Partial Differential Equations Example sheet 3

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3 Elliptic equations

3.1 Introduction and Notation

The equation

$$-\Delta u + u = f \tag{3.1}$$

can be solved for u via the Fourier transform, if $f \in \mathcal{S}(\mathbb{R}^n)$. The solution is the inverse Fourier transform of

$$\hat{u}_f(\xi) = \frac{\hat{f}(\xi)}{1 + \|\xi\|^2}\,; \tag{3.2}$$

this formula defines a Schwartz function, and hence the solution $u = u_f \in \mathcal{S}$ also, and the mapping $f \mapsto u_f$ is continuous in the sense that if f_n is a sequence of Schwartz functions such that $\|f_n - f\|_{\alpha,\beta} \to 0$ for every Schwartz semi-norm $\|\cdot\|_{\alpha,\beta}$, then also $\|u_n - u\|_{\alpha,\beta} \to 0$ for every Schwartz semi-norm, where $u_n = u_{f_n}$, $u = u_f$.

In fact the formula above extends to define a distributional solution u_f for each tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$, i.e. for each $\phi \in \mathcal{S}(\mathbb{R}^n)$ there holds

$$\langle u_f\,, -\Delta\phi + \phi \rangle = \langle f\,, \phi \rangle\,.$$

Using the Fourier transform definition of the Sobolev space one can check that:

$$\|u_f\|^2_{H^{s+2}} = \int_{\mathbb{R}^n} (1 + \|\xi\|^2)^{s+2} |\hat{u}(\xi)|^2\, d\xi = \int_{\mathbb{R}^n} (1 + \|\xi\|^2)^{s} |\hat{f}(\xi)|^2\, d\xi = \|f\|^2_{H^s}\,.$$

Thus the solution operator

$$\begin{aligned} (-\Delta + 1)^{-1} : H^s &\to H^{s+2} \\ f &\mapsto u_f \end{aligned}$$

is bounded, indicating that the solution gains two derivatives, *as measured in* L^2, compared to the inhomogeneous term. This phenomenon goes under the name *elliptic regularity*, and generalizes to wide classes of elliptic equations, as does the *maximum principle* bound

$$\max_{x \in \mathbb{R}^n} |u_f(x)| \le \max_{x \in \mathbb{R}^n} |f(x)|\,, \tag{3.3}$$

which is valid for classical (e.g. Schwartz) solutions, and is an immediate consequence of the calculus necessary conditions for $u \in \mathcal{S}(\mathbb{R}^n)$ to attain a maximum/minimum at a point x_*:

$$\partial_j u(x_*) = 0\,, \quad \partial^2 u(x_*) \geq 0\,, \qquad \text{(minimum)}\,;$$
$$\partial_j u(x_*) = 0\,, \quad \partial^2 u(x_*) \leq 0\,, \qquad \text{(maximum)}\,.$$

The notation indicates definiteness of the symmetric matrices $\partial^2 u(x_*) = \partial^2_{jk} u(x_*)$. This definiteness implies that at a maximum $\Delta u(x_*) = Tr\,\partial^2 u(x_*) \leq 0$ and hence by (3.1) that $\max u = u(x_*) \leq f(x_*) \leq \max|f|$; a similar argument for the minimum completes the derivation of (3.3) for $\mathcal{S}(\mathbb{R}^n)$ solutions. It is clear from the proof just outlined that this result is generalizable, both to more general classical solutions and also to larger classes of equations.

It is the purpose of this chapter to explain the generalizations of the results just discuused from (3.1) to much larger classes of second order elliptic equations.

*Notation:*Let $B_R = \{w : |w| < R\}$ and $\overline{B_R} = \{w : |w| \leq R\}$ be the open and closed balls of radius R and more generally let $B_R(x) = \{w : |w - x| < R\}$ and $\overline{B_R(x)} = \{w : |w - x| \leq R\}$. We write $\partial B_R, \partial B_R(x)$ for the corresponding spheres, i.e. $\partial B_R(x) = \{w : |w - x| = R\}$ etc. In the following $\Omega \subset \mathbb{R}^n$ is always open and bounded unless otherwise stated, $\overline{\Omega}$ is its closure and $\partial\Omega$ is its boundary (always assumed smooth).

3.2 Existence of solutions

In this section it is explained how to formulate and solve elliptic boundary value problems via the Lax-Milgram lemma, starting with the case of periodic boundary conditions.

Definition 3.2.1 *A weak solution of $Pu = f \in L^2([-\pi,\pi]^n)$, with P the operator given by*

$$Pu = -\sum_{jk} \partial_j(a_{jk}\partial_k u) + cu\,, \tag{3.4}$$

with smooth periodic coefficents $a_{jk} = a_{kj} \in C^\infty([-\pi,\pi]^n)$ and $c \in C^\infty([-\pi,\pi]^n)$, is a function $u \in H^1_{per}([-\pi,\pi]^n)$ with the property that

$$\int \sum_{jk} a_{jk}\,\partial_j u \partial_k v + cuv\,dx = \int fv\,dx \tag{3.5}$$

for all $v \in H^1_{per}([-\pi,\pi]^n)$.

Theorem 3.2.2 *Let P be as in* (3.4)*, and assume that the inequalities*

$$m\|\xi\|^2 \leq \sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k \leq M\|\xi\|^2 \tag{3.6}$$

and

$$c(x) \geq c_0 > 0 \tag{3.7}$$

hold everywhere, for some positive constants m, M, c_0 *and all* $\xi \in \mathbb{R}^n$. *Then given* $f \in L^2([-\pi,\pi]^n)$ *there exists a unique weak solution of* $Pu = f$ *in the sense of definition* (3.2.1).

