Lecture Notes on Asymptotics

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1 Introduction

Several interesting functions are defined in terms of integrals.

Example

$$m! = \Gamma(m+1), \quad \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \text{Re } z > 0.$$

What is the behavior of $m!$ as $m \to \infty$? It will be shown later that

$$m! = \sqrt{2\pi m} \left(\frac{m}{e}\right)^m \left[1 + \frac{1}{12}\frac{1}{m} + \cdots\right].$$

Many physical phenomena are modeled by either ODEs or PDEs.

Example The modified Bessel function of order zero, denoted by $I_0(x)$, satisfies the ODE

$$\frac{d^2w}{dx^2} + \frac{1}{x}\frac{dw}{dx} - w = 0,$$

as well as appropriate boundary conditions. It has the following Taylor series expansion about $x = 0$:

$$I_0(x) = \sum_{m=0}^{\infty} \frac{1}{(m!)^2} \left(\frac{x}{2}\right)^{2m}. \tag{1.1}$$

It will be shown that the large x behavior of $I_0(x)$, is given by

$$I_0(x) = \frac{1}{\sqrt{2\pi x}} e^x \left[1 + \frac{1}{1!8x} + \frac{1 \cdot 3^2}{2!(8x)^2} + \frac{1 \cdot 3^2 \cdot 5^2}{3!(8x)^3} + \cdots\right]. \tag{1.2}$$

For fixed x, the Nth term of the above series grows like $N!$, thus it is *not* a convergent series. This is consistent with the fact that $x = \infty$ is an irregular singular point, thus there does *not* exist a Taylor series expansion near $x = \infty$. Actually, in the vicinity of $x = \infty$ there exists the so-called *Stokes phenomenon*:

suppose that $I_0(z)$ is indeed given by (1.2) near $z = \infty$ in some sector in the complex z-plane. Then, in the next sector, $I_0(z)$ is given by (1.2) times an appropriate constant.

An efficient way for obtaining (1.2) is to use that ODEs admit integral representations, see the next example.

Example The Hankel function of the first kind of order v, denoted by $H_v^{(1)}(z)$, satisfies the ODE

$$\frac{d^2w}{dz^2} + \frac{1}{z}\frac{dw}{dz} + \left(1 - \frac{v^2}{z^2}\right) w = 0,$$

as well as appropriate boundary conditions. It admits the integral representation

$$H_v^{(1)}(z) = \frac{1}{\pi}\int_L e^{iz\cos\zeta + iv(\zeta - \frac{\pi}{2})} d\zeta,$$

where the contour L is depicted in figure 1.

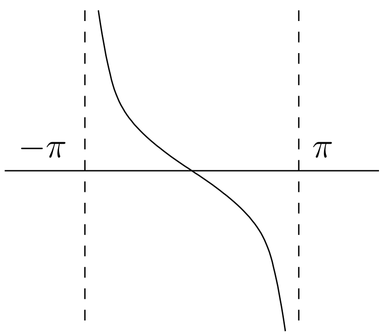

Figure 1:

Using the above integral representation it is straightforward to determine the behavior of $H_v^{(1)}(z)$ as $z \to \infty$.

Example The solution of the initial value problem of the zero-potential Schrödinger equation, formulated on the real line, i.e.

$$i\psi_t + \psi_{xx} = 0, \qquad x \in \mathbb{R}, \quad t > 0, \quad \psi(x,0) = \psi_0(x),$$

is given by

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\psi}_0(k) e^{ikx - ik^2 t} dk,$$

where

$$\hat{\psi}_0(k) = \int_{-\infty}^{\infty} \psi_0(x) e^{-ikx} dx.$$

The study of the large t behavior of $\psi(x,t)$ leads to the investigation of *"Fourier-type" integrals*. Similarly, the expressions obtained using the Laplace transform leads to the investigation of *"Laplace-type" integrals*. In this course we will study the methodologies needed for the study of the above integrals. Furthermore, we will study a certain generalization of these two methodologies, which is called the *steepest descent* method. In certain cases, it is easier to analyze the relevant ODE directly instead of analyzing the associated integral representation of the solution. The relevant method is called the *WKBJ method*. A typical example is given below.

Example The so-called Airy equation is given by

$$\frac{d^2w}{dx^2} + xw = 0.$$

If $x > 0$, we expect that w behaves like $e^{\pm ix}$, while if $x < 0$, we expect that w behaves like $e^{\pm x}$. Thus near $x = 0$, which is called a *turning point*, we expect that the behavior of the solution changes character. The WKBJ method will allow us to *connect* these two different behaviors.

In summary, in this course we will study: Laplace-type integrals, Fourier-type integrals, the steepest descent method, the WKBJ method, and Stokes phenomena. However, it must be noted that many integrals can be studied directly using *integration by parts*, which will be the first technique to be introduced. Before discussing this technique, we first present certain notations and basic notions.

Remark. There exist several important open problems in asymptotics, including the so-called Lindelof's hypothesis.

2 Basic Notions

1. $f(k) = O(g(k))$, $k \to k_0 \leftrightarrow |f(k)| \le M|g(k)|$, $M > 0$.
$f(k)$ is of order $g(k)$ as $k \to k_0$.

Example $2x + 4x^2$ is of order x as $x \to 0$, since $|2x + 4x^2| \le 3|x|$, $x \to 0$.

2. $f(k) \prec g(k), \quad k \to k_0 \leftrightarrow \lim_{k\to k_0} \left|\frac{f(k)}{g(k)}\right| = 0.$

Example $x^2 \prec x, \quad x \to 0$, since $\left|\frac{x^2}{x}\right| = |x| \to 0$ as $x \to 0$.

3. $f(k)$ is an approximation of $I(k)$ correct to order $\delta(k)$ as $k \to k_0 \leftrightarrow \underset{k\to k_0}{lim} \frac{f(k)-\delta(k)}{\delta(k)} = 0$.

Example $1+x$ is an approximation of $1/(1-x)$ correct to order x as $x \to 0$, since

$$\frac{\frac{1}{1-x} - x}{x} = \frac{1 + x + x^2 + \cdots - x}{x} = x \to 0 \quad \text{as} \quad x \to 0.$$

4. The ordered sequence of the functions $\{\delta_j(k)\}$, $j = 1, 2, \cdots$, is called an *asymptotic sequence* as $k \to k_0$, if

$$\delta_{j+1}(k) \prec \delta_j(k), \quad k \to k_0.$$

5. Let $I(k)$ be continuous. Let $\delta_j(k)$ be an asymptotic sequence as $k \to k_0$. The formal series

$$\sum_1^N a_j \delta_j(k), \quad a_j \text{ constants},$$

is called an *asymptotic expansions* of $I(k)$ as $k \to k_0$ valid to order $\delta_N(k)$ as $k \to k_0$, if

$$I(k) = \sum_{j=1}^m a_j \delta_j(k) + O(\delta_{m+1}(k)), \quad k \to k_0,$$

where $m = 1, 2, \cdots, N$.

