1. Obtain the first correction to the Stirling formula in the asymptotic expansion of the Gamma function, i.e.,
\[
\Gamma(x + 1) \sim \sqrt{2\pi x} \left( \frac{x}{e} \right)^x \left( 1 + \frac{1}{12x} + \ldots \right), \quad (x \to +\infty),
\]

2. In the notes “Asymptotic Analysis of Laplace Integrals” and in lectures we derived (essentially) the asymptotic expansion
\[
\int_0^{\pi/2} \exp[x(\sin t)^2] \, dt \sim \frac{e^x}{2} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} - m)} \frac{\Gamma(\frac{1}{2} + m)}{m!} x^{\frac{1}{2} + m}, \quad (x \to +\infty).
\]

By means of a change of variables and the identity\(^1\)
\[
\Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin(\pi z)},
\]
or otherwise, obtain the asymptotic expansion
\[
\int_0^{\pi/2} e^{-x \sin^2 t} \, dt \sim \left( \frac{\pi}{4x} \right)^{1/2} \left\{ 1 + \frac{1}{1! 4x} + \frac{1}{2! (4x)^2} + \ldots + \frac{1}{n!} \frac{1^{2}.3^{2} \ldots (2n - 1)^2}{(4x)^n} + \ldots \right\}.
\]

From this obtain an asymptotic expansion, as \(x \to \infty\), for the Bessel function defined by
\[
I_0(x) = \frac{1}{\pi} \int_0^{\pi} e^{x \cos \theta} \, d\theta.
\]

3. (i) Assume \(a < c < b\) and let \(f(t)\) be a function which is smooth in \((a, c) \cup (c, b)\) but has a discontinuity at \(t = c\). To be precise, assume that for all \(n = 0, 1, 2 \ldots\) the limits of the \(n^{th}\) order derivative \(f^{(n)}(t)\) as \(t \to a+, c-, c+\) and \(b-\) exist and are designated \(f^{(n)}(a+), f^{(n)}(c-), f^{(n)}(c+)\) and \(f^{(n)}(b-)\) respectively. Find the asymptotic expansion as \(|\omega| \to \infty\) of
\[
I(\omega) = \int_a^b f(t) e^{i\omega t} \, dt.
\]

\(^1\)See equation I.2 in the notes “Asymptotic Methods: Notation and Basic Definitions” and surrounding discussion for how to derive this identity.
(ii) By taking the appropriate limits in part (a), find the asymptotic expansion as \( |\omega| \to \infty \) of \( I(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} \, dt \), where
\[
f(t) = \begin{cases} 
-e^t & t < 0 \\
e^{-t} & t \geq 0.
\end{cases}
\]

Compare your result with the exact expression for \( I(\omega) \).

4. Review Stokes’ problem from section II of the Stationary Phase notes. Obtain the leading asymptotic behaviour as \( x \to \infty \) of
\[
\int_{a}^{\infty} f(t) \exp\left(i x (t^3 - t)\right) \, dt,
\]
where \( f \) is smooth and \( f \to 0 \) as \( t \to \pm \infty \) in the two cases: (i) \( a = -\frac{1}{\sqrt{3}} \) and (ii) \( a = 1 \).

5. Show that, as \( x \to +\infty \),
\[
\int_{0}^{\pi} \exp\left(i x (t - \sin t)\right) \, dt \sim e^n \left(\frac{6}{x}\right)^{\frac{1}{3}} \Gamma\left(\frac{4}{3}\right).
\]
How would this result differ if the lower limit of the integral were \(-\pi\)?

6. Find the leading term in the asymptotic approximations, valid as \( x \to \infty \), of
\[
(a) \int_{0}^{1} \cos\left(x t^p\right) \, dt, \quad \text{with } p > 1, \text{ real},
\]
\[
(b) \int_{0}^{\frac{\pi}{2}} \left(1 - \left(\frac{2\theta}{\pi}\right)\right)^{\gamma} \cos\left(x \cos \theta\right) \, d\theta, \quad \text{for } \gamma = 0, \gamma = -\frac{1}{2} \text{ and } \gamma = -\frac{3}{4}.
\]

7. The function \( f(\theta) \) is defined for \( \theta \) real and positive by
\[
f(\theta) = \frac{1}{2\pi i} \int_{\gamma} \exp\left(\theta \left(t + \frac{1}{3} t^3\right)\right) \, dt,
\]
where the path \( \gamma \) begins at \( \infty \) in the sector \(-\frac{\pi}{2} < \arg t < -\frac{\pi}{6}\) and ends at \( \infty \) in the sector \(\frac{\pi}{6} < \arg t < \frac{\pi}{2}\). Find the two saddle points and show that the two paths of steepest descent through these points are
\[
x = + \left(\frac{2 + y}{3 y}\right)^{\frac{1}{2}} (y - 1), \quad y > 0
\]
and
\[
x = - \left(\frac{y - 2}{3 y}\right)^{\frac{1}{2}} (y + 1), \quad y < 0,
\]
where \( t = x + iy \). You should justify carefully your choice of signs for the square roots. Show that, as \( \theta \to \infty \),
\[
f(\theta) = (\pi \theta)^{-\frac{1}{2}} \cos\left(\frac{2\theta}{3} - \frac{\pi}{4}\right) + O(\theta^{-1}).
\]
8. Use the method of steepest descents to obtain the first two non-zero terms in the asymptotic approximation
\[
\int_0^\infty \exp \left( i x \left( \frac{1}{3} t^3 + t \right) \right) dt \sim i \left( \frac{1}{x} + \frac{2}{x^3} + \ldots \frac{a_n}{x^n} + \ldots \right),
\]
as \( x \to +\infty \). Check your answer by doing an integration by parts/stationary phase argument to the integral as it stands.

(*) Find an expression for \( a_n \) for all \( n \).

9. Let
\[ h(t) = i \left( t + t^2 \right). \]
Sketch the path through the point \( t = 0 \) for which \( \text{Im}(h(t)) = \text{const} \). Sketch also the path through the point \( t = 1 \) for which \( \text{Im}(h(t)) = \text{const} \).

By integrating along these paths, show that, as \( \lambda \to \infty \),
\[
\int_0^1 t^{-\frac{3}{2}} \exp \left( i \lambda \left( t + t^2 \right) \right) dt \sim \frac{c_1}{\lambda^{\frac{3}{2}}} + c_2 \frac{e^{2i\lambda}}{\lambda} + \ldots,
\]
where the constants \( c_1 \) and \( c_2 \) are to be determined.

10. (*) Apply the method of steepest descents to the integral
\[
I(k) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\exp[k(z-2z^{1/2})]}{z-c} \, dz,
\]
for the case \( k \to +\infty \). Here the path of integration is parallel to the imaginary axis, and \( \gamma > 1 \) is a real constant. The branch cut for \( \sqrt{z} \) is the negative real axis. Show that the two parameterized curves \( \tau \to z_\pm(\tau) \) given by
\[
z_\pm(\tau) = 1 - \tau^2 \pm 2i\tau, \quad 0 \leq \tau < \infty,
\]
are the steepest descent paths emanating from the saddle-point \( z = 1 \), and show that they form two halves of a parabola crossing the real axis at the saddle point; find the equation of the parabola in real form.

Investigate the asymptotics of \( I(k) \) as \( k \to +\infty \) in the following cases:

(i) \( c \) is real and \( < 1 \);
(ii) \( c \) is real, \( 1 < c < \gamma \);
(iii) \( c = ib \) with \( b \) real and \( b > 2 \).

[You may find it convenient to use \( \tau \) as a variable of integration.]