Proof Define the bilinear form $B(u,v) = \int \sum_{jk} a_{jk}\, \partial_j u \partial_k v + cuv\, dx$ and observe that it obeys the continuity and coercivity conditions in the Lax-Milgram lemma in the Hilbert space H^1_{per}. In particular for continuity take $\|B\| = \|a\|_{L^\infty} + \|c\|_{L^\infty}$, where the norm for the matrix $a(x) = (a_{jk}(x))_{j,k=1}^n$ is the operator norm. For coercivity, notice that (3.6) and (3.7) imply

$$B(u,u) \ \geq\ \min\{m, c_0\}\, \|u\|^2_{H^1}\,. \tag{3.8}$$

The right hand side of (3.5) defines a bounded functional $L(v)$, since

$$|L(v)| = |\int f v dx| \leq \|f\|_{L^2}\|v\|_{L^2} \leq \|f\|_{L^2}\|v\|_{H^1}\,,$$

by the Hölder inequality, and so existence and uniqueness follows from the Lax-Milgram lemma. □

Definition 3.2.1 and theorem 3.2.2 have various generalizations: to obtain the correct definition of weak solution for a given elliptic boundary value problem the general idea is to start with a classical solution and multiply by a test function and integrate by parts using the boundary conditions in their classical format. This will lead to a weak formulation of both the equation and the boundary conditions. For example the weak formulation of the Dirichlet problem

$$Pu = f\,, \qquad u|_{\partial\Omega} = 0\,, \tag{3.9}$$

where

$$Pu = -\sum_{j,k=1}^n \partial_j(a_{jk}\partial_k u) + \sum_{j=1}^n b_j \partial_j u + cu \tag{3.10}$$

for continuous functions $a_{jk} = a_{kj}, b_j$ and c, is to find a function $u \in H^1_0(\Omega)$ such that

$$B(u,v) = L(v)\,, \quad \forall v \in H^1_0(\Omega)\,,$$

where $L(v) = \int f v\, dx$ (a bounded linear map/functional), and B is the bilinear form:

$$B(u,v) = \int \Big(\sum_{jk} a_{jk}\, \partial_j u \partial_k v + \sum b_j \partial_j u v \ + cuv\Big)\, dx\,.$$

By the Lax-Milgram lemma we have

Theorem 3.2.3 *In the situation just described, assume that* (3.6) *and* (3.7) *hold. Then if* $\|b\|_{L^\infty}$ *is sufficiently small, there exists a unique weak solution to* (3.9).

Proof The crucial point is that (3.8) changes into

$$B(u,u) \ \geq\ \min\{m, c_0\}\, \|u\|^2_{H^1} \ -\ \|b\|_{L^\infty}\|u\|^2_{H^1}\,, \tag{3.11}$$

where $\|b\|_{L^\infty} = \sup_x \|b(x)\| = \sup_x(\sum_{j=1}^n b_j(x)^2)^{\frac{1}{2}}$. The remainder of the proof is essentially as above. □

This solution has various regularity properties, the simplest of which is that if in addition $a_{jk} \in C^1(\Omega)$ then in any ball such that $\overline{B_r(y)} \subset \Omega$ there holds for some constant $C > 0$:

$$\|u\|_{H^2(B_r(y))} \le C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) , \qquad \text{(interior } H^2 \text{ regularity)} ,$$

and if in addition all the coefficients are smooth then we have, for arbitrary $s \in \mathbb{N}$ and some $C_s > 0$:

$$\|u\|_{H^{s+2}(B_r(y))} \le C_s(\|f\|_{H^s(\Omega)} + \|u\|_{L^2(\Omega)}) , \qquad \text{(higher interior regularity)} .$$

For the periodic case there is no boundary, and these results hold with the balls $B_r(y)$ replaced by the whole domain of periodicity $[-\pi, \pi]^n$. For example, consider the Poisson equation

$$-\Delta u = f = \sum_{m\in\mathbb{Z}^n} \hat{f}(m)\, e^{im\cdot x} \tag{3.12}$$

with periodic boundary conditions.

Theorem 3.2.4 *(i) If $u \in C^2_{per}$ is a classical solution of* (3.12) *then necessarily* $\hat{f}(0) = 0$.

(ii) If $f \in L^2$ and $\hat{f}(0) = 0$, then there is a unique weak solution of (3.12) *in the Hilbert space*

$$H^1_{per,0} = \{u \in H^1_{per} : \hat{u}(0) = 0\}$$

given by

$$u(x) = \sum_{m\in\mathbb{Z}^n\backslash\{0\}} \frac{\hat{f}(m)}{\|m\|^2}\, e^{im\cdot x} . \tag{3.13}$$

Furthermore, this solution satisfies $\|u\|_{H^{s+2}} \le 2\|f\|_{H^s}$ whenever f also belongs to H^s_{per}.

Proof $H^1_{per,0} \subset H^1_{per}$ is a closed subspace, and is thus a Hilbert space using the same inner product as H^1_{per}. The fact that $\frac{1}{2} \le \frac{\|m\|^2}{1+\|m\|^2} \le 1$ for all $m \in \mathbb{Z}^n\backslash\{0\}$ implies that

$$\begin{aligned} B(u,v) = \int \nabla u \cdot \nabla v\, dx &= (2\pi)^n \sum_{m\in\mathbb{Z}^n\backslash\{0\}} \|m\|^2 \hat{u}(-m)\hat{v}(m) \\ &= (2\pi)^n \sum_{m\in\mathbb{Z}^n\backslash\{0\}} \|m\|^2 \overline{\hat{u}(m)}\hat{v}(m) \end{aligned}$$

satisfies the continuity and coercivity conditions in the Lax-Milgram lemma, applied in the Hilbert space $H^1_{per,0}$. A weak solution in this space means a function $u \in H^1_{per,0}$ such that $B(u,v) = \int f v dx$ for all $v \in H^1_{per,0}$; existence of a unique weak solution in this sense follows. It can be checked directly that this solution is given by (3.13). Using the

Fourier definition of the H^s norm, the same inequality immediately gives the regularity assertion:

$$\|u\|^2_{H^{s+2}} = \sum_{m\in\mathbb{Z}^n\setminus\{0\}} \frac{(1+\|m\|^2)^{s+2}|f(m)|^2}{\|m\|^4} \le 2^2\|f\|^2_{H^2},$$

as claimed. □

Remark 3.2.5 *The significance of the condition $\hat{f}(0)=0$ for weak solutions is this: if $u\in C^2_{per}$ is a weak solution of* (3.12) *and $\hat{f}(0)=0$ then u is in fact a classical solution (i.e. it satisfies* (3.12) *everywhere).*