6.

$$I(k) \sim \eta(k), \quad k \to k_0 \leftrightarrow \lim_{k \to k_0)} \frac{I(k)}{\eta(k)} = 1.$$

3 Integration By Parts

Example

$$I(k) = \int_0^\infty \frac{e^{-kt}}{1+t} dt, \quad k \to \infty.$$

The above integrand is exponentially small as $k \to \infty$, unless $t = 0$, thus we expect the main contribution comes from the neighborhood of $t = 0$. This contribution can be captured using integration by parts:

$$I(k) \sim \frac{e^{-kt}}{-k} \frac{1}{1+t}\bigg|_0^\infty - \frac{1}{k} \int_0^\infty \frac{e^{-kt}}{(1+t)^2} dt \sim \frac{1}{k}.$$

Letting $kt = \tau$ we find

$$I(k) = \varepsilon \int_0^\infty \frac{e^{-\tau}}{1+\varepsilon\tau} d\tau, \quad \varepsilon = \frac{1}{k}.$$

Thus, the following integral can also be evaluated using integration by parts:

Example

$$I(\varepsilon) = \int_0^\infty \frac{e^{-t}}{1+\varepsilon t}dt, \quad \varepsilon \to 0.$$

$$I(\varepsilon) = 1 - \varepsilon \int_0^\infty \frac{e^{-t}}{(1+\varepsilon t)^2}dt = \cdots$$

$$= 1 - \varepsilon + 2!\varepsilon^2 + \cdots (-1)^N N!\varepsilon^N + (-1)^{N+1}(N+1)\varepsilon^{N+1} \int_0^\infty \frac{e^{-t}dt}{(1+\varepsilon t)^{N+2}}.$$

The error can easily be bounded noting that since $\varepsilon > 0$ and $0 < t < \infty$, we find

$$1 + \varepsilon t \geq 1, \quad \text{or} \quad \frac{1}{1+\varepsilon t} \leq 1, \quad \text{or} \quad \int_0^\infty \frac{e^{-t}dt}{(1+\varepsilon t)^{N+2}} \leq \int_{-\infty}^\infty e^{-t} = 1.$$

Thus,

$$\left|(-1)^{N+1}(N+1)\varepsilon^{N+1} \int_0^\infty \frac{e^{-t}dt}{(1+\varepsilon t)^{N+2}}\right| \leq N!\varepsilon^N.$$

For ε fixed, $N!\varepsilon^N \to \infty$ as $N \to \infty$, but for N fixed, $N!\varepsilon^N$ vanishes as $\varepsilon \to 0$.

Example

$$I(k) = \int_k^\infty e^{-t^2}dt, \quad k \to \infty.$$

The largest value of the integrand occurs at $t = k$, thus we expect integration by parts will work:

$$I(k) = \left(-\frac{1}{2}\right)\int_k^\infty \left(-2te^{-t^2}\right)\frac{dt}{t} = \frac{1}{2}\frac{e^{-k^2}}{k} - \int_k^\infty e^{-t^2}\frac{dt}{t^2}$$
$$= \frac{1}{2}\frac{e^{-k^2}}{k} + O\left(\frac{e^{-k^2}}{k^3}\right), \quad k \to \infty.$$

Example

$$I(k) = \int_0^k t^{-\frac{1}{2}}e^{-t}dk, \quad k \to \infty.$$

$$I(k) = \int_0^\infty t^{-\frac{1}{2}}e^{-t}dt - \int_k^\infty t^{-\frac{1}{2}}e^{-t}dt.$$

The first integral can be computed exactly and the second integral can be evaluated using integration by parts. Thus,

$$I(k) = \sqrt{\pi} - \frac{e^{-k}}{\sqrt{k}} + O\left(\frac{e^{-k}}{k^{3/2}}\right), \quad k \to \infty.$$

Example

$$I(\varepsilon) = \int_{\varepsilon}^{\infty} e^{-t^2} dt, \quad \varepsilon \to 0.$$

$$I(\varepsilon) = \int_{0}^{\infty} e^{-t^2} dt - \int_{0}^{\infty} e^{-t^2} dt.$$

The first integral can be computed exactly and the second integral can be computed by employing a Taylor series:

$$\int_{0}^{\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2};$$

also

$$\int_{0}^{\infty} e^{-t^2} dt = \frac{1}{2} \int_{0}^{\infty} e^{-s} s^{-\frac{1}{2}} ds = \frac{1}{2} \Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}.$$

$$\int_{0}^{\varepsilon} e^{-t^2} dt = \int_{0}^{\varepsilon} \left(1 - t^2 + \frac{t^4}{2} + \cdots\right) = -\varepsilon + \frac{\varepsilon^3}{3} + O(\varepsilon^5), \quad \varepsilon \to 0.$$

Thus,

$$I(\varepsilon) = \frac{\sqrt{\pi}}{2} - \varepsilon + \frac{\varepsilon^3}{3} + O(\varepsilon^5), \quad \varepsilon \to 0.$$

4 Laplace Type Integrals

$$I(k) = \int_{a}^{b} f(t) e^{-k\phi(t)} dt, \quad b > a \geq 0, \quad k > 0, \tag{4.1}$$

where f and ϕ are real differentiable functions.

If $\phi(t) = t, a = 0, b = \infty$, then $I(k)$ is the Laplace transform of $f(t)$.

If $\phi(t)$ is monotonic in $[a, b]$ then the maximum of $e^{-k\phi}$ occurs at the boundary, thus we can use integration by parts.

Example

$$I(k) = \int_{0}^{\infty} (1+t)^{-1} e^{-k(t+2)^2} dt, \quad k \to \infty.$$

$$I(k) = \int_{0}^{\infty} \left(\frac{d}{dt} e^{-k(t+2)^2}\right) \frac{(1+t)^{-1}}{-2k(t+2)} dt$$

$$= e^{-k(t+2)^2} \frac{(1+t)^{-1}}{-2k(t+2)}\bigg|_{0}^{\infty} + \frac{1}{2k} \int_{0}^{\infty} e^{-k(t+2)^2} \left(\frac{(1+t)^{-1}}{t+2}\right)'$$

$$= \frac{e^{-4k}}{4k} + O\left(\frac{e^{-4k}}{k^2}\right).$$

Example

$$
\begin{aligned}
I(k) &= \int_1^2 e^{k\cosh t} dt, \quad k \to \infty \\
&= \int_1^2 \left(\frac{d}{dt} e^{k\cosh t}\right) \frac{1}{k\sinh t} \\
&\sim \frac{e^{k\cosh 2}}{k\sinh 2} - \frac{e^{k\cosh 1}}{k\sinh 1} \sim \frac{e^{k\cosh 2}}{k\sinh 2},
\end{aligned}
$$

since $e^{k\cosh 1}$ is exponentially small in comparison to $e^{k\cosh 2}$.

The above examples are generalized in the following lemma.

Lemma (Integration by parts). Consider

$$
I(k) = \int_a^b f(t) e^{-kt} dt, \quad b > a > 0, \quad k > 0. \tag{4.2}
$$

Assume that in $[a,b]$, $f(t)$ has $N+1$ continuous derivatives and that $f^{(N+2)}(t)$ is piecewise continuous. Furthermore if $b = \infty$ assume that $f(t) = O(e^{at})$, as $t \to \infty$. Then,

$$
I(k) \sim \sum_{m=0}^{N} \frac{e^{-ka}}{k^{m+1}} f^{(m)}(a), \quad t \to \infty.
$$

Proof See A+F, page 425.

Example

$$
I(\varepsilon) = \int_0^\infty (1+\varepsilon t)^{-1} e^{-t} dt, \quad \varepsilon \to 0,
$$

$$
\varepsilon t = \tau, \quad I(k) = k \int_0^\infty (1+\tau)^{-1} e^{-\tau k}, \quad k \to \infty, \quad k = 1/\varepsilon.
$$

Thus,

$$
f(\tau) = (1+\tau)^{-1}, \quad f^{(m)}(0) = (-1)^m m!
$$

Hence,

$$
I(\varepsilon) \sim \sum_{m=0}^{N} (-1)^m m! \varepsilon^m, \quad \varepsilon \to 0.
$$

Remark. If $\phi(t)$ is monotonic, then the change of variables $z = \phi(t)$ maps (4.1) to (4.2). However, if $\phi'(t)$ vanishes in $[a,b]$, then the above change of variables yields a function $f(t)$ with an integrable singularity, as illustrated by the example below.

