Asymptotic Methods: Example Sheet 2

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Two more integrals of Laplace type

1. Obtain the first correction to the Stirling formula in the asymptotic expansion of the Gamma function, i.e.,

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x} + \cdots\right), \quad \text{as } x \to \infty.$$

2. Derive the expansion

$$\int_0^{\pi/2} e^{x \sin^2 t}\, dt \sim \frac{e^x}{2} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(1/2)}{\Gamma(1/2-m)\, m!} \frac{\Gamma(1/2+m)}{x^{m+1/2}}, \quad \text{for } x \to \infty.$$

By means of a change of variables $t \mapsto \pi/2 - t$ and the identity[1]

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

or otherwise, show that

$$\int_0^{\pi/2} e^{-x \sin^2 t}\, dt \sim \left(\frac{\pi}{4x}\right)^{1/2} \left(1 + \frac{1}{1!}\frac{1^2}{4x} + \frac{1}{2!}\frac{1^2 \cdot 3^2}{(4x)^2} + \cdots + \frac{1}{n!}\frac{1^2 \cdot 3^2 \cdots (2n-1)^2}{(4x)^n} + \cdots\right),$$

as $x \to \infty$. Use this to find an asymptotic expansion for the Bessel function, defined by

$$I_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{x \cos\theta}\, d\theta,$$

in the same limit.

Oscillatory integrals and Stationary Phase

3. (a) Assume $a < c < b$ and let $f(t)$ be a function that is smooth in $(a, c) \cup (c, b)$ but has a discontinuity at $t = c$. To be precise, assume that, for all $n \in \{0, 1, 2, \dots\}$, the limits of the nth order derivative $f^{(n)}(t)$ as $t \to a^+$, c^-, c^+, and b^- exist and are designated $f^{(n)}(a^+)$, $f^{(n)}(c^-)$, $f^{(n)}(c^+)$, and $f^{(n)}(b^-)$ respectively. Find the asymptotic expansion, as $|\omega| \to \infty$, of

$$I(\omega) = \int_a^b f(t) e^{i\omega t}\, dt.$$

[1] See the course on Further Complex Methods to derive this identity.

(b) By taking the appropriate limits in part (a), find the asymptotic expansion of $I(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$, as $|\omega| \to \infty$, when

$$f(t) = \begin{cases} -e^{t} & t < 0 \\ e^{-t} & t \geq 0. \end{cases}$$

Compare your result with the exact expression for $I(\omega)$.

4. Review Stokes' problem from the lectures on stationary phase (also see p. 22 in Stuart's notes). Obtain the leading asymptotic behaviour of

$$\int_{a}^{\infty} f(t) \exp\big(ix(t^3 - t)\big)\, dt,$$

as $x \to \infty$, where f is smooth and $f \to 0$ as $t \to \pm\infty$, in the two cases: $a = -1/\sqrt{3}$, and $a = 1$.

5. Show that, as $x \to +\infty$,

$$\int_{0}^{\pi} \exp[ix(t - \sin t)]\, dt \sim e^{i\pi/6} \left(\frac{6}{x}\right)^{1/3} \Gamma\left(\frac{4}{3}\right).$$

How would this result differ if the lower limit of the integral were $-\pi$?

6. Taking $x \to \infty$, find the leading term in the asymptotic approximations of

(a) $\displaystyle\int_{0}^{1} \cos(xt^{p})\, dt,$ with $p > 1$ real,

(b) $\displaystyle\int_{0}^{\pi/2} \left(1 - \frac{2\theta}{\pi}\right)^{\gamma} \cos(x \cos\theta)\, d\theta,$ for $\gamma = 0$, $\gamma = -1/2$, and $\gamma = -3/4$.

Method of Steepest Descent

7. For θ real and positive, the function $f(\theta)$ is defined by

$$f(\theta) = \frac{1}{2\pi i} \int_{\mathcal{C}} \exp\left(\theta\left(t + \frac{1}{3}t^3\right)\right) dt,$$

where the path $\mathcal{C}$ lies in the complex plane, beginning at ∞ in the sector $-\pi/2 < \arg(t) < -\pi/6$ and ending at ∞ in the sector $\pi/6 < \arg(t) < \pi/2$. Find the two saddle points and show that the two paths of steepest *descent* through these points are

$$x = +\left(\frac{2 + y}{3y}\right)^{1/2} (y - 1), \quad y > 0$$

and

$$x = -\left(\frac{y - 2}{3y}\right)^{1/2} (y + 1), \quad y < 0,$$

where $t = x + iy$. You should justify carefully your choice of signs for the square roots. Show that, as $\theta \to \infty$,

$$f(\theta) = (\pi\theta)^{-1/2} \cos\left(\frac{2}{3}\theta - \frac{\pi}{4}\right) + O(\theta^{-1}).$$

8. Use the method of steepest descent to obtain the first two non-zero terms in the asymptotic approximation

$$\int_0^\infty \exp\left(ix\left(\frac{1}{3}t^3+t\right)\right)dt \sim i\left(\frac{1}{x}+\frac{2}{x^3}+\cdots+\frac{a_n}{x^n}+\cdots\right),$$

as $x \to +\infty$. Check your answer via an integration by parts argument.

$(*)$ Find an expression for a_n, for all n.

9. Let $h(t) = i(t+t^2)$. Sketch the path through the point $t = 0$ for which $\mathrm{Im}(h(t)) = \text{const}$. Also sketch the path through the point $t = 1$ for which $\mathrm{Im}(h(t)) = \text{const}$.

By integrating along these paths, show that

$$\int_0^1 t^{-1/2}\exp\big(i\lambda(t+t^2)\big)\,dt \sim \frac{c_1}{\lambda^{1/2}} + c_2\frac{e^{2i\lambda}}{\lambda} + \cdots,$$

as $\lambda \to \infty$, where the constants c_1 and c_2 are to be determined.

10. $(*)$ Apply the method of steepest descent to the integral

$$I(k) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\exp(k(z-2z^{1/2}))}{z-c}\,dz,$$

for the case $k \to +\infty$. Here the path of integration is parallel to the imaginary axis, and $\gamma > 1$ is a real constant. The branch cut for $z^{1/2}$ is the negative real axis. Show that the two parametrised curves $\tau \mapsto z_\pm(\tau)$, given by

$$z_\pm(\tau) = 1-\tau^2 \pm 2i\tau, \quad 0 \le \tau < \infty,$$

are the steepest descent paths emanating from the saddle-point $z = 1$. Verify that they form two halves of a parabola crossing the real axis at the saddle point and find the equation of the parabola in real form.

Investigate the asymptotics of $I(k)$ as $k \to +\infty$ in the following cases:

(a) c is real and < 1;

(b) c is real and $1 < c < \gamma$;

(c) $c = ib$, with b real and $b > 2$.

[You may find it convenient to use τ as a variable of integration.]