For the case of a domain with boundary, as in theorem 3.2.3, to get regularity right up to the boundary it is necessary to assume that the boundary itself is smooth: in this case the interior regularity estimate for the weak solution of (3.9) can be improved to

$$\|u\|_{H^2(\Omega)} \le C'(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}), \qquad \text{(boundary } H^2 \text{ regularity)}.$$

3.3 Stability in Sobolev spaces

Weak solutions to elliptic boundary value problems obtained via the Lax-Milgram lemma inherit a stability (well-posedness) property in the space H^1. For example in the periodic case:

Theorem 3.3.1 *Let $a_{jk}=a_{kj}\in C^\infty([-\pi,\pi]^n)$ and $c\in C^\infty([-\pi,\pi]^n)$ be smooth periodic coefficents for the elliptic operator*

$$Pu = -\sum_{jk}\partial_j(a_{jk}\partial_k u) + cu$$

and assume (3.6) *and* (3.7) *hold for some poitive constants m, M, c_0. Assume $Pu=f$ with $f\in L^2([-\pi,\pi]^n)$, then there exists a number L such that then*

$$\|u\|_{H^1} \le L\|f(x)\|_{L^2}.$$

If $Pu_j=f_j$ are two such solutions then

$$\|u_1-u_2\|_{H^1} \le L\|f_1-f_2\|$$

(stability or well-posedness in H^1).

Proof This can be proved directly by integration by parts. □

Alternatively, this type of result is an immediate and general consequence of coercivity and the Lax-Milgram formulation. Indeed, assume that $B(u_j,v)=L_j(v)$ for $j=1,2$ with the bilinear form B continuous and coercive as in the Lax-Milgram lemma with coercivity constant γ, and with L_1, L_2 bounded linear functionals. Then subtracting the two equations, and choosing as test function $v=u_1-u_2$, we deduce that

$$\gamma\|u_1-u_2\|^2 \le B(u_1-u_2,u_1-u_2) = |(L_1-L_2)(u_1-u_2)| \le \|L_1-L_2\|\|u_1-u_2\|.$$

Here the norm on linear functionals $L : X \to \mathbb{R}$ on a Hilbert space X is the dual norm

$$\|L\| = \sup_{u\in X, u\neq 0} \frac{\|L_j u\|}{\|u\|}$$

This gives the general stability estimate

$$\|u_1 - u_2\| \leq \gamma^{-1} \|L_1 - L_2\| \tag{3.14}$$

for Lax-Milgram problems.

3.4 The maximum principle

In the previous two sections we devloped techniques based on the weak formulation, which involves integration by parts ("energy" methods). For this reason it was convenient to work with operators in the form (3.4), (3.10) in which the principal term is a divergence. In the present section this is no longer particularly convenient, so the divergence form for the pricipal term will be dropped, and variable coefficient operators of the form (3.15) and (3.16) will be considered. Throughout this section the coefficients $a_{jk}(x) = a_{kj}(x)$ are continuous and will be again assumed to satisfy the uniform ellipticity condition (3.6) for some positive constants m, M and all $\xi \in \mathbb{R}^n$.

Recall that $\Omega \subset \mathbb{R}^n$ is always open and bounded unless otherwise stated, $\overline{\Omega}$ is its closure and $\partial\Omega$ is its boundary (always assumed smooth). The proofs of the following results are all similar to the proof of the first, which is given. In all proofs we use the following fact from linear algebra. (Recall that a symmetric matrix A is non-negative if $\xi^T A \xi \geq 0 \,\forall\, \xi \in \mathbb{R}^n$.)

Lemma 3.4.1 *If A, B are real symmetric non-negative matrices. Then* $Tr(AB) = \sum_{jk} A_{jk} B_{jk} \geq 0$.

Theorem 3.4.2 (Weak maximum principle I) *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfy $Pu = 0$ where*

$$Pu = -\sum_{j,k=1}^{n} a_{jk} \partial_j \partial_k u + \sum_{j=1}^{n} b_j \partial_j u \tag{3.15}$$

is an elliptic operator with continuous coefficients and (3.6) *holds, then*

$$\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x) .$$

Proof Let $R > 0$ be chosen such that $mR > \|b\|_{L^\infty}$, and define $u^\epsilon = u + \epsilon e^{x_1 R}$ for $\epsilon > 0$. Then $Pu^\epsilon = (-a_{11}R^2 + b_1 R)\epsilon e^{x_1 R} < 0$ since $a_{11} \geq m$ everywhere inside Ω by assumption. Now for contradiction assume there exists an interior point x_* at which u^ϵ attains a maximum point. Then at this point $\partial_j u^\epsilon(x_*) = 0$ and $\partial^2_{jk} u^\epsilon(x_*) \leq 0$ (as a symmetric matrix) and hence lemma 3.4.1 implies that $Pu^\epsilon(x_*) \geq 0$, giving a contradiction. It follows that there can never be an interior maximum, i.e. $\max_{\overline{\Omega}} u^\epsilon = \max_{\partial\Omega} u^\epsilon$. Since this holds for all $\epsilon > 0$ the result follows by taking the limit $\epsilon \downarrow 0$. □

Theorem 3.4.3 (Weak maximum principle II) *Let* $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ *satisfy* $Pu = 0$ *where*

$$Pu = -\sum_{j,k=1}^{n} a_{jk}\partial_j\partial_k u + \sum_{j=1}^{n} b_j \partial_j u + cu \tag{3.16}$$

is an elliptic operator with continuous coefficients and (3.6) *holds and* $c \geq 0$ *everywhere, then* $\max_{x\in\overline{\Omega}} u(x) \leq \max_{x\in\partial\Omega} u^+(x)$ *where* $u^+ = \max\{u, 0\}$ *is the positive part of the function* u*.*

In these theorems the phrase *weak* maximum principle is in contrast to the *strong* maximum principle (proved for harmonic functions in the next section) which asserts that if a maximum is attained at an interior point the harmonic function is (locally) constant.