Example

$$I(k) = \int_0^5 (t^2+2t)^{-\frac{1}{2}} e^{-kt} dt, \quad k \to \infty. \tag{4.3}$$

Since e^{-kt} attains its maximum value at $t=0$, we expect that the main contribution comes from the neighborhood of $t=0$. Thus we expect that

$$I(k) \sim \int_0^5 (2t)^{-\frac{1}{2}} e^{-kt} dt, \quad k \to \infty. \tag{4.4}$$

But the above integral has an integrable singularity at $t=0$, thus integration by parts fails. However, the integral (4.4) *can* be evaluated using Watson's lemma. Actually, in order to justify (4.4) we split (4.3) as follows:

$$I(k) = \int_0^R (t^2+2t)^{-\frac{1}{2}} e^{-kt} dt + \int_R^5 (t^2+2t)^{-\frac{1}{2}} e^{-kt} dt, \tag{4.5}$$

where R is sufficiently small so that we can expand $(t^2+2t)^{-\frac{1}{2}}$ near $t=0$, i.e. $R<2$. Denoting the integrals in (4.5) by I_1 and I_2 we find the following:

$$I_1 = \int_0^R (2t)^{-\frac{1}{2}} \left(1+\frac{t}{2}\right)^{-\frac{1}{2}} e^{-kt} dt = \int_0^R (2t)^{-\frac{1}{2}} \left(1-\frac{t}{4}+O(t^2)\right) e^{-kt} dt$$

$$= \int_0^\infty (2t)^{-\frac{1}{2}} \left(1-\frac{t}{4}+O(t^2)\right) e^{-kt} dt - \int_R^\infty (2t)^{-\frac{1}{2}} \left(1-\frac{t}{4}+O(t^2)\right) e^{-kt} dt.$$

The second integral above as well as the integral $I_2(k)$ can be evaluated using integration by parts. Thus we expect that

$$I(k) \sim \int_0^\infty (2t)^{-\frac{1}{2}} e^{-kt} dt \underset{kt=\tau}{=} \frac{1}{(2k)^{\frac{1}{2}}} \int_0^\infty \tau^{-\frac{1}{2}} e^{-\tau} d\tau = \frac{\Gamma(\frac{1}{2})}{\sqrt{2k}}.$$

The above is rigorously justified by Watson's lemma.

Watson's Lemma Consider

$$I(k) = \int_0^b f(t) e^{-kt} dt, \quad k > 0.$$

Assume that

(i)

$$f(t) \sim t^\alpha \sum_{m=0}^\infty a_m t^{\beta m}, \quad t \to 0^+, \quad \alpha > -1, \quad \beta > 0, \quad a_m \text{ constants.}$$

(ii) The integral converges throughout its range for all sufficiently large k.

Proof. See A+F, page 427.

Example

$$I(k) = \int_0^\infty (t^2+2t)^{-\frac{1}{2}} e^{-kt} dt, \quad k \to \infty.$$

$$f(t) = (2t)^{-\frac{1}{2}} \left(1+\frac{t}{2}\right)^{-\frac{1}{2}}.$$

But, the formula

$$(1+z)^\alpha = \sum_{m=0}^\infty c_m z^m, \quad c_m = \frac{\Gamma(\alpha+1)}{(m+1)!\Gamma(\alpha+1-m)},$$

implies

$$\left(1+\frac{t}{2}\right)^{-\frac{1}{2}} = \sum_{m=0}^\infty c_m \left(\frac{t}{2}\right)^m, \quad c_m = \frac{\Gamma(\frac{1}{2})}{(m+1)!\Gamma(\frac{1}{2}-m)}.$$

Thus,

$$f(t) = \frac{1}{\sqrt{2}} t^{-\frac{1}{2}} \sum_{m=0}^\infty t^m \frac{c_m}{2^m},$$

hence, $\alpha = -\frac{1}{2}$, $\beta = 1$, thus

$$I(k) \sim \sum_{m=0}^\infty a_m \frac{\Gamma(\frac{1}{2}+m)}{k^{\frac{1}{2}+m}}, \quad k \to \infty.$$

Lemma (Laplace's Method) Consider

$$I(k) = \int_a^b f(t) e^{-k\phi(t)} dt, \quad b > a \geq 0, \quad k > 0.$$

Assume:

(i) $\phi'(c) = 0$, $\phi''(c) > 0$, $a < c < b$.
(ii) $\phi'(t) \neq 0$ for every $t \in [a,b]$ except $t = c$.
(iii) $\phi \in c^4[a,b]$, $f \in c^2[a,b]$.
Then,

$$I(k) = e^{-k\phi(c)} f(c) \sqrt{\frac{2\pi}{k\phi''(c)}} + O\left(\frac{e^{-k\phi(c)}}{k^{3/2}}\right).$$

Furthermore, if c is an end point, then

$$I(k) = e^{-k\phi(c)} f(c) \sqrt{\frac{\pi}{2k\phi''(c)}} + O\left(\frac{e^{-k\phi(c)}}{k}\right).$$

Proof See A+F, page 431.

Example Consider

$$I(k) = \int_a^b f(t) e^{k\phi(t)} dt, \quad k \to \infty, \tag{4.6}$$

where $\phi(t)$ has a unique maximum at $t = 0$. Then, using the change of variables

$$\tau = (t-c)\sqrt{\frac{-k\phi''(c)}{2}},$$

we find

$$I(k) \sim f(c) e^{k\phi(c)} \sqrt{\frac{2\pi}{k|\phi''(c)|}}. \tag{4.7}$$

Example Consider the L_p norm of g, i.e.

$$||g||_p = \left(\int_a^b |g(t)|^p dt \right)^{\frac{1}{p}}.$$

Assume that $|g(t)| \in c^4$ and that it has a unique maximum at $t = c$. Then, using (4.7) with

$$\phi(t) = \ln|g(t)|, \quad f = 1, \quad \phi' = \frac{g'}{g}, \quad \phi'' = \frac{g''}{g} - \frac{g'^2}{g^2}, \quad \phi'(c) = 0,$$

we find

$$\int_a^b |g(t)|^p dt \sim \sqrt{\frac{2\pi}{p} \frac{|g(c)|}{|g''(c)|}} e^{p \ln|g(c)|} \left[1 + O\left(\frac{1}{p^{3/2}} \right) \right]$$
$$= A p^{-\frac{1}{2}} |g(c)|^p \left[1 + O\left(\frac{1}{p^{3/2}} \right) \right].$$

Hence,

$$||g||_p \sim A^{\frac{1}{p}} |g(c)| p^{-\frac{1}{2p}}.$$

Thus, using

$$p^{-\frac{1}{2p}} = e^{-\frac{1}{2p}\ln p} \sim 1 + O\left(\frac{\ln p}{p} \right), \quad A^{\frac{1}{p}} = e^{\frac{1}{p}\ln A} = 1 + O\left(\frac{1}{p} \right),$$

we find

$$||g||_p \sim |g(c)|.$$

Example

$$I(k)=\int_0^R \sin t e^{-k(\sinh t)^4}dt.$$

Thus,

$$I(k)\sim\int_0^R te^{-kt^4}dt=\frac{1}{\sqrt{k}}\int_0^{k^{\frac{1}{4}}R}\tau e^{-\tau^4}d\tau$$
$$=\frac{1}{2\sqrt{k}}\int_0^\infty e^{-x^2}dx=\frac{1}{4}\sqrt{\frac{\pi}{k}},\quad k\to\infty,$$

where $\tau^2=x$.

