Asymptotic Methods: Example Sheet 1

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The first two questions on background material are not for supervision.

1. Read Section II.1 in the notes and calculate \( \Gamma(\frac{1}{2}) \).

2. Calculate \( \lim_{R \to +\infty} \int_{0}^{R} e^{ix^2} \, dx \), with justification for any change of variables used.

Asymptotic expansions - basic properties

3. Suppose that the functions \( f \) and \( g \) have the asymptotic expansions

\[
f(z) \sim \sum_{n=0}^{\infty} a_n z^{-n}, \quad \text{and} \quad g(z) \sim \sum_{n=0}^{\infty} b_n z^{-n}
\]

as \( z \to \infty \). Show that

\[
f(z) \, g(z) \sim \sum_{n=0}^{\infty} c_n z^{-n},
\]

as \( z \to \infty \), where \( c_n = \sum_{k=0}^{n} a_{n-k} b_k \).

4. (a) Show that if a function admits as asymptotic expansion \( f(x) \sim \sum_{n=0}^{\infty} a_n x^n \) as \( x \to 0^+ \), then the \( a_n \) are determined uniquely by \( f \).

(b) Consider the function \( e(x) = \exp(-1/x) \) for \( x > 0 \). Show that, in an asymptotic expansion of the form

\[
e(x) \sim \beta_0 + \beta_1 x + \beta_2 x^2 + \ldots,
\]

valid as \( x \to 0^+ \), all the coefficients \( \beta_0, \beta_1, \beta_2, \ldots \) are zero. Deduce that in (a) the coefficients \( \{a_n\} \) do not determine \( f \) uniquely.

5. (a) Taking \( \delta \) to be a positive constant, show that as \( z \to \infty \) in the complex plane (not necessarily along a ray)

\[
cosh(z) \sim \frac{1}{2} e^{z}
\]

in the sector \( -\frac{\pi}{2} + \delta < \arg z < \left( \frac{\pi}{2} - \delta \right) \) and

\[
cosh(z) \sim \frac{1}{2} e^{-z}
\]
in the sector \( \left( \frac{\pi}{2} + \delta \right) < \arg z < \left( \frac{3\pi}{2} - \delta \right) \). Is this still true if \( \delta = 0 \)?

(b) Find asymptotic expansions for \( \tanh z \) as \( z \rightarrow \infty \) in the complex plane, stating in which sectors they hold and specifying the Stokes lines.

**Asymptotic Expansions of Real Integrals.**

6. (a) Show that the Stieltjes integral

\[
F(x) = \int_0^\infty \frac{\rho(t)}{1 + xt} \, dt
\]

admits the asymptotic expansion \( F(x) \sim \sum (-1)^n a_n x^n \), \( x \rightarrow 0^+ \), where \( a_n = \int t^n \rho(t) \, dt \), under the assumption that the continuous function \( \rho \) satisfies \( \rho(t) \leq Ce^{-\epsilon t} \) for some positive \( C \), \( \epsilon \) and all \( t \geq 0 \). Deduce that \( F(x) = \int_0^\infty \frac{xe^{-t}}{1+xt} \, dt \) admits the expansion \( F(x) \sim \sum_{n=0}^\infty (-1)^n n! x^{n+1} \) as \( x \rightarrow 0^+ \). Show similarly that

\[
G(x) = \int_0^\infty \frac{e^{-t}}{(1+xt)^2} \, dt \sim \sum_{n=0}^\infty (-1)^n (n+1)! x^n , \quad (x \rightarrow 0^+).
\]

(b) Differentiating through the integral show that \( F' = G \) and comment on the relation between the two asymptotic series you just obtained. Give an example of a smooth function \( H : (0, \infty) \rightarrow (0, \infty) \) with the property that \( H \) admits an asymptotic expansion \( \sum \alpha_n x^n \) as \( x \rightarrow 0^+ \), but term-by-term differentiation does not give an asymptotic expansion for \( H' \). Show however, that if in this situation \( H' \) is continuous on \([0, \infty)\) and admits an asymptotic expansion \( \sum \beta_n x^n \) as \( x \rightarrow 0^+ \), then necessarily this expansion is given by term-by-term differentiation, i.e. \( \beta_n = (n+1)\alpha_{n+1} \).

(c) For a given small positive value of \( x \), find the value(s) of \( n \) giving the term(s) of smallest magnitude in the asymptotic expansion for \( G \). Hence, use optimal truncation to obtain an estimate of the ‘exact’ value \( G(0.1) = 0.84366660602... \) [By convention **optimal truncation** of an asymptotic expansion means keeping all terms in the expansion up to the one BEFORE the smallest.]

(d) For the case \( \rho(t) = e^{-t} \) recall from lectures that \( F(x) = \sum_{n=0}^N (-1)^n n! x^n + \text{Err}_N \) with error bound \( |\text{Err}_N| \leq (N + 1)! x^{N+1} \). Using this to define optimal truncation by \( N + 1 = \lfloor x^{-1} \rfloor \), the integer part of \( x^{-1} \), use Stirling’s formula to show that the resulting “optimal error bound” is \( O(\lfloor x^{-1} \rfloor^\frac{1}{2} \exp( -\lfloor x^{-1} \rfloor )) = o(x^M) \), as \( x \rightarrow 0^+ \) for every positive integer \( M \).

7. (a) Use integration by parts to find an asymptotic expansion, valid as \( x \rightarrow \infty \), for the exponential integral

\[
E_1(x) = \int_x^\infty \frac{e^{-t}}{t} \, dt \sim e^{-x} \left( b_1 x^{-1} + b_2 x^{-2} + b_3 x^{-3} + \ldots \right),
\]

for suitable constants \( b_1, b_2, b_3, \ldots, \). Show that the remainder is \( O( e^{-x} x^{-N-1} ) \) as \( x \rightarrow \infty \), for suitable \( N \).

(b) Check your answer by making the substitution \( t = x(1+s) \) in the integral and applying Watson’s Lemma.

(c) Obtain an asymptotic expansion of \( E_1(x) \) as \( x \rightarrow 0^+ \) by considering \( \frac{d}{dx} (E_1(x) + \ln x) \) and integrating.
8. Find asymptotic expansions as $x \to \infty$ of

$$I_1(x) = \int_0^1 e^{-xt(1-t)^2} \, dt \quad \text{and} \quad I_2(x) = \int_0^\infty e^{-xt(1-t)^2} \, dt,$$

giving all terms up to and including $O(x^{-1})$.

9. By means of Laplace’s method, show that the first two terms in an asymptotic expansion as $x \to \infty$ of

$$I(x) = \int_0^{\pi/2} \exp(-xt^3 \cos t) \, dt$$

are given by

$$I(x) \sim \frac{1}{3x^{1/3}} \Gamma\left(\frac{1}{3}\right) + \left(\frac{1}{6} + \frac{8}{\pi^3}\right) \frac{1}{x} + \ldots.$$

Find the next term in the expansion.

10. Show that

$$\int_0^{\pi/4} \exp\left[x \cos \sqrt{t}\right] \, dt \sim e^x \left(\frac{2}{x} + \frac{2}{3x^2} + \ldots\right)$$

as $x \to \infty$ and obtain the corresponding asymptotic expansion when the upper limit is replaced by $4\pi^2$. 