ and $x-y = Y$, and in the new coordinates X, Y the equation takes the form $U_{XY} = 0$. This gives the general solution as $F(X) + G(Y)$, or as $F(x+y) + G(x-y)$ in the original coordinate system, for arbitrary F, G.

1.5 Some worked problems

1. Consider the two-dimensional domain

$$G := \{(x,y) \mid R_1^2 < x^2 + y^2 < R_2^2\},$$

where $0 < R_1 < R_2 < \infty$. Solve the Dirichlet boundary value problem for the Laplace equation

$$\Delta u = 0 \text{ in } G,$$
$$u = u_1(\varphi),\ r = R_1,$$
$$u = u_2(\varphi),\ r = R_2,$$

where (r, φ) are polar coordinates. Assume that u_1, u_2 are smooth 2π-periodic functions on the real line.

Discuss the convergence properties of the series so obtained.

[*Hint: Use separation of variables in polar coordinates ($u = R(r)\Phi(\varphi)$), with perodic boundary conditions for the function Φ of the angle variable. Use an ansatz of the form $R(r) = r^\alpha$ for the radial function.*]

Answer As the hint suggests, we use radial coordinates and transform the Laplacian. Using this, our PDE becomes

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi} = 0.$$

Using separation of variables as the hint suggests yields

$$R''\Phi + \frac{1}{r}R'\Phi + \frac{1}{r^2}R\Phi'' = 0$$

We multiply this equation by $\frac{r^2}{R\Phi}$ and rearrange to obtain an equality between an expression which depends only on r and an expression which depends only on φ. This implies that the two equations are equal to a constant (denoted λ^2):

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = \frac{-\Phi''(\varphi)}{\Phi} = \lambda^2$$

We require a 2π periodicity for each value of the radial coordinate, so we require that Φ be 2π periodic, and thus obtain:

$$\Phi(\varphi) = A\sin(\lambda\varphi) + B\cos(\lambda\varphi),$$

with positve constants A, B. Since we assume the solution to be 2π-periodic it follows that λ must be an integer, and w.l.o.g. λ is non-negative.

For the other equation we use the Ansatz $R(r) = r^\alpha$ and obtain

$$\alpha(\alpha - 1)r^\alpha + \alpha r^\alpha - \lambda^2 r^\alpha = 0$$

and thus

$$\alpha = \pm\lambda.$$

Therefore

$$R(r) = Cr^\lambda + Dr^{-\lambda}, \quad \text{if } \lambda > 0$$

and by considering the case $\lambda = 0$ separately

$$R(r) = E + F\ln(r), \quad \text{if } \lambda = 0.$$

As the equation is linear, the most general solution is

$$\begin{aligned} u(r,\varphi) = &\sum_{\lambda=1}^{\infty}(A_\lambda r^\lambda + B_\lambda r^{-\lambda})\sin(\lambda\varphi) + (C_\lambda r^\lambda + D_\lambda r^{-\lambda})\cos(\lambda\varphi) \\ &+ E_0 + F_0\ln(r) \end{aligned}$$

To account for the boundary conditions, we expand u_1 and u_2 as Fourier series:

$$\begin{aligned} u_1(\varphi) &= \frac{a_0}{2} + \sum_{\lambda=1}^{\infty} a_\lambda \sin(\lambda\varphi) + b_\lambda \cos(\lambda\varphi), \\ u_2(\varphi) &= \frac{c_0}{2} + \sum_{\lambda=1}^{\infty} c_\lambda \sin(\lambda\varphi) + d_\lambda \cos(\lambda\varphi). \end{aligned}$$