Moving Critical Points

$$I(k)=\int_0^\infty e^{-\frac{1}{t}}e^{-kt}dt,\quad k\to\infty.$$

e^{-kt} has a maximum at $t=0$, but $e^{-\frac{1}{t}}$ vanishes exponentially at $t=0$. Thus, we look at the maximum of $e^{-(kt+\frac{1}{t})}$. The minimum of $kt+\frac{1}{t}$ occurs at

$$k-\frac{1}{t^2}=0,\quad\text{or}\quad t=\frac{1}{\sqrt{k}},$$

thus the critical point is *not* fixed but it depends on k. In order to fix this critical point we let $t=s/\sqrt{k}$. Thus,

$$I(k)=\frac{1}{\sqrt{k}}\int_0^\infty e^{-\sqrt{k}\left(s+\frac{1}{s}\right)}ds.$$

Now,

$$\phi(s)=s+\frac{1}{s},\quad \phi'(s)=1-\frac{1}{s^2},$$

thus, $s=1$ is the critical point. Letting $s=1+\tau$, we find

$$I(k)=\frac{1}{\sqrt{k}}\int_{-1}^\infty e^{-\sqrt{k}(1+\tau+1-\tau+\tau^2+O(\tau^3))}d\tau$$

$$\sim\frac{e^{-2\sqrt{k}}}{\sqrt{k}}\int_{-1}^\infty e^{-\sqrt{k}\tau^2}d\tau\underset{k^{\frac{1}{4}}\tau=x}{=}\frac{e^{-2\sqrt{k}}}{k^{3/4}}\int_{-\infty}^\infty e^{-x^2}dx=\frac{\sqrt{\pi}e^{-2\sqrt{k}}}{k^{3/4}},\quad k\to\infty.$$

Example

$$\Gamma(k+1)=\int_0^\infty e^{-t}t^kdt=\int_0^\infty e^{-t+k\ln t}dt.$$

$-1+\frac{k}{t}=0$, or $k=t$, thus $t=ks$. Hence

$$\Gamma(k+1)=\int_0^\infty e^{-ks}k^ks^kds=k^{k+1}\int_0^\infty e^{-ks+k\ln s}ds.$$

Using $s = 1 + \tau$, we find

$$\Gamma(k+1) = k^{k+1} \int_{-1}^{\infty} e^{-k(1+\tau)+k(\tau - \frac{\tau^2}{2} + O(\tau^3))} dt \sim e^{-k} k^{k+1} \int_{-1}^{\infty} e^{-\frac{k\tau^2}{2}} d\tau =$$

$$\sqrt{2\pi k} \left(\frac{k}{e}\right)^k, \quad k \to \infty,$$

where

$$\sqrt{\frac{k}{2}}\tau = x.$$

5 Fourier Type Integrals

Consider

$$I(k) = \int_a^b f(t) e^{ik\phi(t)} dt, \quad b > a \geq 0, \quad k > 0 \qquad (5.1),$$

where $f(t)$ and $\phi(t)$ are real valued functions. If $f \in L_1$, then under minimal conditions on $\phi(t)$ it can be shown that $I(k) \to 0$ as $k \to \infty$. This is the consequence of the Riemann Lebesgue Lemma, which is given in the Appendix.

If $\phi(t)$ is a monotonic function in $[a, b]$, then using a simple change of variables, $I(k)$ can be evaluated using the following result:

Lemma (Integration by Parts)
Consider

$$I(k) = \int_a^b f(t) e^{ikt} dt, \quad b > a \geq 0, \quad b < \infty, \quad k > 0.$$

Assume that in $[a, b]$, $f(t)$ has $N + 1$ continuous derivatives and that $f^{(N+2)}$ is piecewise continuous. Then,

$$I(k) \sim \sum_{m=0}^{N} \frac{(-1)^m}{(ik)^{m+1}} \left[f^{(m)}(b) e^{ikb} - f^{(m)}(a) e^{ika} \right], \quad k \to \infty.$$

Proof Similar with the analogous proof for Laplace type integrals. **QED**

Recall that for Laplace type integrals, if $\phi(t)$ vanishes in $[a, b]$, then after a simple transformation we obtain a function $f(t)$ which has an integrable singularity. This case can be analyzed using Watson's lemma, which in turn is based on the Gamma function. Now, instead of the Gamma function we obtain an integral which involves $\exp(it)$ (instead of $\exp(-t)$). This integral *can* be mapped to the Gamma function via complexification. However, this requires employing Jordan's lemma, which for completeness is stated below.

Jordan's Lemma Let C_R denote the semi-circle of radius R in the upper half complex z-plane centered at the origin, i.e.

$$C_R : \{z = Re^{i\theta}, \quad 0 < \theta < \pi, \quad R > 0\}.$$

Let $f(z)$ be an analytic function which vanishes on C_R as $R \to \infty$, i.e.

$$|f(z)| < K(R), \quad z \in C_R; \quad K(R) \to 0 \text{ as } R \to \infty.$$

Then

$$\lim_{R\to\infty} \int_{C_R} e^{iz\alpha} f(z)dz = 0, \quad \alpha > 0.$$

Remark Recall that without $\exp(iza)$, in order for the integral of $f(z)$ to vanish, we require $|f(z)| < k(R)/R$.

Preliminary Example 1

$$\int_0^\infty t^{\gamma-1} e^{it} dt = e^{\frac{i\pi\gamma}{2}} \Gamma(\gamma), \quad 0 < \gamma < 1. \tag{5.2}$$

The requirements of $\gamma > 0$ and $\gamma < 1$ ensure that the above integral converges at $t = 0$ and $t = \infty$ respectively.

In order to derive the above result we apply Jordan's lemma in the first quadrant of the complex z-plane for the analytic function $z^{\gamma-1}e^{iz}$:

If $J(\gamma)$ denotes the LHS of (5.2) we find

$$\int_0^\infty \left(e^{\frac{i\pi}{2}}\rho\right)^{\gamma-1} e^{-\rho} e^{\frac{i\pi}{2}} d\rho = e^{\frac{i\pi\gamma}{2}} \int_0^\infty \rho^{\gamma-1} e^{-\rho} d\rho,$$

which is equation (5.2).

Preliminary Example 2

$$\int_0^\infty t^\gamma e^{ivt^p} dt = \left(\frac{1}{|v|}\right)^{\frac{\gamma+1}{p}} \Gamma\left(\frac{\gamma+1}{p}\right) e^{\frac{i\pi}{2}(\gamma+1)\frac{sgn(v)}{p}}, \quad p > 0, -1 < \gamma < 0, v \in R. \tag{5.3}$$

In order to derive (5.3) we first assume that $v > 0$ and apply Jordan's lemma in the sector of the upper half complex plane bounded by the real axis and the ray $\rho e^{\frac{i\pi}{2p}}$. Thus, if $J(\gamma, v, p)$ denotes the LHS of (5.3) we find the following:

$$J = \int_0^\infty \left(e^{\frac{i\pi}{2}}\rho\right)^\gamma e^{-v\rho^p} e^{\frac{i\pi}{2p}} d\rho = e^{\frac{i\pi}{2p}(\gamma+1)} \int_0^\infty \rho^\gamma e^{-v\rho^p} d\rho. \tag{5.4}$$

Using the change of variables $v\rho^p = u$, we find

$$\int_0^\infty \rho^\gamma e^{-v\rho^p} d\rho = \frac{1}{v^{\frac{\gamma+1}{p}}} \frac{1}{p} \int_0^\infty u^{\frac{\gamma+1}{p}-1} e^{-u},$$

and then (5.4) becomes (5.3) for $v > 0$.

The case $v < 0$ can be analyzed in a similar way, where the relevant ray is now $\rho e^{-i\pi/2p}$.