Corollary 3.4.4 *In the situation of theorem 3.4.3* $\max_{x\in\overline{\Omega}} |u(x)| = \max_{x\in\partial\Omega} |u(x)|$.

Theorem 3.4.5 (Maximum principle bound for inhomogeneous problems) *Let* $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ *satisfy* $Pu = f$ *with Dirichlet data* $u|_{\partial\Omega} = 0$*, where*

$$Pu = -\sum_{j,k=1}^{n} a_{jk}\partial_j\partial_k u + \sum_{j=1}^{n} b_j \partial_j u + cu \tag{3.17}$$

is an elliptic operator with continuous coefficients and (3.6) *holds and*

$$c(x) \geq c_0 > 0$$

everywhere, for some constant $c_0 > 0$*, and* $f \in C(\overline{\Omega})$*, then*

$$\max_{x\in\overline{\Omega}} u(x) \leq \frac{1}{c_0} \max_{x\in\overline{\Omega}} |f(x)| \,.$$

If $Pu_j = f_j$ *are two such solutions then* $\max |u_1 - u_2| \leq \max |f_1 - f_2|/c_0$ *(stability or well-posedness in uniform norm).*

3.5 Harmonic functions

Definition 3.5.1 *A function* $u \in C^2(\Omega)$ *which satisfies* $\Delta u(x) = 0$ *(resp.* $\Delta u(x) \geq 0$*, resp.* $\Delta u(x) \leq 0$*) for all* $x \in \Omega$*, for an open set* $\Omega \subset \mathbb{R}^n$*, is said to be harmonic (resp. subharmonic, resp. superharmonic) in* Ω*.*

Theorem 3.5.2 *Let* u *be harmonic in* $\Omega \subset \mathbb{R}^n$ *and assume* $\overline{B_R(x)} \subset \Omega$*. Then for* $0 < r \leq R$*:*

$$u(x) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(y)\, dy \,, \qquad \textit{(mean value property)}\,. \tag{3.18}$$

Proof This is a consequence of the Green identity

$$\int_{\rho<|w-x|<r} (v\Delta u - u\Delta v)\, dw = \int_{|w-x|=r} (v\partial_\nu u - u\partial_\nu v)\, d\Sigma - \int_{|w-x|=\rho} (v\partial_\nu u - u\partial_\nu v)\, d\Sigma\,,$$

(where $\partial_\nu = n\cdot\nabla$ just means the normal derivative on the boundary) with the choice of $v(w) = N(w-x)$, where N is the fundamental solution for Δ:

$$\begin{aligned} N(x) &= \frac{|x|^{2-n}}{(2-n)\omega_n}\,, && (n>2) \\ &= \frac{1}{2\pi}\ln|x|\,, && (n=2)\,. \end{aligned}$$

Here $\omega_n = \int_{|x|=1} d\Sigma(x) = 2\pi^{\frac{n}{2}}/\Gamma(n/2)$ is the area of the unit sphere in $\mathbb{R}^n$. Thus on $\partial B_r(x)$ we have $v = r^{2-n}/(2-n)\omega_n, n>2$ or $v = (\ln r)/(2\pi), n=2$ - in other words v is constant on $\partial B_r(x)$, which implies that

$$\int_{|w-x|=r} v\partial_\nu u\, d\Sigma = v(r)\int_{|w-x|\le r} \Delta u\, dx = 0$$

by the divergence theorem, and the harmonicity of u. Together with the corresponding formula for the normal derivative, $\partial_\nu v = r^{1-n}/\omega_n$ on $\partial B_r(x)$, this implies that

$$\lim_{\rho\to 0}\int_{|w-x|=\rho} v\partial_\nu u\, d\Sigma\ = 0\,,\ \text{and}\ \lim_{\rho\to 0}\int_{|w-x|=\rho} u\partial_\nu v\, d\Sigma\ = u(x)$$

where we have used also the continuity of u to take the latter limit:

$$\begin{aligned} \Big|u(x) - \int_{|w-x|=r} u\partial_\nu v\, d\Sigma\Big| &= \Big|\frac{1}{\omega_n r^{n-1}}\int_{|w-x|=\rho} \big(u(x)-u(w)\big)\, d\Sigma(w)\Big| \\ &\le \sup_{|w-x|=\rho} |u(w)-u(x)| \to 0 \end{aligned}$$

as $\rho\to 0$. Substituting these into the Green identity above in the limit $\rho\to 0$ gives (3.19).

Corollary 3.5.3 *If u is a C^2 harmonic function in an open set Ω then $u\in C^\infty(\Omega)$. In fact if u is any C^2 function in Ω for which the mean value property* (3.19) *holds whenever $\overline{B_r(x)}\subset\Omega$ then u is a smooth harmonic function.*

Corollary 3.5.4 (Strong maximum principle for harmonic functions) *Let $\Omega\subset\mathbb{R}^n$ be a connected open set and $u\in C(\overline{\Omega})$ harmonic in Ω with $M = \sup_{x\in\overline{\Omega}} u(x) < \infty$. Then either $u(x) < M$ for all $x\in\Omega$ or $u(x) = M$ for all $x\in\Omega$. (In words, a harmonic function cannot have an interior maximum unless it is constant on connected components).*

Corollary 3.5.5 *Let $\Omega \subset \mathbb{R}^n$ be open with bounded closure $\overline{\Omega}$, and let $u_j \in C(\overline{\Omega})$, $j = 1,2$ be two harmonic functions in Ω with boundary values $u_j|_{\partial\Omega} = f_j$. Then*

$$\sup_{x\in\Omega} |u_1(x) - u_2(x)| \leq \sup_{x\in\partial\Omega} |f_1(x) - f_2(x)| , \qquad \textit{(stability or well-posedness).}$$

In particular if $f_1 = f_2$ then $u_1 = u_2$.

Corollary 3.5.6 *A harmonic function $u \in C^2(\mathbb{R}^n)$ which is bounded is constant.*

Another consequence of the Green identity is the following. Let $N(x,y) = N(|x - y|)$ where N is the fundamental solution defined above.

