Asymptotic Methods: Example Sheet 2

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Two more Integrals of Laplace type

1. Obtain the first correction to the Stirling formula in the asymptotic expansion of the Gamma function, i.e.,

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(1+\frac{1}{12x}+\dots\right), \quad (x \to +\infty),$$

2. Derive the expansion:

$$\int_0^{\frac{\pi}{2}} \exp[x\,(\sin t)^2]\,dt \sim \frac{e^x}{2}\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}-m)\,m!}\,\frac{\Gamma(\frac{1}{2}+m)}{x^{\frac{1}{2}+m}}, \quad (x \to +\infty).$$

By means of a change of variables $t \to \frac{\pi}{2} - t$ and the identity[1]

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

or otherwise, obtain the asymptotic expansion

$$\int_0^{\pi/2} e^{-x\sin^2 t}\,dt \sim \left(\frac{\pi}{4x}\right)^{1/2}\left\{1+\frac{1}{1!}\frac{1^2}{4x}+\frac{1}{2!}\frac{1^2.3^2}{(4x)^2}+\dots+\frac{1}{n!}\frac{1^2.3^2.\cdots.(2n-1)^2}{(4x)^n}+\cdots\right\}.$$

From this obtain an asymptotic expansion, as $x \longrightarrow \infty$, for the Bessel function defined by

$$I_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\theta}\,d\theta.$$

Oscillatory Integrals and Stationary Phase

3. (i) Assume $a < c < b$ and let $f(t)$ be a function which is smooth in $(a,c)\cup(c,b)$ but has a discontinuity at $t = c$. To be precise, assume that for all $n = 0, 1, 2\dots$ the limits of the n^{th} order derivative $f^{(n)}(t)$ as $t \to a+, c-, c+$ and $b-$ exist and are designated $f^{(n)}(a+)$, $f^{(n)}(c-)$, $f^{(n)}(c+)$ and $f^{(n)}(b-)$ respectively. Find the asymptotic expansion as $|\omega| \longrightarrow \infty$ of

$$I(\omega) = \int_a^b f(t)e^{i\omega t}\,dt.$$

[1] See the course on Further Complex Methods to derive this identity.

(ii) By taking the appropriate limits in part (a), find the asymptotic expansion as $|\omega| \longrightarrow \infty$ of $I(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\, dt$, where

$$f(t) \;=\; \begin{cases} -e^{t} & t < 0 \\ e^{-t} & t \geq 0\,. \end{cases}$$

Compare your result with the exact expression for $I(\omega)$.

4. Review Stokes' problem from the Stationary Phase notes. Obtain the leading asymptotic behaviour as $x \longrightarrow \infty$ of

$$\int_a^{\infty} f(t)\; \exp\big(i\,x\,(t^3 - t)\big)\, dt\,,$$

where f is smooth and $f \longrightarrow 0$ as $t \longrightarrow \pm\infty$ in the two cases: (i) $a = -\frac{1}{\sqrt{3}}$ and (ii) $a = 1$.

5. Show that, as $x \longrightarrow +\infty$,

$$\int_0^{\pi} \exp\big(i\,x\,(t - \sin t)\big)\, dt \quad \sim \quad e^{\frac{i\pi}{6}} \left(\frac{6}{x}\right)^{\frac{1}{3}} \Gamma\!\left(\frac{4}{3}\right).$$

How would this result differ if the lower limit of the integral were $-\pi$?

6. Find the leading term in the asymptotic approximations, valid as $x \longrightarrow \infty$, of

(a) $\displaystyle\int_0^1 \cos\big(x\,t^p\big)\, dt\,, \qquad \text{with } p > 1\,, \text{ real},$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} \left(1 - \left(\frac{2\theta}{\pi}\right)\right)^{\gamma} \cos\big(x \cos\theta\big)\, d\theta\,,$ for $\gamma = 0\,, \gamma = -\dfrac{1}{2}$ and $\gamma = -\dfrac{3}{4}$.

Method of Steepest Descent

7. The function $f(\theta)$ is defined for θ real and positive by

$$f(\theta) \;=\; \frac{1}{2\pi i} \int_{\gamma} \exp\left(\theta\left(t + \frac{1}{3}t^3\right)\right) dt\,,$$

where the path γ, in the complex plane, begins at ∞ in the sector $-\frac{\pi}{2} < \arg t < -\frac{\pi}{6}$ and ends at ∞ in the sector $\frac{\pi}{6} < \arg t < \frac{\pi}{2}$. Find the two saddle points and show that the two paths of steepest *descent* through these points are

$$x \;=\; +\left((2 + y)/3y\right)^{\frac{1}{2}} (y - 1)\,, \qquad y > 0$$

and

$$x \;=\; -\left((y - 2)/3y\right)^{\frac{1}{2}} (y + 1)\,, \qquad y < 0\,,$$

where $t = x + i\,y$. You should justify carefully your choice of signs for the square roots. Show that, as $\theta \longrightarrow \infty$,

$$f(\theta) = (\pi\,\theta)^{-\frac{1}{2}} \cos\left(\frac{2\,\theta}{3} - \frac{\pi}{4}\right) + O(\theta^{-1}).$$

8. Use the method of steepest descent to obtain the first two non-zero terms in the asymptotic approximation

$$\int_0^\infty \exp\left(i\,x\left(\frac{1}{3}\,t^3 + t\right)\right) dt \sim i\left(\frac{1}{x} + \frac{2}{x^3} + \ldots \frac{a_n}{x^n} + \ldots\right),$$

as $x \longrightarrow +\infty$. Check your answer by doing an integration by parts/stationary phase argument to the integral as it stands.

(∗) Find an expression for a_n for all n.

9. Let

$$h(t) = i\left(t + t^2\right).$$

Sketch the path through the point $t = 0$ for which $\mathrm{Im}\big(h(t)\big) = const.$ Sketch also the path through the point $t = 1$ for which $\mathrm{Im}\big(h(t)\big) = const.$

By integrating along these paths, show that, as $\lambda \longrightarrow \infty$,

$$\int_0^1 t^{-\frac{1}{2}} \exp\Big(i\,\lambda\left(t + t^2\right)\Big)\,dt \sim \frac{c_1}{\lambda^{\frac{1}{2}}} + c_2\,\frac{e^{2i\lambda}}{\lambda} + \ldots,$$

where the constants c_1 and c_2 are to be determined.

10. (∗) Apply the method of steepest descent to the integral

$$I(k) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\exp[k(z - 2z^{1/2})]}{z - c}\,dz,$$

for the case $k \to +\infty$. Here the path of integration is parallel to the imaginary axis, and $\gamma > 1$ is a real constant. The branch cut for $\sqrt{z}$ is the negative real axis. Show that the two paramterized curves $\tau \to z_\pm(\tau)$ given by

$$z_\pm(\tau) = 1 - \tau^2 \pm 2i\tau, \qquad 0 \le \tau < \infty,$$

are the steepest descent paths emanating from the saddle-point $z = 1$, and show that they form two halves of a parabola crossing the real axis at the saddle point; find the equation of the parabola in real form.

Investigate the asymptotics of $I(k)$ as $k \to +\infty$ in the following cases:

(i) c is real and < 1;
(ii) c is real, $1 < c < \gamma$;
(iii) $c = ib$ with b real and $b > 2$.

[You may find it convenient to use τ as a variable of integration.]