The Formal Derivation Suppose that in $[a,b]$, $\phi'(t)$ vanishes at the single point c, which is an interior point. *Assuming* that the main contribution to the integral $I(k)$ defined in (5.1) comes from the neighborhood of c we find:

$$I(k) \sim \int_{C-R}^{C+R} f(c) e^{ik[\phi(c)+\frac{(t-c)^2}{2}\phi''(c)]} dt, \quad k \to \infty.$$

Letting $v = sgn\phi''(c)$, and employing the change of variables

$$v\tau^2 = (t-c)^2 \frac{\phi''(c)k}{2}, \quad \text{or} \quad \tau = (t-c)\sqrt{\frac{|\phi''(c)|k}{2}},$$

we find

$$I(k) \sim \sqrt{\frac{2}{k|\phi''(c)|}} e^{ik\phi(c)} \int_{-\infty}^{\infty} e^{iv\tau^2} d\tau.$$

But,

$$\int_{-\infty}^{\infty} e^{iv\tau^2} d\tau = 2\int_0^{\infty} e^{iv\tau^2} d\tau = \sqrt{\pi} e^{\frac{i\pi v}{4}},$$

where we have used (5.3). Hence,

$$I(k) \sim \sqrt{\frac{2\pi}{k|\phi''(c)|}} e^{ik\phi(c)} e^{\frac{i\pi}{4} sgn\phi''(c)}, \quad k \to \infty. \tag{5.5}$$

By imposing appropriate restrictions of the functions $f(t)$ and $\phi(t)$ it is possible to justify rigorously the above formula. A generalization of equation (5.5), where $f(t)$ is allowed to be singular, is presented below (the proof is similar with the analogous result valid for Laplace type integrals).

In many applications the functions $f(t)$ and $\phi(t)$ are analytic and then it is possible to map the relevant integral to a Laplace type integral, via complexification, see section 6.

Lemma Consider

$$I(k) = \int_a^b f(t) e^{ik\phi(t)} dt, a < b < \infty.$$

Assume that:

1. In the interval (a,b): $\phi^{'}(t)$ and $f(t)$ are continous, $\phi'(t) > 0$, and $\phi^{''}(t)$, $f'(t)$ become discontinous on infinity at most a finite number of points.

2. As $t \to a_+ : \phi(t) - \phi(a) \sim (t-a)^{\mu}\Phi$, $f(t) \sim (t-a)^{\lambda-1}F$, $\lambda < \mu$, where Φ, F, λ, μ are positive constants, the first condition is twice differentiate, and the second condition is differentiable.

3. $\int_R^b |\frac{f(t)}{\phi'(t)}| dt$ is finite for all $R \in (a,b)$.

4. As $t \to b_-$: $\frac{f(t)}{\phi'(t)}$ tends to a finite limit.

Then,

$$I(k) \sim e^{\frac{i\pi\lambda}{2\mu}} \frac{F}{\mu} \Gamma\left(\frac{\lambda}{\mu}\right) \frac{e^{ik\phi(a)}}{(\Phi k)^{(\lambda/\mu)}}, \qquad k \to \infty. \tag{5.6}$$

5.1 Sketch of Proof

Assumption (2) implies convergence at a, whereas (3) and (4) imply convergence at b. Then, local consideration in the neighbourhood of a, imply equation (5.6) via the change of variables $\tau = \phi(t) - \phi(a)$.

Example Consider

$$I(k) = \int_0^{\frac{\pi}{2}} e^{ik\cos t} dt, \quad k \to \infty.$$

Here $\phi'(t) = \sin t$, thus the main contribution comes from the neighborhood of $t = 0$. Hence,

$$I(k) = \int_0^{\frac{\pi}{2}} e^{ik(1-\frac{t^2}{2}+O(t^4))} dt \sim e^{ik} \int_0^{\frac{\pi}{2}} e^{-\frac{ikt^2}{2}} dt, \quad k \to \infty.$$

Using the substitution

$$\frac{kt^2}{2} = \tau^2, \quad \text{or} \quad \sqrt{\frac{k}{2}} t = \tau,$$

we find (see (5.4))

$$I(k) \sim e^{ik} \sqrt{\frac{2}{k}} \int_0^{\infty} e^{-i\tau^2} d\tau = \sqrt{\frac{\pi}{2k}} e^{i(k-\frac{\pi}{4})}, \quad k \to \infty.$$

Example Consider the Bessel function of order n:

$$J_n(x) = \frac{1}{\pi} Re \int_0^{\pi} e^{i(x\sin t - nt)} dt.$$

Hence

$$J_n(n) = \frac{1}{\pi} Re \int_0^{\pi} e^{in(\sin t - t)} dt.$$

In order to evaluate the large n behavior of the above function we note that $\phi(t) = \sin t - t$, thus again the main contribution comes from the neighborhood of $t = 0$:

$$J_n(n) = \frac{1}{\pi} Re \int_0^{\pi} e^{in(-\frac{t^3}{6}+O(t^5))} dt.$$

Using the substitution

$$\frac{nt^3}{6} = \tau^3, \quad \text{or} \quad \left(\frac{n}{6}\right)^{\frac{1}{3}} t = \tau,$$

we find

$$J_n(n) \sim \frac{1}{\pi}\left(\frac{6}{n}\right)^{\frac{1}{3}} \int_0^\infty e^{-i\tau^3} d\tau, \quad n \to \infty.$$

Using (5.4) we find

$$\int_0^\infty e^{-i\tau^3} d\tau = \frac{1}{3} e^{-\frac{i\pi}{6}} \Gamma\left(\frac{1}{3}\right).$$

Thus,

$$J_n(n) \sim \frac{1}{3\pi} \cos\left(\frac{\pi}{6}\right) \Gamma\left(\frac{1}{3}\right) \left(\frac{6}{n}\right)^{\frac{1}{3}}, \quad n \to \infty.$$

6 The Method of Steepest Descent

Consider

$$I(k) = \int_C f(z) e^{k\phi(z)} dz, \tag{6.1}$$

where $f(z)$ and $\phi(z)$ are analytic functions. Let

$$\phi(z) = u(x, y) + iv(x, y),$$

where u and v are real functions. The key step of the method of steepest descent is to use Cauchy's theorem to deform C to a new contour C' on which v is constant,

$$v(x, y) = v(x_0, y_0). \tag{6.2}$$

Then (6.1) becomes

$$I(k) = e^{ikv(x_0,y_0)} \int_{C'} f(z) e^{ku(x,y)} dz. \tag{6.3}$$

Thus, the method of steepest descent provides the extension of the Laplace method to the complex plane.

It turns out that curves along which v is constant are also steepest curves, namely curves along which the change of $u(x, y)$ is maximal:

(i) The curves defined by (6.2) are steepest curves.

Indeed, the direction orthogonal to these curves is given by ∇v. But $\nabla v = (v_x, v_y) = (-u_y, u_x)$, where we have used the Cauchy Riemann relations. Thus, the direction along these curves is given by $(u_x, y_y) = \nabla u$. However, the direction of the vector ∇u is the direction of the maximal change of $u(x, y)$.

(ii) The steepest curves are given by (6.2).

Let

$$\delta\phi = \phi(z) - \phi(z_0) = \delta u + i\delta v.$$

Thus,

$$|\delta\phi|^2 = |\delta u|^2 + |\delta v|^2, \quad \text{or} \quad |\delta u| \leq |\delta\phi|,$$

and equality is achieved when $\delta v = 0$.