(This is possible by the assumptions on $u_{1,2}$.) To enforce

$$u(R_1,\varphi) = u_1(\varphi), \quad u(R_2,\varphi) = u_2(\varphi)$$

comparison of the coefficients leads to

$$\begin{aligned} A_\lambda R_1^\lambda + B_\lambda R_1^{-\lambda} = a_\lambda, &\quad A_\lambda R_2^\lambda + B_\lambda R_2^{-\lambda} = c_\lambda, \\ C_\lambda R_1^\lambda + D_\lambda R_1^{-\lambda} = b_\lambda, &\quad C_\lambda R_2^\lambda + D_\lambda R_2^{-\lambda} = d_\lambda \end{aligned}$$

and

$$\begin{aligned} E_0 + F_0\ln(R_1) &= \frac{a_0}{2}, \\ E_0 + F_0\ln(R_2) &= \frac{c_0}{2}. \end{aligned}$$

The first two equations result in

$$A_\lambda = \frac{R_1^\lambda a_\lambda - R_2^\lambda c_\lambda}{R_1^{2\lambda} - R_2^{2\lambda}}, \quad B_\lambda = \frac{R_1^{-\lambda} a_\lambda - R_2^{-\lambda} c_\lambda}{R_1^{-2\lambda} - R_2^{-2\lambda}},$$

and

$$C_\lambda = \frac{R_1^\lambda b_\lambda - R_2^\lambda d_\lambda}{R_1^{2\lambda} - R_2^{2\lambda}}, \quad D_\lambda = \frac{R_1^{-\lambda} b_\lambda - R_2^{-\lambda} d_\lambda}{R_1^{-2\lambda} - R_2^{-2\lambda}}.$$

From the last equation we obtain

$$E_0 = \frac{a_0 \ln(R_2) - c_0 \ln(R_1)}{2(\ln(R_2) - \ln(R_1))}\,, \quad F_0 = \frac{c_0 - a_0}{2(\ln(R_2) - \ln(R_1))}\,.$$

Since the $u_{1,2}$ are smooth and periodic, their Fourier coeficients are rapidly decreasing, i.e.

$$\sup_{\lambda \in \mathbb{N}} \lambda^N (|a_\lambda| + |b_\lambda| + |c_\lambda| + |d_\lambda|) < \infty$$

for any positive N. Now let $\rho = R_1/R_2 = (R_2/R_1)^{-1} \in (0,1)$, then the above formulae can be written as

$$A_\lambda = R_2^{-\lambda} \frac{c_\lambda - \rho^\lambda a_\lambda}{1 - \rho^{2\lambda}}\,, \quad B_\lambda = R_1^\lambda \frac{a_\lambda - \rho^\lambda c_\lambda}{1 - \rho^{2\lambda}}\,,$$

and

$$C_\lambda = R_2^{-\lambda} \frac{d_\lambda - \rho^\lambda b_\lambda}{1 - \rho^{2\lambda}}\,, \quad D_\lambda = R_1^\lambda \frac{b_\lambda - \rho^\lambda d_\lambda}{1 - \rho^{2\lambda}}\,.$$

Since $\rho \in (0,1)$ it follows from these formulae that $|A_\lambda| \le R_2^{-\lambda}(|a_\lambda| + |c_\lambda|)/(1-\rho)$ and $|B_\lambda| \le R_1^\lambda(|a_\lambda| + |c_\lambda|)/(1-\rho)$ while $|C_\lambda| \le R_2^{-\lambda}(|b_\lambda| + |d_\lambda|)/(1-\rho)$ and $|D_\lambda| \le R_1^\lambda(|b_\lambda| + |d_\lambda|)/(1-\rho)$. As a consequence

$$\sup_{\lambda \in \mathbb{N}} \sup_{R_1 \le r \le R_2} \lambda^N (r^\lambda |A_\lambda| + r^{-\lambda}|B_\lambda| + r^\lambda |C_\lambda| + r^{-\lambda}|D_\lambda|) < \infty$$

for any positive N. Therefore the series

$$\begin{aligned} u(r,\varphi) = &\sum_{\lambda=1}^{\infty} (A_\lambda r^\lambda + B_\lambda r^{-\lambda}) \sin(\lambda\varphi) + (C_\lambda r^\lambda + D_\lambda r^{-\lambda}) \cos(\lambda\varphi) \\ &+ E_0 + F_0 \ln(r) \end{aligned}$$

converges absolutely and uniformly in the *closed* annulus $\overline{G}$, to define a continuous function $u \in C(\overline{G})$, which agrees with the given data on the boundary ∂G. Furthermore u is smooth in the open annulus G where it solves $\Delta u = 0$.

