Asymptotic Methods: Example Sheet 1

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The first two questions on background material are not for supervision.

1. Read Section II.1 in the notes and calculate $\Gamma(\frac{1}{2})$.

2. Calculate $\lim_{R\to+\infty}\int_0^R e^{ix^2}dx$, with justification for any change of variables used.

   Asymptotic expansions - basic properties

3. Suppose that the functions f and g have the asymptotic expansions

$$f(z) \sim \sum_{n=0}^{\infty} a_n\, z^{-n}, \qquad \text{and} \quad g(z) \sim \sum_{n=0}^{\infty} b_n\, z^{-n}$$

   as $z \longrightarrow \infty$. Show that

$$f(z)\, g(z) \sim \sum_{n=0}^{\infty} c_n\, z^{-n},$$

   as $z \longrightarrow \infty$, where $c_n = \sum_{k=0}^{n} a_{n-k}\, b_k$.

4. (a) Show that if a function admits as asymptotic expansion $f(x) \sim \sum_{n=0}^{\infty} a_n x^n$ as $x \to 0^+$, then the a_n are determined uniquely by f .
   (b) Consider the function

$$e(x) = \exp\big(-1/x\big)$$

   for $x > 0$. Show that, in an asymptotic expansion of the form

$$e(x) \sim \beta_0 + \beta_1 x + \beta_2 x^2 + \dots ,$$

   valid as $x \longrightarrow 0^+$, all the coefficients $\beta_0, \beta_1, \beta_2, \dots$ are zero. Deduce that in (a) the coefficients $\{a_n\}$ do not determine f uniquely.

5. (a) Taking δ to be a positive constant, show that as $z \longrightarrow \infty$ in the complex plane (not necessarily along a ray)

$$\cosh(z) \sim \frac{1}{2}\, e^{z}$$

   in the sector $\left(-\frac{\pi}{2}+\delta\right) < \arg z < \left(\frac{\pi}{2}-\delta\right)$ and

$$\cosh(z) \sim \frac{1}{2}\, e^{-z}$$

in the sector $\left(\frac{\pi}{2}+\delta\right) < \arg z < \left(\frac{3\pi}{2}-\delta\right)$. Is this still true if $\delta = 0$?

(b) Find asymptotic expansions for $\tanh z$ as $z \to \infty$ in the complex plane, stating in which sectors they hold and specifying the Stokes lines.

Asymptotic Expansions of Real Integrals.

6. (a) Show that the Stieltjes integral

$$F(x) \quad = \quad \int_0^\infty \frac{\rho(t)}{1+xt}\,dt$$

admits the asymptotic expansion $F(x) \sim \sum (-1)^n a_n x^n$, $(x \to 0^+)$, where $a_n = \int t^n \rho(t)\,dt$, under the assumption that the continuous function ρ satisfies $\rho(t) \le Ce^{-\epsilon t}$ for some positive C, ϵ and all $t \ge 0$. Deduce that $F(x) = \int_0^\infty \frac{xe^{-t}}{1+xt}\,dt$ admits the expansion $F(x) \sim \sum_{n=0}^\infty\ (-1)^n\ n!\ x^{n+1}$ as $x \to 0^+$. Show similarly that

$$G(x) = \int_0^\infty \frac{e^{-t}}{\left(1+xt\right)^2}\,dt \sim \sum_{n=0}^\infty\ (-1)^n\ (n+1)!\ x^n\,, \quad (x \to 0^+)\,.$$

(b) Differentiating through the integral show that $F' = G$ and comment on the relation between the two asymptotic series you just obtained. Give an example of a smooth function $H : (0,\infty) \to (0,\infty)$ with the property that H admits an asymptotic expansion $\sum \alpha_n x^n$ as $x \to 0^+$, but term-by-term differentation does not give an asymptotic expansion for H'. Show however, that if in this situation H' is continuous on $[0,\infty)$ and admits an asymptotic expansion $\sum \beta_n x^n$ as $x \to 0^+$, then necessarily this expansion is given by term-by-term differentiation, i.e. $\beta_n = (n+1)\alpha_{n+1}$.

(c) For a given small positive value of x, find the value(s) of n giving the term(s) of smallest magnitude in the asymptotic expansion for G. Hence, use optimal truncation to obtain an estimate of the 'exact' value $G(0.1) = 0.84366660602...$ [By convention *optimal truncation* of an asymptotic expansion means keeping all terms in the expansion up to the one BEFORE the smallest.]

(d) For the case $\rho(t) = e^{-t}$ recall from lectures that $F(x) = \sum_{n=0}^N (-1)^n n! x^n + \mathrm{Err}_N$ with error bound $|\mathrm{Err}_N| \le (N+1)! x^{N+1}$. Using this to define optimal truncation by $N+1 = [x^{-1}]$, the integer part of x^{-1}, use Stirling's formula to show that the resulting "optimal error bound" is $O\big([x^{-1}]^{\frac{1}{2}}\exp(-[x^{-1}])\big) = o(x^M)$, as $x \to 0^+$ for every positive integer M.

7. (a) Use integration by parts to find an asymptotic expansion, valid as $x \longrightarrow \infty$, for the **exponential integral**

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt \sim e^{-x}\left(b_1 x^{-1} + b_2 x^{-2} + b_3 x^{-3} + \ \ldots\right),$$

for suitable constants $b_1, b_2, b_3, \ \ldots$. Show that the remainder is $O\big(e^{-x}\,x^{-N-1}\big)$ as $x \longrightarrow \infty$, for suitable N.

(b) Check your answer by making the substitution $t = x\,(1+s)$ in the integral and applying Watson's Lemma.

(c) Obtain an asymptotic expansion of $E_1(x)$ as $x \to 0^+$ by considering $\frac{d}{dx}(E_1(x) + \ln x)$ and integrating.

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8. Find asymptotic expansions as $x \to \infty$ of

$$I_1(x) \;=\; \int_0^1 e^{-\,xt(1-t)^2}\,dt \qquad \text{and} \;\; I_2(x) \;=\; \int_0^\infty e^{-\,xt(1-t)^2}\,dt\,,$$

giving all terms up to and including $O\big(x^{-1}\big)$.

9. By means of Laplace's method, show that the first two terms in an asymptotic expansion as $x \longrightarrow \infty$ of

$$I(x) \quad = \quad \int_0^{\frac{\pi}{2}} \exp(-\,x\,t^3\cos t)\,dt$$

are given by

$$I(x) \quad \sim \quad \frac{1}{3\,x^{1/3}}\,\Gamma\!\left(\frac{1}{3}\right) \;+\; \left(\frac{1}{6}+\frac{8}{\pi^3}\right)\frac{1}{x} \;+\; \ldots\,.$$

*Find the next term in the expansion.

10. Show that

$$\int_0^{\frac{\pi^2}{4}} \exp\!\left[x\cos\sqrt{t}\,\right]dt \quad \sim \quad e^x\left(\frac{2}{x}+\frac{2}{3x^2}+\ldots\right)$$

as $x \to \infty$ and obtain the corresponding asymptotic expansion when the upper limit is replaced by $4\pi^2$.

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