Saddle Points The point z_0 is called a saddle point of order N if the first N derivatives of $\phi(z)$ vanishes at z_0 and the $N+1$ derivative is different than zero:

$$\left.\frac{d^m\phi(z)}{dz^m}\right|_{z=z_0} = 0, \quad m = 1, \cdots, N,$$

$$\left.\frac{d^{N+1}\phi(z)}{dz^{N+1}}\right|_{z=z_0} = ae^{i\alpha}, \quad a > 0, \quad \alpha \in \mathbb{R}.$$

If $N = 1$, z_0 is a *simple saddle point*. The directions of the steepest descent and of steepest ascent are defined respectively by

$$\theta = -\frac{\alpha}{N+1} + (2m+1)\frac{\pi}{N+1}, \quad m = 0, 1, ..., N$$

and

$$\theta = -\frac{\alpha}{N+1} + 2m\frac{\pi}{N+1}, \quad m = 0, 1, \cdots, N.$$

Indeed,

$$\phi(z) - \phi(z_0) \sim \frac{(z-z_0)^{N+1}}{(N+1)!}\left.\frac{d^{N+1}\phi(z)}{dz^{N+1}}\right|_{z=z_0} = \frac{(\rho e^{i\theta})^{N+1}}{(N+1)!}ae^{i\alpha}, \quad z \to z_0.$$

Thus,

$$\phi(z) - \phi(z_0) \sim \frac{\rho^{N+1}a}{(N+1)!}\left[\cos(\alpha + (N+1)\theta) + i\sin(\alpha + (N+1)\theta)\right], \quad z \to z_0.$$

Hence, the paths of steepest descent are defined by

$$\sin(\alpha + (N+1)\theta) = 0, \quad \cos(\alpha + (N+1)\theta) < 0,$$

whereas for the paths of steepest descent the cos above is positive. If $N = 1$, then the paths of steepest descent and ascent are given respectively by

$$\theta = -\frac{\alpha}{2} + \frac{\pi}{2}, \quad -\frac{\alpha}{2} + \frac{3\pi}{2}$$

and

$$\theta = -\frac{\alpha}{2}, \quad -\frac{\alpha}{2} + \pi.$$

Example

$$\phi(z) = z - \frac{z^3}{3}, \quad \phi'(z) = 1 - z^2, \quad \phi''(z) = -2z.$$

$$z = 1 : \phi''|_1 = -z = 2e^{i\pi}, \quad \alpha = \pi,$$

thus, the paths of steepest descent are $\theta = 0$ and $\theta = \pi$.

$$z = -1 : \phi''|_{-1} = z = 2, \quad \alpha = 0,$$

thus, the paths of steepest descent are $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$.

Laplace's Method for Complex Contours

Let us assume that

$$f(z) \sim f_0(z - z_0)^{\beta - 1}, \quad Re\beta > 0, \quad z \to z_0.$$

We introduce the change of variables

$$\phi(z) - \phi(z_0) = -t, t > 0. \tag{6.4}$$

Using

$$\phi(z) - \phi(z_0) = -(z - z_0)^{N+1}\Phi(z),$$

where $\Phi(z)$ is analytic in the neighborhood of z_0, we can use the implicit function theorem to solve the equation

$$(z - z_0)^{N+1}\Phi(z) = t$$

for t as a function of z. In what follows we only concentrate on the leading order term:

$$\frac{(z - z_0)^{N+1}}{(N+1)!} \left|\phi^{N+1}(z_0)\right| e^{i\alpha} = -t. \tag{6.5}$$

Taking the absolute value of this equation and then solving for $|z - z_0|$ we find

$$|z - z_0| = \left(\frac{(N+1)!}{|\phi^{N+1}(z_0)|}\right)^{\frac{1}{N+1}} t^{\frac{1}{N+1}}. \tag{6.6}$$

Using in (6.3) the change of variables (6.1) we find

$$I(k) \sim -e^{k\phi(z_0)} \int_0^\infty \frac{f(z)}{\phi'(z)} e^{-kt} dt, \quad k \to \infty. \tag{6.7}$$

But,

$$\frac{f(z)}{\phi'(z)} \sim \frac{f_0(z - z_0)^{\beta - 1}}{\frac{(z - z_0)^N}{N!}\phi^{N+1}(z_0)} = N! \frac{f_0(z - z_0)^{\beta - N - 1}}{\phi^{N+1}(z_0)}$$

$$= \frac{N! f_0(|z - z_0|e^{i\theta})^{\beta - N - 1}}{|\phi^{N+1}(z_0)|e^{i\alpha}} = \frac{N! f_0}{|\phi^{N+1}(z_0)|} \frac{e^{i\theta(\beta - N - 1)}}{e^{i\alpha}} \left(\frac{(N+1)!}{|\phi^{N+1}(z_0)|}\right)^{\frac{\beta - (N+1)}{N+1}} t^{\frac{\beta - (N+1)}{tN+1}}.$$

Using $N! = (N+1)!/(N+1)$ and simplifying we find

$$\frac{f(z)}{\phi'(z)} \sim \frac{f_0}{N+1}\left(\frac{(N+1)!}{|\phi^{N+1}(z_0)|}\right)^{\frac{\beta}{N+1}} t^{\frac{\beta}{N+1}-1} e^{i[\beta\theta-(\alpha+(N+1)\theta)]}.$$

Using

$$e^{-i(\alpha+(N+1)\theta)} = -1$$

and substituting the resulting expression for $f(z)/\phi'(z)$ in (6.7) we find

$$I(k) \sim e^{k\phi(z_0)} \frac{f_0}{N+1}\left(\frac{(N+1)!}{|\phi^{N+1}(z_0)|}\right)^{\frac{\beta}{N+1}} e^{i\beta\theta} \frac{\Gamma(\frac{\beta}{N+1})}{k^{\frac{\beta}{N+1}}}, \quad k \to \infty, \tag{6.8}$$

where we have employed the identity

$$\int_0^\infty e^{-kt} t^{\frac{\beta}{N+1}-1} dt = \frac{\Gamma(\frac{\beta}{N+1})}{k^{\frac{\beta}{N+1}}}.$$

In general, the implementation of the steepest descent method involves the following steps:

(i) Identify the relevant critical points, namely saddle points, end points, and singularities.

(ii) Deform C to the steepest contour (or contours). For the computation of the leading behavior it is sufficient to deform to asymptotically steepest descent contours, namely to contours which coincide with the steepest descent contours only in the neighborhood of the saddle points.

(iii) Apply the Laplace method in the complex plane.

Example Consider

$$I(k) = \int_0^1 \ln t e^{ikt} dt$$

The complexification of the above integral suggests

$$\tilde{I}(k) = \int_C \ln z e^{ikz} dz.$$

Thus $\phi(z) = z$, $\phi'(z) = 1$, and hence there do not exist saddle points. Thus we expect the main contribution will come from the end points. If $z = i\rho$ and $z = 1 + i\rho$ we obtain Laplace type integrals, thus we take C to be the union of the contours shown in Figure 2. Hence,

$$I(k) = i\int_0^R \ln(i\rho)e^{-k\rho}d\rho + \int_0^1 \ln(x+iR)e^{ik(x+iR)}dx - ie^{ik}\int_0^R \ln(1+i\rho)^{-k\rho}d\rho.$$

Letting $R \to \infty$ we find

$$I(k) = i\int_0^\infty \ln(i\rho)e^{ik\rho}d\rho - ie^{ik}\int_0^\infty \ln(1+i\rho)e^{-k\rho}d\rho = I_1(k) + I_2(k).$$

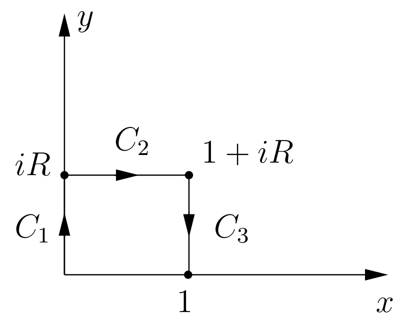

Figure 2:

In I_1 we use the substitution $k\rho = \tau$, thus

$$I_1(k) = \frac{i}{k}\int_0^\infty \ln\left(\frac{i\tau}{k}\right) e^{-\tau} d\tau = \frac{i}{k}\left[\ln\left(\frac{i}{k}\right)\int_0^\infty e^{-\tau} d\tau + \int_0^\infty \ln\tau e^{-\tau} d\tau\right].$$