As a final comment on the method of solution, an alternative to separation of variables is to say that any *smooth* function $u(r,\varphi)$ *which is* 2π *periodic* in φ can be decomposed as

$$u(r,\varphi) = u_0(r) + \sum_{\lambda=1}^{\infty} \alpha_\lambda(r) \sin(\lambda\varphi) + \beta_\lambda(r) \cos(\lambda\varphi)\,,$$

with $\alpha_\lambda, \beta_\lambda$ rapidly decreasing so that term by term differentiation is allowed. Then substitute this into the equation to obtain equations for $u_0(r), \alpha_\lambda(r), \beta_\lambda(r)$ and the same answer will follow.

2. (i) State the local existence theorem for real-valued solutions of the first order quasi-linear partial differential equation

$$\sum_{j=1}^{n} a_j(x,u) \frac{\partial u}{\partial x_j} = b(x,u) \tag{1.17}$$

with data specified on a hypersurface S, including a definition of "non-characteristic" in your answer. Also define the characteristic curves for (1.17) and briefly explain their use in obtaining the solution.

(ii) For the linear constant coefficient case (i.e. all the functions $a_1, \ldots, a_n$, are real constants and $b(x, u) = cu + d$ for some real numbers c, d) and with the hypersurface S taken to be the hyperplane $x \cdot \nu = 0$ explain carefully the relevance of the non-characteristic condition to obtaining a solution via the method of characteristics.

(iii) Solve the equation

$$\frac{\partial u}{\partial y} + u \frac{\partial u}{\partial x} = 0,$$

with initial data $u(0, y) = -y$ prescribed on $x = 0$, for a real valued function. Describe the domain on which your solution is C^1 and comment on this in relation to the theorem stated in (i).

Answer (i)

Theorem 1.5.1 *Let $\mathcal{S}$ be a C^1 hypersurface, and assume that the a_j, b, ϕ are all C^1 functions. Assume the non-characteristic condition:*

$$\Sigma_{j=1}^n a_j(x_0, \phi(x_0)) n_j(x_0) \neq 0$$

holds at a point $x_0 \in \mathcal{S}$. Then there is an open set $\mathcal{O}$ containing x_0 in which there exists a unique C^1 solution of (1.17) *which also satisfies*

$$u(x) = \phi(x)\,, \quad x \in \mathcal{S} \cap \mathcal{O}\,. \tag{1.18}$$

If the non-characteristic condition holds at all points of $\mathcal{S}$, then there is a unique solution of (1.17)-(1.11) *in an open neighbourhood of $\mathcal{S}$.*

The characteristic curves are obtained as the x component of the integral curves of the characteristic ode:

$$\frac{dx_j}{ds} = a_j(x, z)\,, \quad \frac{dz}{ds} = b(x, z) \tag{1.19}$$

with data $x_j(\sigma, 0) = g_j(\sigma), z(\sigma, 0) = \phi(g(\sigma))$; let $(X(\sigma, s), Z(\sigma, s)) \in \mathbb{R}^n \times \mathbb{R}$ be this solution. The characteristic curves starting at $g(\sigma)$ are the curves $s \mapsto X(\sigma, s)$. They are useful because the non-characteristic condition implies (via the inverse function theorem) that the "restricted flow map" which takes $(\sigma, s) \mapsto X(\sigma, s) = x$ is locally invertible, with inverse $\sigma_j = \Sigma_j(x), s = S(x)$ and this allows one to obtain the solution by tracing along the characteristic curve using the z component of the characteristic ode above. This gives the final formula: $u(x) = Z(\Sigma(x), S(x))$.