Theorem 3.5.7 *Let u be harmonic in Ω with $\overline{\Omega}$ bounded and $u \in C^1(\overline{\Omega})$. Then*

$$u(x) = \int_{\partial\Omega} \Big[u(y)\partial_{\nu_y} N(x,y) - N(x,y)\partial_{\nu_y} u(y)\Big]\, d\Sigma(y) \, ,$$

where $\partial_{\nu_y} = n\cdot\nabla_y$ just means the normal derivative in y, while ∂_ν is the normal in x. In fact the same formula holds with $N(x,y)$ replaced by any function $G(x,y)$ such that $G(x,y) - N(x,y)$ is harmonic in $y \in \Omega$ and C^1 for $y \in \overline{\Omega}$ for each $x \in \Omega$.

It is known from above that u is determined by its boundary values - to determine a harmonic function u from $u|_{\partial\Omega}$ is the *Dirichlet problem.* (The corresponding problem of determining u from its normal derivative $\partial_\nu u|_{\partial\Omega}$ is called the *Neumann problem.* To get a formula for (or understand) the solution of these problems it is sufficient to get a formula for (or understand) the correponding Green function:

Definition 3.5.8 *(i) A function $G_D = G_D(x,y)$ defined on $G_D : \Omega\times\overline{\Omega} - \{x = y\} \to \mathbb{R}$ such that (a) $G_D(x,y) - N(|x-y|)$ is harmonic in $y \in \Omega$ and continuous for $y \in \overline{\Omega}$ for each x, and (b) $G_D(x,y) = 0$ for $y \in \partial\Omega$, is a Dirichlet Green function.*

(ii) A function $G_N = G_N(x,y)$ defined on $G_N : \Omega\times\overline{\Omega} - \{x = y\} \to \mathbb{R}$ such that (a) $G_N(x,y) - N(|x-y|)$ is harmonic in $y \in \Omega$ and continuous for $y \in \overline{\Omega}$ for each x, and (b) $\partial_{\nu_y} G_N(x,y) = 0$ for $y \in \partial\Omega$, is a Neumann Green function.

Given such functions we obtain representation formulas:

$$\Delta u = 0\,, \quad u|_{\partial\Omega} = f \implies u(x) = \int_{\partial\Omega} f(y)\partial_{\nu_y} G_D(x,y)\, d\Sigma(y)\,,$$

and

$$\Delta u = 0\,, \quad \partial_\nu u|_{\partial\Omega} = g \implies u(x) = -\int_{\partial\Omega} g(y) G_N(x,y)\, d\Sigma(y)\,,$$

for $f, g \in C(\partial\Omega)$.

The function $P(x,y) = \partial_{\nu_y} G_D(x,y)$, defined for $(x,y) \in \Omega\times\partial\Omega$ is called the Poisson kernel, and is known explicitly for certain simple domains. For example, for the unit ball $\Omega = B_1$, the Poisson kernel is $P(x,y) = (1 - \|x\|^2)/\omega_n\|x-y\|^n$ and the solution of the Dirichlet problem on the unit ball is

$$u(x) = \int_{\partial B_1} f(y)\, \frac{(1-\|x\|^2)}{\omega_n\|x-y\|^n}\, d\Sigma(y)\,.$$

The formula for the half-space $\Omega = \{(x,y) : x \in \mathbb{R}^n, y > 0\}$ can also be computed explicitly (exercise).

3.6 Worked problems

1. (i) Write down the fundamental solution of the operator $-\Delta$ on $\mathbb{R}^3$ and state precisely what this means.

   (ii) State and prove the mean value property for harmonic functions on $\mathbb{R}^3$.

   (iii) Let $u \in C^2(\mathbb{R}^3)$ be a harmonic function which satisfies $u(p) \geq 0$ at every point p in an open set $\Omega \subset \mathbb{R}^3$. Show that if $B(z,r) \subset B(w,R) \subset \Omega$, then

$$u(w) \geq (\frac{r}{R})^3 u(z).$$

   Assume that $B(x,4r) \subset \Omega$. Deduce, by choosing $R = 3r$ and w, z appropriately, that

$$\inf_{B(x,r)} u \geq 3^{-3} \sup_{B(x,r)} u.$$

   [In (iii) $B(z,\rho) = \{x \in \mathbb{R}^3 : \|x - z\| < \rho\}$ is the ball of radius $\rho > 0$ centered at $z \in \mathbb{R}^3$.]

Answer (i) the distribution $\mathbf{N} \in \mathcal{S}'(\mathbb{R}^3)$ defined by the integrable function $(4\pi|\mathbf{x}|)^{-1}$ is the fundamental solution, and the precise meaning is that

$$-\int_{\mathbb{R}^3} (4\pi|\mathbf{x}|)^{-1} \Delta\phi(\mathbf{x}) d^3x = \phi(0)$$

for every Schwarz function $\phi \in \mathcal{S}(\mathbb{R}^3)$.

(ii) *Let u be harmonic in $\Omega \subset \mathbb{R}^n$ and assume $\overline{B_R(x)} \subset \Omega$. Then for $0 < r \leq R$:*

$$u(x) \;=\; \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(y)\, dy\,, \qquad \textit{(mean value property)}\,. \tag{3.19}$$

This is a consequence of the Green identity

$$\int_{\rho<|w-x|<r} (v\Delta u - u\Delta v)\, dw = \int_{|w-x|=r} (v\partial_\nu u - u\partial_\nu v)\, d\Sigma - \int_{|w-x|=\rho} (v\partial_\nu u - u\partial_\nu v)\, d\Sigma\,,$$

(where $\partial_\nu = n \cdot \nabla$ just means the normal derivative on the boundary) with the choice of $v(w) = \mathbf{N}(w - x)$, where $\mathbf{N}$ is as in (i). On $\partial B_r(x)$ we have $v = \frac{1}{4\pi r}$ - in particular v is constant on the sphere $\partial B_r(x)$, which implies that

$$\int_{|w-x|=r} v\partial_\nu u\, d\Sigma = v(r) \int_{|w-x|\leq r} \Delta u\, dx = 0$$

by the divergence theorem, and the harmonicity of u. Together with the corresponding formula for the normal derivative, $\partial_\nu v = -\frac{1}{4\pi r^2}$ on $\partial B_r(x)$, we have:

$$\lim_{\rho\to 0} \int_{|w-x|=\rho} v\partial_\nu u\, d\Sigma \;=\; 0\,, \text{ and } \lim_{\rho\to 0} \int_{|w-x|=\rho} u\partial_\nu v\, d\Sigma \;=\; -u(x)$$

(having used also the continuity of u to take the latter limit.) Substituting these into the Green identity above in the limit $\rho \to 0$ gives (3.19).