Thus,

$$I_1(k) = \frac{i}{k}\left[\frac{i\pi}{2} - \ln k - \gamma\right] = -\frac{i\ln k}{k} - \frac{\left(i\gamma + \frac{\pi}{2}\right)}{k}.$$

The integral I_2 can be computed via Watson's lemma:

$$\ln(1 + i\rho) = -\sum_{m=1}^{\infty}\frac{(-i\rho)^m}{m}.$$

Thus

$$I_2(k) \sim ie^{ik}\sum_{m=1}^{\infty}(-i)^m\frac{(m-1)!}{k^{m+1}}.$$

Hence,

$$I(k) \sim -\frac{i\ln k}{k} - \frac{i\gamma + \pi/2}{k} + ie^{ik}\sum_{m=1}^{\infty}(-i)^m\frac{(m-1)!}{k^{m+1}}, \quad k \to \infty.$$

Example Consider

$$I(k) = \int_C e^{k(\mathrm{sech}\ v \sinh z - z)} dz, \quad v \neq 0, \quad k > 0,$$

where C is the so-called Sommerfeld contour depicted in Figure 3
It turns out that

$$I(k) = \frac{J_k(k\mathrm{sech}\ v)}{2i\pi},$$

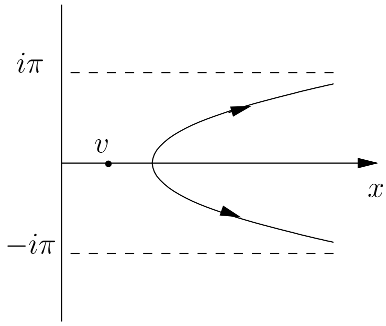

Figure 3:

where J_k is the Bessel function of order k. Debye in 1954 rediscovered the method of steepest descent by studying the above integral (this method was introduced earlier by Riemann in connection with the hypergeometric function).

In the above case

$$\phi(z) = \text{sech}\ v \sinh z - z, \quad \phi'(z) = \text{sech}\ v \cosh z - 1, \quad z_0 = v.$$

Thus, we expect that the main contribution will come from the neighborhood of $z = v$. Letting $z = v + \rho e^{i\theta}$, we find

$$\phi(z) = \text{sech}\ v[\sinh v \cosh(\rho e^{i\theta}) + \cosh v \sinh(\rho e^{i\theta})] - \rho e^{i\theta} - v.$$

But

$$\cosh \varepsilon = \frac{e^{\varepsilon} + e^{-\varepsilon}}{2} = 1 + \frac{\varepsilon^2}{2} + O(\varepsilon^4), \quad \sinh \varepsilon = \varepsilon + O(\varepsilon^3), \quad \varepsilon \to 0.$$

Hence,

$$\phi(z) = -v + \tanh v \left[1 + \frac{\rho^2}{2} e^{2i\theta} + O(\rho^4)\right] + \rho e^{i\theta} + O(\rho^3) - \rho e^{i\theta}$$

$$= -v + \tanh v + \tanh v \frac{\rho^2}{2}(\cos 2\theta + i \sin 2\theta) + O(\rho^3), \quad \rho \to 0.$$

The conditions $\sin 2\theta = 0$, $\cos 2\theta < 0$, imply $\theta = \pi/2$ and $\theta = 3\pi/2$. For $\theta = \pi/2$ we find the following contribution:

$$ie^{k(\tanh v - v)} \int_0^R e^{-k \tanh v \frac{\rho^2}{2}} d\rho \sim ie^{k(\tanh v - v)} \sqrt{\frac{2}{k \tanh v}} \int_0^\infty e^{-\tau^2} d\tau, \quad k \to \infty,$$

whee we have used the substitution $k \tanh v \rho^2/2 = \tau^2$. Adding to the above the contribution from $\theta = 3\pi/2$ we find

$$I(k) \sim i\sqrt{\frac{2\pi}{k \tanh v}} e^{k(\tanh v - v)}, \quad k \to \infty.$$

Example Consider the Hankel function of order n defined by

$$H_v^{(1)}(k) = \frac{1}{\pi}\int_C e^{ik\cos z} e^{iv(z-\frac{\pi}{2})} dz, \quad v \in \mathbb{R}, \quad k > 0,$$

where C is depicted in Figure 1.

In this case

$$\phi(z) = i\cos z, \quad \phi'(z) = -i\sin z, \quad z_0 = 0.$$

Letting $z = \rho e^{i\theta}$, we find

$$e^{ik\cos z} = e^{ik + \frac{k\rho^2}{2}(\sin 2\theta - i\cos 2\theta) + O(\rho^4)}.$$

The conditions $\cos 2\theta = 0$, $\sin 2\theta < 0$, imply $\theta = 3\pi/4$ and $\theta = -\pi/4$. For $\theta = 3\pi/4$ we find to the leading order the contribution

$$-\frac{e^{ik}}{\pi}\int_0^R e^{-\frac{k\rho^2}{2}} e^{iv\left[\rho e^{3i\pi/4} - \frac{\pi}{2}\right]} e^{\frac{3i\pi}{4}} d\rho \sim \frac{e^{i(k-\frac{\pi}{4}-\frac{v\pi}{2})}}{\pi}\sqrt{\frac{2}{k}}\int_0^\infty e^{-\tau^2} d\tau, \quad k \to \infty.$$

Adding to the above, the contribution from $\theta = -\pi/4$, we find

$$H_v^{(1)}(k) \sim \sqrt{\frac{2}{\pi k}} e^{i\left(k - \frac{\pi}{4} - \frac{v\pi}{2}\right)}, \quad k \to \infty.$$

In what follows we indicate briefly how to obtain higher order terms (and how to justify rigorously the above formal analysis). The steepest paths going through $z_0 = 0$ are defined by

$$Im[i\cos(x+iy)] = Im[i].$$

Thus

$$\text{Re}\,[\cos(x)\cos(iy) - \sin(x)\sin(iy)] = 1,$$

or using

$$\cos(iy) = \cosh y \quad \text{and} \quad \sin(iy) = i\sin hy,$$

$$\cos x \cosh y = 1.$$

For small x, y we find

$$\left(1 - \frac{x^2}{2} + O(x^4)\right)\left(1 + \frac{y^2}{2} + O(y^4)\right) = 1, \quad x \to 0, \quad y \to 0.$$

Hence,

$$\frac{y^2}{2} - \frac{x^2}{2} \sim 0, \quad x \to 0, \quad y \to 0$$

and the steepest descent curve is $y \sim -x$. The equation

$$\cos x = \frac{2}{e^y + e^{-y}},$$

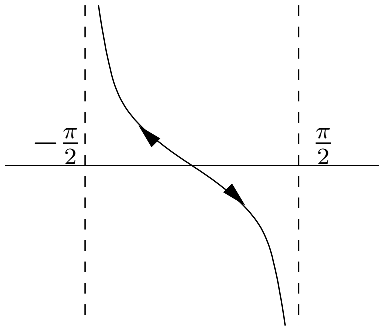

Figure 4:

implying $\cos x \sim 2e^{-y}$ as $y \to +\infty$, hence $x \sim -\pi/2$, $y \to \infty$. Thus the steepest descent curve denoted by C' is depicted in Figure 4

The global change of variables $\phi(z) - \phi(z_0) = -t$, $t > 0$, implies

$$i(\cos z - 1) = -t,$$

or

$$1 - \cos z = e^{-\frac{i\pi}{2}} t, \quad \text{or} \quad \frac{z^2}{2}\left(1 - \frac{z^2}{12} + O(z^4)\right) = e^{-\frac{i\pi}{2}} t, \quad z \to 0.$$

