Asymptotic Methods: Example Sheet 1

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The first two questions on background material are not for supervision.

1. Based on the definition of the Gamma Function in the first lecture, calculate $\Gamma(\frac{1}{2})$.

2. Calculate $\lim_{R\to+\infty}\int_0^R e^{\pm ix^2}dx$, with justification for any change of variables used.

Asymptotic expansions - basic properties

3. Determine whether the sequence of functions $\phi_n(x) = 1-\cosh x^n$, for $n = 1, 2, \cdots$, is an asymptotic sequence as $x \to 0$.

4. Suppose that the functions f and g have the asymptotic expansions

$$f(z) \sim \sum_{n=0}^{\infty} a_n\, z^{-n}\,, \qquad \text{and} \quad g(z) \sim \sum_{n=0}^{\infty} b_n\, z^{-n}$$

as $z \longrightarrow \infty$. Show that

$$f(z)\, g(z) \sim \sum_{n=0}^{\infty} c_n\, z^{-n}\,,$$

as $z \longrightarrow \infty$, where $c_n = \sum_{k=0}^{n} a_{n-k}\, b_k$.

5. Suppose that

$$f(z) \quad \sim \quad a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots\,,$$

as $z \longrightarrow \infty$, where $a_0 \neq 0$. Show by induction, or otherwise, that

$$\frac{1}{f(z)} \quad \sim \quad \frac{1}{a_0}\sum_{n=0}^{\infty}\frac{d_n}{z^n}\,,$$

as $z \to \infty$, where the sequence $\{d_n\}$ is defined by $d_0 = 1$ and, for $n \geq 1$,

$$\sum_{k=0}^{n} d_{n-k}\, a_k \quad = \quad 0\,.$$

6. (a) Show that if a function admits an asymptotic expansion $f(x) \sim \sum_{n=0}^{\infty} a_n x^n$ as $x \to 0^+$, then the a_n are determined uniquely by f.

   (b) Consider the function
$$e(x) = \exp\big(-1/x\big)$$
for $x > 0$. Show that, in an asymptotic expansion of the form
$$e(x) \sim \beta_0 + \beta_1 x + \beta_2 x^2 + \ldots ,$$
valid as $x \longrightarrow 0^+$, all the coefficients $\beta_0, \beta_1, \beta_2, \ldots$ are zero. Deduce that in (a) the coefficients $\{a_n\}$ do not determine f uniquely.

7. (a) Taking δ to be a positive constant, show that as $|z| \longrightarrow \infty$ in the complex plane (not necessarily along a ray)
$$\cosh(z) \sim \frac{1}{2} e^z$$
in the sector $\left(-\frac{\pi}{2} + \delta\right) < \arg z < \left(\frac{\pi}{2} - \delta\right)$ and
$$\cosh(z) \sim \frac{1}{2} e^{-z}$$
in the sector $\left(\frac{\pi}{2} + \delta\right) < \arg z < \left(\frac{3\pi}{2} - \delta\right)$. Is this still true if $\delta = 0$? (b) Find asymptotic expansions for $\tanh z$ as $|z| \to \infty$ in the complex plane, stating in which sectors they hold and specifying the Stokes lines. [Revisit this part by the end of the term when we talk about Stokes lines].

8. Suppose that for every $n = 0, 1, 2, \ldots$, $\phi_n(x)$ is an asymptotic sequence as $x \to x_0$, and that $\phi_n(x) \neq 0$ for $x \neq x_0$. Show that for any $N = 0, 1, 2, \cdots$, the set of functions $\{\phi_0(x), \phi_1(x), \cdots, \phi_N(x)\}$ are linearly independent near $x = x_0$.

9. Show that, if f is continuous and $f(x) = o\{\phi(x)\}$ as $x \to \infty$, where ϕ is a continuous, positive non-decreasing function of x, then
$$\int_a^x f(t)\, dt = o\{x\, \phi(x)\}$$
as $x \longrightarrow +\infty$.

Asymptotic Expansions of Real Integrals.