(iii) To start with integrate (3.19) with respect to r to obtain:

$$u(x) = \frac{1}{|B_r|} \int_{B_r(x)} u(y)\, dy\,. \tag{3.20}$$

For the first bit observe that non-negativity of u implies that $\int_{B(w,R)} u \geq \int_{B(z,r)} u$ and then apply (3.20) to get:

$$|B_R| u(w) = \int_{B(w,R)} u \geq \int_{B(z,r)} u = |B_r| u(z)\,.$$

Dividing by $\frac{4\pi R^3}{3} = |B_R|$ gives $u(w) \geq (\frac{r}{R})^3 u(z)$. For the second part, consider any two points w, z in the ball $B(x,r)$. Then $\|w - z\| < 2r$, and therefore $B(z,r) \subset B(w,3r) \subset \Omega$ by the triangle inequality. It follows that $u(w) \geq 3^{-3} u(z)$ and since w, z are arbitrary in $B(x,r)$ that $\inf_{B(x,r)} u \geq 3^{-3} \sup_{B(x,r)} u$. (The result is called a Harnack inequality.)

2. In this question Ω is a bounded open set in $\mathbb{R}^n$ with smooth boundary, and ν is the outward pointing unit normal vector and $\partial_\nu = \nu \cdot \nabla$.

[a] (i) Let $u \in C^4(\overline{\Omega})$[1] solve

$$\Delta^2 u = f \text{ in } \Omega\,,$$
$$u = \partial_\nu u = 0 \text{ on } \partial\Omega\,,$$

for some continuous function f. Show that if $v \in C^4(\overline{\Omega})$ also satisfies $v = \partial_\nu v = 0$ on $\partial\Omega$ then

$$\int_\Omega \Delta u\, \Delta v\, dx = \int_\Omega f\, v\, dx\,.$$

Use this to formulate a notion of weak solution to the above boundary value problem in the space $H^2_\partial(\Omega) \subset H^2(\Omega)$ which is formed by taking the closure of $C_0^\infty(\Omega)$ in the Sobolev space:

$$H^2(\Omega) = \{u \in L^2(\Omega) : \|u\|^2_{H^2} = \sum_{\alpha:|\alpha|\leq 2} \|\partial^\alpha u\|^2_{L^2} < \infty\}\,.$$

[a](ii) State the Lax-Milgram lemma. Use it to prove that there exists a unique function u in the space $H^2_\partial(\Omega)$ which is a weak solution of the boundary value problem above for $f \in L^2(\Omega)$.

[*Hint: Use regularity of the solution of the Dirichlet problem for the Poisson equation.*]

[b] Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $u \in H^1(\Omega)$ and denote

$$\bar{u} = \int_\Omega u\, d^n x \Big/ \int_\Omega d^n x\,.$$

The following Poincaré-type inequality is known to hold

$$\|u - \bar{u}\|_{L^2} \leq C \|\nabla u\|_{L^2}\,,$$

where C only depends on Ω. Use the Lax-Milgram lemma and this Poincaré-type inequality to prove that the Neumann problem

$$\Delta u = f \text{ in } \Omega\,, \qquad \partial_\nu u = 0 \text{ on } \partial\Omega\,,$$

[1] This means all partial derivatives up to order 4 exist inside the open set Ω, and they have continuous extensions to the closure $\overline{\Omega}$.

has a unique weak solution in the space

$$H^1_-(\Omega) \,=\, H^1(\Omega) \cap \{u : \Omega \to \mathbb{R}\,;\, \bar{u} \,=\, 0\}\,,$$

for arbitrary $f \in L^2$ such that $\bar{f} = 0$. Show also that if this weak solution has regularity $u \in C^2(\overline{\Omega})$ then it is a classical solution of the Neumann problem if $\bar{f} = 0$.

Show also that if there exists a classical solution $u \in C^2(\overline{\Omega})$ to this Neumann problem then necessarily $\bar{f} = 0$.

Answer [a](i) For u, v as described the Green identity gives:

$$\begin{aligned}\int_\Omega vf\,dx \,=\, \int_\Omega v\Delta^2 u\,dx &= -\int_\Omega \nabla\Delta u\cdot\nabla v\,dx + \int_{\partial\Omega} v\nabla\Delta u\cdot\nu\,dS\\ &= \int_\Omega \Delta u\cdot\Delta v\,dx - \int_{\partial\Omega}\Delta u\nabla v\cdot\nu\,dS\,,\end{aligned}$$

which gives the result. Define the bilinear form

$$B : H^2_\partial(\Omega)\times H^2_\partial(\Omega) \to \mathbb{R}$$

by $B[u,v] := \int_\Omega \Delta u\Delta v\,dx$. Then call a weak solution of the problem a function $u \in H^2_\partial(\Omega)$ such that $B[u,v] = \int_\Omega vf\,dx$ for all functions $v \in H^2_\partial$.

[a](ii) We assume for this section H is a real Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. We let $\langle\cdot,\cdot\rangle$ denote the pairing of H with its dual space.
Lax-Milgram Lemma: Assume that

$$B : H \times H \to \mathbb{R}$$

is a bilinear mapping, for which there exists constants $\alpha, \beta > 0$ such that

$$|B[u,v]| \le \alpha\|u\|\|v\| \ (u, v \in H)$$

and

$$\beta\|u\|^2 \le B[u,u] \ (u \in H).$$

Finally, let $f : H \to \mathbb{R}$ be a bounded linear functional on H. Then there exists a unique element $u \in H$ such that

$$B[u,v] = \langle f, v\rangle$$

for all $v \in H$.