(ii) In the linear constant coefficient case the non-characteristic condition reads $a \cdot \nu \neq 0$, and the characteristic curves are lines with tangent vector $a = (a_1, \ldots a_n)$, obtained by integrating the characteristic ode:

$$\frac{dx_j}{ds} = a_j\,, \quad \frac{dz}{ds} = b(x, z) = cz + d\,, \tag{1.20}$$

and taking the "x component". The flow map is the smooth function $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Phi(s, x) = x + sa,$$

i.e. the solution of the characteristic ode starting at x. Parametrize the initial hyperplane as $x = \sum_{j=1}^{n-1} \sigma_j \gamma_j$, where the $\gamma_j \in \mathbb{R}^n$ are a linearly independent set of vectors in the plane (i.e. satisfying $\gamma_j \cdot \nu = 0$). The restricted flow map is just the restriction of the flow map to the initial hypersurface, i.e.

$$X(s,\sigma) = \sum_{j=1}^{n-1} \sigma_j \gamma_j + sa = J \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_{n-1} \\ s \end{pmatrix} .$$

Notice that here J, the Jacobian of the linear mapping $(s,\sigma) \mapsto X(s,\sigma)$, is precisely the *constant* $n \times n$ matrix whose columns are $\{\gamma_1, \gamma_2, \dots \gamma_{n-1}, a\}$. But the non-characteristic condition $a \cdot \nu \neq 0$ is equivalent to invertibility of this matrix and consequently $X(s, \sigma_1, \dots, \sigma_{n-1}) = x$ is uniquely solvable for

$$s = S(x), \sigma_1 = \Sigma_1(x), \dots, \sigma_{n-1} = \Sigma_{n-1}(x)$$

as functions of x (i.e. X is a linear bijection). This means that given any point $x \in \mathbb{R}^n$ there is a unique characteristic passing through it which intersects the initial hyperplane at exactly one point. This detemines the solution u uniquely at x since by the chain rule $z(s, \sigma_1, \dots, \sigma_{n-1}) = u(X(s, \sigma_1, \dots, \sigma_{n-1}))$ satisfies

$$\frac{dz}{ds} = cz + d,$$

so the evolution of u along the characteristic curves is known.

(iii) The characteristic ode are

$$\frac{dx}{ds} = -z\,, \ \frac{dy}{ds} = 1\,, \ \frac{dz}{ds} = 0\,.$$

The initial hypersurface can be parametrized as $(x(\sigma), y(\sigma) = (0, \sigma)$ and the solutions of the characteristic ode with initial data $z(0,\sigma) = -\sigma$ are $x(s,\sigma) = -s\sigma$, $y(s,\sigma) = \sigma + s$ and $z(s,\sigma) = z(0,\sigma) = -\sigma$. The restricted flow map is therefore $X(s,\sigma) = (-s\sigma, s + \sigma)$. Inverting this leads to a quadratic and the solution is given explicitly as:

$$u(x,y) = -\frac{1}{2}y - \frac{1}{2}\sqrt{(y^2+4x)} \quad y > 0,$$

$$u(x,y) = -\frac{1}{2}y + \frac{1}{2}\sqrt{(y^2+4x)} \quad y < 0,$$

where $\sqrt{a}$ means positive square root of a. Both of these formulae define C^1 (even smooth) functions in the region $\{y^2 + 4x > 0\}$, and can be verified to solve $u_y + uu_x = 0$ there. The region $\{y^2 + 4x > 0\}$ includes open neighbourhoods of every point on the initial hypersurface $x = 0$ except for the point $x = 0 = y$: this fits in with the statement of the theorem since it is at this point, and only this point, that the non-characterisitic condition fails to hold. To solve the Cauchy problem it is necessary to match the initial data: *notice that the signs of the square roots in the solution given above are chosen to ensure that the initial data are taken on correctly.* It is necessary to choose one of the "branches", depending upon how the initial hypersurface $\{x = 0\}$ is approached. This means the solution is no longer globally smooth - it is discontinuous along the half line $\{x > 0, y = 0\}$. As in complex analysis this line of discontinuity (like a "branch-cut") could be chosen differently, e.g. the half line $\{y = x, x > 0\}$.