10. (a) Show that the Stieltjes integral
$$F(x) = \int_0^\infty \frac{\rho(t)}{1 + xt}\, dt$$
admits the asymptotic expansion $F(x) \sim \sum (-1)^n a_n x^n$, $(x \to 0^+)$, where $a_n = \int t^n \rho(t)\, dt$, under the assumption that the continuous function ρ satisfies $0 \leq \rho(t) \leq$

$Ce^{-\epsilon t}$ for some positive C, ϵ and all $t \geq 0$. Deduce that $F(x) = \int_0^\infty \frac{xe^{-t}}{1+xt}\,dt$ admits the expansion $F(x) \sim \sum_{n=0}^\infty (-1)^n\ n!\ x^{n+1}$ as $x \to 0^+$. Show similarly that

$$G(x) = \int_0^\infty \frac{e^{-t}}{\left(1+xt\right)^2}\,dt \sim \sum_{n=0}^\infty (-1)^n\ (n+1)!\ x^n\,, \quad (x \to 0^+)\,.$$

(b) Differentiating through the integral show that $F' = G$ and comment on the relation between the two asymptotic series you just obtained. Give an example of a smooth function $H : (0,\infty) \to (0,\infty)$ with the property that H admits an asymptotic expansion $\sum \alpha_n x^n$ as $x \to 0^+$, but term-by-term differentation does not give an asymptotic expansion for H'. Show however, that if in this situation H' is continuous on $[0,\infty)$ and admits an asymptotic expansion $\sum \beta_n x^n$ as $x \to 0^+$, then necessarily this expansion is given by term-by-term differentiation, i.e. $\beta_n = (n+1)\alpha_{n+1}$.

(c) For a given small positive value of x, find the value(s) of n giving the term(s) of smallest magnitude in the asymptotic expansion for G. Hence, use optimal truncation to obtain an estimate of the 'exact' value $G(0.1) = 0.84366660602...$ [By convention *optimal truncation* of an asymptotic expansion means keeping all terms in the expansion up to the one BEFORE the smallest.]

(d) For the case $\rho(t) = e^{-t}$ we find $F(x) = \sum_{n=0}^N (-1)^n n! x^n + \mathrm{Err}_N$ with error bound $|\mathrm{Err}_N| \leq (N+1)! x^{N+1}$ (see Section 1 in Chapling's Lecture Notes). Using this to define optimal truncation by $N+1 = [x^{-1}]$, the integer part of x^{-1}, use Stirling's formula to show that the resulting "optimal error bound" is $O\big([x^{-1}]^{\frac{1}{2}} \exp(-[x^{-1}])\big) = o(x^M)$, as $x \to 0^+$ for every positive integer M.

11. (a) Use integration by parts to find an asymptotic expansion, valid as $x \longrightarrow \infty$, for the **exponential integral**

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt \sim e^{-x}\left(b_1\,x^{-1} + b_2\,x^{-2} + b_3\,x^{-3} + \ \ldots\right),$$

for suitable constants $b_1, b_2, b_3, \ \ldots$. Show that the remainder is $O\big(e^{-x}\,x^{-N-1}\big)$ as $x \longrightarrow \infty$, for suitable N.

(b) Check your answer by making the substitution $t = x\,(1+s)$ in the integral and applying Watson's Lemma.

(c) Obtain an asymptotic expansion of $E_1(x)$ as $x \to 0^+$ by considering $\frac{d}{dx}(E_1(x) + \ln x)$ and integrating.

12. Find asymptotic expansions as $x \to \infty$ of

$$I_1(x) = \int_0^1 e^{-xt(1-t)^2}\,dt \qquad \text{and} \quad I_2(x) = \int_0^\infty e^{-xt(1-t)^2}\,dt\,,$$

giving all terms up to and including $O\big(x^{-1}\big)$.

13. By means of Laplace's method, show that the first two terms in an asymptotic expansion as $x \longrightarrow \infty$ of

$$I(x) = \int_0^{\frac{\pi}{2}} \exp(-x\,t^3 \cos t)\,dt$$

are given by

$$I(x) \quad \sim \quad \frac{1}{3\,x^{1/3}}\,\Gamma\!\left(\frac{1}{3}\right) \;+\; \left(\frac{1}{6} + \frac{8}{\pi^3}\right)\frac{1}{x} \;+\; \dots .$$

*Find the next term in the expansion.

14. Show that

$$\int_0^{\frac{\pi^2}{4}} \exp\!\left[x\cos\sqrt{t}\right]\,dt \quad \sim \quad e^x\left(\frac{2}{x} + \frac{2}{3x^2} + \dots\right)$$

as $x \to \infty$ and obtain the corresponding asymptotic expansion when the upper limit is replaced by $4\pi^2$.