To apply this lemma to [a](i) first notice that $|B[u,v]| \le \|u\|_{H^2_\partial(\Omega)}\|v\|_{H^2_\partial(\Omega)}$ trivially, so it is a matter to check the second (coercivity) condition. Considering the hint, regularity of the Dirichlet problem for the Poisson equation

$$\begin{aligned}\Delta u &= f \text{ in } \Omega\\ u &= 0 \text{ on } \partial\Omega\end{aligned}$$

with $f \in L^2(\Omega)$ asserts that the unique weak solution $u \in H_0^1(\Omega)$ is actually in L^2 and verifies:

$$\|u\|^2_{H^2(\Omega)} \leq K\|f\|^2_{L^2(\Omega)} = K\|\Delta u\|^2_{L^2(\Omega)}\,.$$

Apply this to our problem: clearly any $u \in H^2_\partial$ also lies in H^1_0, and so:

$$\frac{1}{K}\|u\|^2_{H^2_\partial(\Omega)} \leq B[u,u].$$

Therefore by Lax-Milgram there exists a unique $u \in H^2_\partial(\Omega)$ such that

$$B[u,v] = \int_\Omega fv\,dx \tag{3.21}$$

for all $v \in H^2_\partial(\Omega)$ i.e. u is a weak solution.

[b] Define

$$B : H^1_-(\Omega) \times H^1_-(\Omega) \to \mathbb{R}$$

by $B[u,v] := \int_\Omega \nabla u \nabla v\,dx$.

As in a) $|B[u,v]| \leq \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$. Moreover, by the Poincare inequality with $\bar{u} = 0$, we have for $u \in H^1_-(\Omega)$:

$$\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \leq (C^2+1)B[u,u].$$

Also, since $\bar{f} = 0$:

$$\Big|\int fu dx\Big| \;=\; \Big|\int f(u-\bar{u})dx\Big| \;\leq\; C\|f\|_{L^2}\|\nabla u\|_{L^2}\,.$$

Thus the functional $v \mapsto \int_\Omega fv\,dx$ is bounded on $H^1_-(\Omega)$. Therefore by Lax-Milgram there exists a unique $u \in H^1_-(\Omega)$ such that

$$B[u,v] = -\int_\Omega fv\,dx$$

for all $v \in H^1_-(\Omega)$ i.e. u is a weak solution.

Now if $u \in C^2(\overline{\Omega})$ then f is also continuous; choosing as test function $\phi - \bar{\phi}$ for arbitrary $\phi \in C^1(\overline{\Omega})$, and integrating by parts, we obtain

$$\int_{\partial\Omega} \partial_\nu u(\phi - \bar{\phi})\,dS \;-\; \int_\Omega \Delta u\,(\phi-\bar{\phi})\,dx \;=\; -\int_\Omega f\,(\phi-\bar{\phi})\,dx\,.$$

Now the divergence theorem gives $\int_{\partial\Omega} \partial_\nu u\,dS \;=\; \int_\Omega \Delta u\,dx$, so that the terms with $\bar{\phi}$ on the left cancel, leaving:

$$\int_{\partial\Omega} \partial_\nu u\,\phi\,dS \;-\; \int_\Omega \Delta u\,\phi\,dx \;=\; -\int_\Omega f\,(\phi-\bar{\phi})\,dx\,.$$

If $\bar{f} = 0$ then $\int_\Omega f\,\bar{\phi}\,dx = 0$ and so we obtain

$$-\int_\Omega \Delta u\,\phi\,dx \;=\; -\int_\Omega f\,\phi\,dx\,, \quad \text{for all } \phi \in C_0^1(\Omega)$$

which implies that $\Delta u = f$ (under the assumption $\bar{f} = 0$).

For the last part we assume we have a solution classical solution u of the Neumann problem. Then we integrate the Poisson equation over Ω to obtain:

$$\int_\Omega f\,dx = \int_\Omega \Delta u\,dx = \int_{\partial\Omega} \nabla u \cdot \nu\,dx = 0$$

by the divergence theorem. Therefore $\bar{f} = 0$.

3.7 Example sheet 3

1. Recall that if $u \in C^2(\mathbb{R}^3)$ and $\Delta u \geq 0$ then u is called subharmonic. State and prove a mean value property for subharmonic functions. Also state the analogous result for superharmonic functions, i.e. those C^2 functions which satisfy $\Delta u \leq 0$.

2. Let $\phi \in C(\mathbb{R}^n)$ be absolutely integrable with $\int \phi(x)dx = 1$. Assume $f \in C(\mathbb{R}^n)$ is bounded with $\sup |f(x)| \leq M < \infty$. Define $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$ and show

$$\phi_\epsilon * f(x) - f(x) = \int \big(f(x - \epsilon w) - f(x)\big)\phi(w)dw$$

(where the integrals are over $\mathbb{R}^n$). Now deduce the *approximation lemma*:

$$\phi_\epsilon * f(x) \to f(x) \quad \text{as } \epsilon \to 0$$

and uniformly if f is uniformly continuous. (Hint: split up the w integral into an integral over the ball $B_R = \{w : |w| < R\}$ and its complement B_R^c for large R).*Prove that if $f \in L^p(\mathbb{R}^n), 1 \leq p < \infty$ then $\lim_{\epsilon \to 0} \|\phi_\epsilon * f(x) - f(x)\|_{L^p} = 0$. (Hint: use the Minkowski inequality).

*By computing the Fourier transform of the function $\gamma_{\epsilon,a}(\xi) = \exp[i\xi \cdot a - \epsilon\|\xi\|^2]$ deduce the Fourier inversion theorem from the identity $(\hat{u}, \gamma_{\epsilon,a})_{L^2} = (u, \hat{\gamma}_{\epsilon,a})_{L^2}$.