1.6 Example sheet 1

1. Write out the multinomial expansion for $(x_1 + \ldots x_n)^N$ and the n-dimensional Taylor expansion using multi-index notation.
2. Consider the problem of solving the heat equation $u_t = \Delta u$ with data $u(x,0) = f(x)$. Is the non-characteristic condition satisfied? How about for the wave equation $u_{tt} = \Delta u$ with data $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$? For which of these problems, and for which data, does the Cauchy-Kovalevskaya theorem ensure the existence of a local solution? How about the Cauchy problem for the Schrödinger equation?
3. (a) Find the characteristic vectors for the operator $P = \partial_1\partial_2$ $(n = 2)$. Is it elliptic? Do the same for $P = \sum_{j=1}^{m} \partial_j^2 - \sum_{j=m+1}^{n} \partial_j^2$ $(1 < m < n)$.
   (b) Let $\Delta = \sum_{j=1}^{n-1} \partial_j^2$ be the laplacian. For which vectors $a \in \mathbb{R}^{n-1}$ is the operator $P = \partial_t^2 u + \partial_t \sum_{j=1}^{n-1} a_j \partial_j u - \Delta u$ hyperbolic?
4. Solve the linear PDE $x_1 u_{x_2} - x_2 u_{x_1} = u$ with boundary condition $u(x_1, 0) = f(x_1)$ for f a C^1 function. Where is your solution valid? Classify the f for which a global C^1 solution exists. (Global solution here means a solution which is C^1 on all of $\mathbb{R}^2$.)
5. Solve Cauchy problem for the semi-linear PDE $u_{x_1} + u_{x_2} = u^4$, $u(x_1, 0) = f(x_1)$ for f a C^1 function. Where is your solution C^1?
6. For the quasi-linear Cauchy problem $u_{x_2} = x_1 u u_{x_1}$, $u(x_1, 0) = x_1$
   (a) Verify that the Cauchy-Kovalevskaya theorem implies existence of an analytic solution in a neighbourhood of all points of the initial hypersurface $x_2 = 0$ in $\mathbb{R}^2$,
   (b) Solve the characteristic ODE and discuss invertibility of the restricted flow map $X(s,t)$ (this may not be possible explicitly),
   (c) give the solution to the Cauchy problem (implicitly).
7. For the quasi-linear Cauchy problem $Au_{x_1} - (B - x_1 - u)u_{x_2} + A = 0$, $u(x_1, 0) = 0$:
   (a) Find all points on the intial hypersurface where the Cauchy-Kovalevskaya theorem can be applied to obtain a local solution defined in a neighbourhood of the point.
   (b) Solve the characteristic ODE and invert (where possible) the restricted flow map, relating your answer to (a).
   (c) Give the solution to the Cauchy problem, paying attention to any sign ambiguities that arise.
   (In this problem take A, B to be positive real numbers).
8. For the Cauchy problem
$$u_{x_1} + 4u_{x_2} = \alpha u \qquad u(x_1, 0) = f(x_1), \tag{1.21}$$
   with C^1 initial data f, obtain the solution $u(x_1, x_2) = e^{\alpha x_2/4} f(x_1 - x_2/4)$ by the method of characteristics. For fixed x_2 write $u(x_2)$ for the function $x_1 \mapsto u(x_1, x_2)$ i.e. the solution restricted to "time" x_2. Derive the following *well-posedness* properties for solutions $u(x_1, x_2)$ and $v(x_1, x_2)$ corresponding to data $u(x_1, 0)$ and $v(x_1, 0)$ respectively:
   (a) for $\alpha = 0$ there is *global well-posedness* in the supremum (or L^∞) norm *uniformly in time* in the sense that if for fixed x_2 the distance between u and v is taken to be
$$\|u(x_2) - v(x_2)\|_{L^\infty} \equiv \sup_{x_1} |u(x_1, x_2) - v(x_1, x_2)|$$
   then
$$\|u(x_2) - v(x_2)\|_{L^\infty} \leq \|u(0) - v(0)\|_{L^\infty} \qquad \text{for all } x_2.$$
   Is the inequality ever strict?
   (b) for all α there is *well-posedness* in supremum norm *on any finite time interval* in the sense that for any time interval $|x_2| \leq T$ there exists a number $c = c(T)$ such that
$$\|u(x_2) - v(x_2)\|_{L^\infty} \leq c(T) \|u(0) - v(0)\|_{L^\infty}.$$

