Asymptotic Methods: Example Sheet 2

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Two more Integrals of Laplace type

1. Obtain the first correction to the Stirling formula in the asymptotic expansion of the Gamma function, i.e.,

\[
\Gamma(x + 1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( 1 + \frac{1}{12x} + \ldots \right), \quad (x \to +\infty),
\]

2. Derive the expansion:

\[
\int_0^{\pi/2} \exp[x (\sin t)^2] \, dt \sim \frac{e^x}{2} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(1/2)}{\Gamma(1/2 - m) m!} \frac{\Gamma(1/2 + m)}{x^{1/2 + m}}, \quad (x \to +\infty).
\]

By means of a change of variables \( t \to \pi/2 - t \) and the identity

\[
\Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin(\pi z)},
\]

or otherwise, obtain the asymptotic expansion

\[
\int_0^{\pi/2} e^{-x\sin^2 t} \, dt \sim \left( \frac{\pi}{4x} \right)^{1/2} \left\{ 1 + \frac{1}{1! 4x} + \frac{1}{2! (4x)^2} + \ldots + \frac{1}{n!} \frac{1}{(4x)^n} \cdots (2n - 1)^2 + \ldots \right\}.
\]

From this obtain an asymptotic expansion, as \( x \to +\infty \), for the Bessel function defined by

\[
I_0(x) = \frac{1}{\pi} \int_0^{\pi} e^{x \cos \theta} \, d\theta.
\]

Oscillatory Integrals and Stationary Phase

3. (i) Assume \( a < c < b \) and let \( f(t) \) be a function which is smooth in \((a, c) \cup (c, b)\) but has a discontinuity at \( t = c \). To be precise, assume that for all \( n = 0, 1, 2 \ldots \) the limits of the \( n^{th} \) order derivative \( f^{(n)}(t) \) as \( t \to a+, c-, c+ \) and \( b- \) exist and are designated \( f^{(n)}(a+), f^{(n)}(c-), f^{(n)}(c+) \) and \( f^{(n)}(b-) \) respectively. Find the asymptotic expansion as \( |\omega| \to +\infty \) of

\[
I(\omega) = \int_a^b f(t) e^{i\omega t} \, dt.
\]

\(^1\)See the course on Further Complex Methods to derive this identity.
(ii) By taking the appropriate limits in part (a), find the asymptotic expansion as $|\omega| \to \infty$ of $I(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$, where

$$f(t) = \begin{cases} 
-e^t & t < 0 \\
-e^{-t} & t \geq 0 
\end{cases}$$

Compare your result with the exact expression for $I(\omega)$.

4. Review Stokes’ problem from the Stationary Phase notes. Obtain the leading asymptotic behaviour as $x \to \infty$ of

$$\int_{a}^{\infty} f(t) \exp\left(i x (t^3 - t)\right) dt,$$

where $f$ is smooth and $f \to 0$ as $t \to \pm \infty$ in the two cases: (i) $a = -\frac{1}{\sqrt{3}}$ and (ii) $a = 1$.

5. Show that, as $x \to +\infty$,

$$\int_{0}^{\pi} \exp\left(i x (t - \sin t)\right) dt \sim e^{i\pi} \left(\frac{6}{x}\right)^{\frac{1}{3}} \Gamma\left(\frac{4}{3}\right).$$

How would this result differ if the lower limit of the integral were $-\pi$?

6. Find the leading term in the asymptotic approximations, valid as $x \to \infty$, of

\begin{align*}
(a) & \quad \int_{0}^{1} \cos(x t^p) dt, \quad \text{with } p > 1, \ \text{real}, \\
(b) & \quad \int_{0}^{\frac{\pi}{2}} \left(1 - \left(\frac{2\theta}{\pi}\right)^\gamma\right) \cos(x \cos \theta) d\theta, \quad \text{for } \gamma = 0, \gamma = -\frac{1}{2} \text{ and } \gamma = -\frac{3}{4}.
\end{align*}

\underline{Method of Steepest Descent}

7. The function $f(\theta)$ is defined for $\theta$ real and positive by

$$f(\theta) = \frac{1}{2\pi i} \int_{\gamma} \exp\left(\theta \left(t + \frac{1}{3} t^3\right)\right) dt,$$

where the path $\gamma$ begins at $\infty$ in the sector $-\frac{\pi}{2} < \arg t < -\frac{\pi}{6}$ and ends at $\infty$ in the sector $\frac{\pi}{6} < \arg t < \frac{\pi}{2}$. Find the two saddle points and show that the two paths of steepest descent through these points are

$$x = + \left(\frac{2 + y}{3y}\right)^{\frac{1}{2}} (y - 1), \quad y > 0$$

and

$$x = - \left(\frac{y - 2}{3y}\right)^{\frac{1}{2}} (y + 1), \quad y < 0.$$
where \( t = x + iy \). You should justify carefully your choice of signs for the square roots. Show that, as \( \theta \to \infty \),

\[
f(\theta) = \left( \pi \theta \right)^{-1/2} \cos \left( \frac{2\theta}{3} - \frac{\pi}{4} \right) + O(\theta^{-1}).
\]

8. Use the method of steepest descent to obtain the first two non-zero terms in the asymptotic approximation

\[
\int_0^\infty \exp \left( i x \left( \frac{1}{3} t^3 + t \right) \right) \, dt \sim i \left( \frac{1}{x} + \frac{2}{x^3} + \ldots \frac{a_n}{x^n} + \ldots \right),
\]

as \( x \to +\infty \). Check your answer by doing an integration by parts/stationary phase argument to the integral as it stands.

(*) Find an expression for \( a_n \) for all \( n \).

9. Let

\[ h(t) = i \left( t + t^2 \right). \]

Sketch the path through the point \( t = 0 \) for which \( \text{Im}(h(t)) = \text{const.} \). Sketch also the path through the point \( t = 1 \) for which \( \text{Im}(h(t)) = \text{const.} \).

By integrating along these paths, show that, as \( \lambda \to \infty \),

\[
\int_0^1 t^{-1/2} \exp \left( i \lambda \left( t + t^2 \right) \right) \, dt \sim \frac{c_1}{\lambda^{1/2}} + c_2 \frac{e^{2i\lambda}}{\lambda} + \ldots,
\]

where the constants \( c_1 \) and \( c_2 \) are to be determined.

10. (*) Apply the method of steepest descent to the integral

\[ I(k) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\exp[k(z - 2z^{1/2})]}{z - c} \, dz, \]

for the case \( k \to +\infty \). Here the path of integration is parallel to the imaginary axis, and \( \gamma > 1 \) is a real constant. The branch cut for \( \sqrt{z} \) is the negative real axis. Show that the two parameterized curves \( \tau \to z_\pm(\tau) \) given by

\[ z_\pm(\tau) = 1 - \tau^2 \pm 2i\tau, \quad 0 \leq \tau < \infty, \]

are the steepest descent paths emanating from the saddle-point \( z = 1 \), and show that they form two halves of a parabola crossing the real axis at the saddle point; find the equation of the parabola in real form.

Investigate the asymptotics of \( I(k) \) as \( k \to +\infty \) in the following cases:

(i) \( c \) is real and \( < 1 \);

(ii) \( c \) is real, \( 1 < c < \gamma \);

(iii) \( c = ib \) with \( b \) real and \( b > 2 \).

[You may find it convenient to use \( \tau \) as a variable of integration.]