3. Starting with the mean value property for harmonic $u \in C^2(\mathbb{R}^3)$ deduce that if $\phi \in C_0^\infty(\mathbb{R}^3)$ has total integral $\int \phi(x)dx = 1$ and is radial $\phi(x) = \psi(|x|)\,, \psi \in C_0^\infty(\mathbb{R})$ then $u = \phi_\epsilon * u$ where $\phi_\epsilon(x) = \epsilon^{-3}\phi(x/\epsilon)$. Deduce that harmonic functions $u \in C^2(\mathbb{R}^3)$ are in fact C^∞. Also for $u \in C^2(\Omega)$ harmonic in an open set $\Omega \in \mathbb{R}^3$ deduce that u is smooth in the interior of Ω (interior regularity). *Prove that if ϕ is a continuous function on $\mathbb{R}^n$ which satisfies the mean value property, then it is a smooth harmonic function.

4. If $u_1, u_2 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ are harmonic in Ω and agree on the boundary $\partial\Omega$, show in two different ways that $u_1 = u_2$ thoroughout Ω.

5. (i) Using the Green identities show that if f_1, f_2 both lie in $\mathcal{S}(\mathbb{R}^n)$ then the corresponding Schwartzian solutions u_1, u_2 of the equation $-\Delta u + u = f$, i.e.

$$(-\Delta + 1)u_1 = f_1 \qquad (-\Delta + 1)u_2 = f_2$$

satisfy

$$(*) \qquad \int |\nabla(u_1 - u_2)|^2 + |u_1 - u_2|^2 \leq c \int |f_1 - f_2|^2$$

where the integrals are over $\mathbb{R}^n$. (This is interpreted as implying the equation $-\Delta u + u = f$ is well-posed in the H^1 norm (or "energy" norm) defined by the left hand side of (*).) Now try to improve the result so that the H^2 norm:

$$\|u\|_{H^2}^2 \equiv \sum_{|\alpha| \leq 2} \int |\partial^\alpha u|^2 dx,$$

appears on the left. (The sum is over all multi-indices of order less than or equal to 2).

(ii) Prove a maximum principle bound for u in terms of f and deduce that $\sup_{\mathbb{R}^n} |u_1 - u_2| \leq \sup_{\mathbb{R}^n} |f_1 - f_2|$.

(iii) Verify that for $f \in \mathcal{S}'(\mathbb{R}^n)$ the formula for u_f in (3.2) remains valid, i.e. for each $\phi \in \mathcal{S}(\mathbb{R}^n)$ there holds

$$\langle\, u_f \,,\, -\Delta\phi + \phi \,\rangle \,=\, \langle\, f \,,\, \phi \,\rangle\,.$$

6. Prove a maximum principle for solutions of $-\Delta u + V(x)u = 0$ (on a bounded domain Ω with smooth boundary $\partial\Omega$) with $V > 0$: if $u|\partial\Omega = 0$ then $u \leq 0$ in Ω. (Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Hint: exclude the possibility of u having a strictly positive interior maximum).

   What does the maximum principle reduce to for one dimensional harmonic functions i.e. C^2 functions such that $u_{xx} = 0$?

7. Write down the definition of a weak H^1 solution for the equation $-\Delta u + u + V(x)u = f \in L^2(\mathbb{R}^3)$ on the domain $\mathbb{R}^3$. Assuming that V is real valued, continuous, bounded and $V(x) \geq 0$ for all x prove the existence and uniqueness of a weak solution. Formulate and prove well posedness (stability) in H^1 for this solution.

   How about the case that V is pure imaginary valued?

8. The Dirichlet problem in half-space:
   Let $H = \{(x, y) : x \in \mathbb{R}^n, y > 0\}$ be the half-space in $\mathbb{R}^{n+1}$. Consider the problem $\Delta_x u + \partial_y^2 u = 0$, where Δ_x is the Laplacian in the x variables only) and $u(x, 0) = f(x)$ with f a bounded and uniformly continuous function on $\mathbb{R}^n$. Define

$$u(x, y) = P_y * f(x) = \int_{\mathbb{R}^n} P_y(x - z) f(z) dz$$

   where $P_y(x) = \frac{2y}{c_n(|x|^2+y^2)^{\frac{n+1}{2}}}$ for $x \in \mathbb{R}^n$ and $y > 0$. Show that for an appropriate choice of c_n the function u is harmonic on H and is equal to f for $y = 0$. This is the *Poisson kernel* for half-space.
   (Hint: first differentiate carefully under the integral sign; then note that $P_y(x) = y^{-n} P_1(\frac{x}{y})$ where $P_1(x) = \frac{2}{c_n(1+|x|^2)^{\frac{n+1}{2}}}$, i.e. an approximation to the identity) and use the approximation lemma to obtain the boundary data).

   (ii) Assume that $n = 1$ and $f \in \mathcal{S}(\mathbb{R})$. Take the Fourier transform in the x variables to prove the same result.

9. Formulate and prove a maximum principle for a 2nd order elliptic equation $Pu = f$ in the case of periodic boundary conditions. Take $Pu = -\sum_{jk=1}^n a_{jk}\partial_{jk}^2 u + \sum_{j=1}^n b_j \partial_j u + cu$ with $a_{jk} = a_{kj}, b_j, c$ and f all continuous and 2π - periodic in each variable and assume u is a C^2 function with same periodicity. Assume uniform ellipticity (3.6) and $c(x) \geq c_0 > 0$ for all x. Formulate and prove well-posedness for $Pu = f$ in the uniform norm.

10. Formulate a notion of weak H^1 solution for the Sturm-Liouville problem $Pu = f$ on the unit interval $[0, 1]$ with inhomogeneous Neumann data: assume $Pu = -(pu')' + qu$ with $p \in C^1([0, 1])$ and $q \in C([0, 1])$ and assume there exist constants m, c_0 such that $p \geq m > 0$ and $q \geq c_0 > 0$ everywhere, and consider boundary conditions $u'(0) = \alpha$ and $u'(1) = \beta$. (Hint : start with a classical solution, multiply by a test function $v \in C^1([0, 1])$ and integrate by parts). Prove the existence and uniqueness of a weak H^1 solution for given $f \in L^2$. (*) Show that a weak solution $u \in C^2((0, 1))$ whose first derivative u' extends continuously up to the boundary of the interval, is in fact a classical solution which satisfies $u'(0) = \alpha$ and $u'(1) = \beta$.