and find $c(T)$. Also, for different α, when can c be assumed independent of time for positive (respectively negative) times x_2?
(c) Try to do the same for the L^2 *norm*, i.e. the norm defined by

$$\|u(x_2) - v(x_2)\|^2_{L^2(dx_1)} = \int |u(x_1, x_2) - v(x_1, x_2)|^2 dx_1.$$

9. For which real numbers a can you solve the Cauchy problem

$$u_{x_1} + u_{x_2} = 0 \qquad u(x_1, ax_1) = f(x_1)$$

for any C^1 function f. Explain both in terms of the non-characteristic condition and by explicitly trying to invert the (restricted) flow map, interpreting your answer in relation to the line $x_2 = ax_1$ on which the initial data are given.

10. (a). Consider the equation

$$u_{x_1} + nu_{x_2} = f \tag{1.22}$$

where n is an integer and f is a smooth function which is 2π- periodic in both variables:

$$f(x_1 + 2\pi, x_2) = f(x_1, x_2 + 2\pi) = f(x_1, x_2).$$

Apply the method of characteristics to find out for which f there is a solution which is also 2π- periodic in both variables:

$$u(x_1 + 2\pi, x_2) = u(x_1, x_2 + 2\pi) = u(x_1, x_2).$$

(b) Consider the problem in part (a) using fourier series representations of f and u (both 2π- periodic in both variables) and compare your results.
(c)* What can you say about the case when n is replaced by an *irrational* number ω? [Hint: look in *http://en.wikipedia.org/wiki/Diophantine_approximation* for the definition of Liouville number, and use this as a condition to impose on ω and investigate the consequences for solving (1.22).]

11. Define, for non-negative s, the norm $\|\cdot\|_s$ on the space of smooth 2π-periodic function of x by

$$\|f\|_s^2 \equiv \sum_{m\in\mathbb{Z}} (1 + m^2)^s |\hat{f}(m)|^2$$

where $\hat{f}(n)$ are the fourier coefficients of f. (This is called the Sobolev H^s norm).
(i) What are these norms if $s = 0$? Write down a formula for these norms for $s = 0, 1, 2\ldots$ in terms of $f(x)$ and its derivatives directly. (Hint Parseval).
(ii) If $u(t, x)$ is the solution for the heat equation with 2π-periodic boundary conditions, then for $t > 1$ and $s = 0, 1, 2, , ,$ find a number $C_s > 0$ such that

$$\|u(t, \cdot)\|_s \leq C_s \|u(0, \cdot)\|_0 \ .$$

(iii) Show that there exists a number $\gamma_1 > 0$ which does not depend on f so that $\max|f(x)| \leq \gamma_1 \|f\|_1$ for all smooth 2π-periodic f. For which $s > 0$ is it also true that there exists $\gamma_s > 0$ such that $\max|f(x)| \leq \gamma_s \|f\|_s$ for all smooth 2π-periodic functions f?
(iv) Generalize the inequality in the last sentence of (iii) to periodic functions $f = f(x_1, \ldots x_n)$ of n variables. Find a number $\sigma(n)$ such that the inequality holds if and only if $s > \sigma(n)$?