Hence,

$$\frac{z}{\sqrt{2}}\left(1 - \frac{z^2}{24} + O(z^4)\right) = e^{-\frac{i\pi}{4}} t^{\frac{1}{2}}, \quad t > 0, \quad z \to 0.$$

Hence,

$$z = \sqrt{2} e^{-\frac{i\pi}{4}} t^{\frac{1}{2}} \left(1 + \frac{z^2}{24} + O(z^4)\right), \quad z \to 0.$$

Thus

$$z = \sqrt{2} e^{-\frac{i\pi}{4}} t^{\frac{1}{2}} \left(1 + \frac{2e^{-\frac{i\pi}{2}} t}{24} + \cdots\right), \quad t \to 0,$$

or

$$z = \sqrt{2} e^{-\frac{i\pi}{4}} t^{\frac{1}{2}} + \frac{\sqrt{2}}{12} e^{-\frac{3i\pi}{4}} t^{\frac{3}{2}} + \cdots, \quad t \to 0. \tag{6.9}$$

Hence,

$$H_v^{(1)}(k) \sim \frac{1}{\pi} \int_{C'} e^{ik - kt - \frac{iv\pi}{2}} e^{ivz(t)} \frac{dz(t)}{dt} dt$$
$$= \frac{1}{\pi} e^{i\left(k - \frac{v\pi}{2}\right)} \int_{C'} e^{-kt} \left[1 + ivz + \frac{(ivz)^2}{2!} + \frac{(ivz)^3}{3!} + O(z^4)\right] \frac{dz}{dt} dt.$$

Splitting the steepest descent contour into two parts and using (6.9) to replace z by t we find

$$H_v^{(1)}(k) \sim \frac{2}{\pi} e^{i\left(k - \frac{v\pi}{2}\right)} \int_0^{\infty} e^{-kt} \left[c_0 t^{-\frac{1}{2}} + c_1 + c_2 t^{\frac{1}{2}} + c_3 t + O\left(t^{\frac{3}{2}}\right)\right] dt,$$

where $\{C_j\}_0^3$ are particular constants. Thus,

$$H_v^{(1)}(k) \sim \frac{2}{\pi} e^{i(k-\frac{v\pi}{2})} \left[\sqrt{\frac{\pi}{2}} \frac{e^{-\frac{i\pi}{4}}}{k^{\frac{1}{2}}} + \frac{v}{k} + \frac{\sqrt{2\pi}}{4} \frac{(\frac{1}{4}-v^2)e^{-\frac{3i\pi}{4}}}{k^{\frac{3}{2}}} + \frac{iv(v^2-1)}{3k^2} + O\left(\frac{1}{k^{\frac{5}{2}}}\right) \right], k \to \infty,$$

where we have used

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right).$$

7 The WKB(J) Method

It was shown earlier that the steepest descent method provides an effective method for evaluating the asymptotics of the Hankel and the Bessel functions. On the other hand, these function satisfy certain second order linear ODEs. Thus, the following question arises: is it possible to obtain similar results by analyzing directly the given ODE instead of the relevant integral representation? The answer to this question is affirmative. In order to introduce the associated method, called WKB(J) we consider the Airy function:

$$Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(xs+\frac{1}{3}s^3)} ds, \quad x \in \mathbb{R}. \tag{7.1}$$

Let us complexify the integrand of the above integral, $z = |z|\exp(i\theta)$. For large $|z|$ convergence requires $\sin(3\theta) \geq 0$, or

$$0 \leq \theta \leq \frac{\pi}{3} \quad \text{and} \quad \frac{2\pi}{3} \leq \theta \leq \pi.$$

Thus, we can deform the contour along the real axis to a contour which for large $|z|$ lies in the shaded domains below

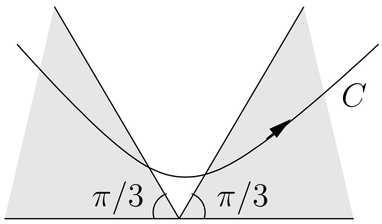

Figure 5:

Noting that

$$\left(\frac{d^2}{dx^2} - x\right) Ai(x) = \frac{1}{2\pi} \int_C -(z^2+x) e^{i\left(xz+\frac{1}{3}z^3\right)} dz = \frac{i}{2\pi} \int_C \left(\frac{d}{dz} e^{i\left(xz+\frac{z^2}{3}\right)}\right) dz,$$

it follows that $Ai(x)$ satisfies the Airy equation

$$-\frac{d^2y}{dx^2} + xy = 0. \tag{7.2}$$

The point $x = 0$ is an ordinary point and the only singular point of (7.2) is $x = \infty$. Thus, there exists a Taylor series at $x = 0$ with infinite radius of convergence:

$$y(x) = a_0 \left(1 + \frac{x^3}{3.2} + \frac{x^6}{6.5.3.2} + \cdots\right) + a_1 \left(x + \frac{x^4}{4.3} + \frac{x^7}{7.6.4.3} + \cdots\right). \tag{7.3}$$

In order to computer the large x asymptotics of $Ai(x)$, we note that

$$\phi(z) = i\left(xz + \frac{z^3}{3}\right), \quad \phi'(z) = i(x + z^2).$$

Thus, $z_0^{\pm} = \pm i x^{\frac{1}{2}}$ are moving small points. Hence, letting $z = x^{\frac{1}{2}}\zeta$ we find

$$Ai(x) = \frac{x^{\frac{1}{2}}}{2\pi}\int_C e^{ix^{\frac{3}{2}}\left(\zeta + \frac{\zeta^3}{3}\right)} d\zeta.$$

Now, $\zeta_0^{\pm} = \pm i$, and we deform the contour to pass through $\zeta_0 = i$:

$$\zeta = i + \rho e^{i\theta}.$$

Thus,

$$\zeta + \frac{\zeta^3}{3} = i + \rho e^{i\theta} + \frac{1}{3}\left(i^3 + 3i^2\rho e^{i\theta} + 3i\rho^2 e^{2i\theta}\right) + O(\rho^3), \quad \rho \to 0$$

$$= \frac{2i}{3} + \rho^2(i\cos 2\theta - \sin 2\theta) + O(\rho^3), \quad \rho \to 0.$$

The conditions

$$\sin 2\theta = 0, \quad \cos 2\theta > 0,$$

imply $\theta = 0$¡ $\theta = \pi$. The contribution from $\theta = 0$ yields

$$\frac{x^{\frac{1}{2}}}{2\pi} e^{-\frac{2}{3}x^{\frac{3}{2}}} \int_0^\infty e^{-x^{\frac{3}{2}}\rho^2} d\rho = \frac{x^{\frac{1}{2}}}{x^{\frac{3}{4}}}\frac{1}{2\pi} e^{-\frac{2}{3}x^{\frac{3}{2}}} \int_0^\infty e^{-\tau^2} d\tau = \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{4x^{\frac{1}{2}}}\frac{1}{\sqrt{\pi}}.$$

Thus,

$$Ai(x) \sim \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{2\sqrt{\pi}x^{\frac{1}{4}}}, \quad x \to +\infty. \tag{7.4}$$

Similarly,

$$Ai(x) \sim \frac{\cos\left(\frac{2}{3}|x|^{\frac{3}{2}} - \frac{\pi}{4}\right)}{\sqrt{\pi}|x|^{\frac{1}{4}}}, \quad x \to -\infty. \tag{7.5}$$

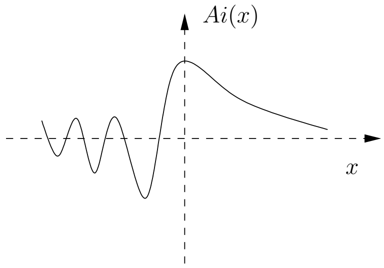

Figure 6:

Thus
Equation (7.2) can be interpreted as the Schrödinger equation of potential $V(x) = x$ with energy $E = 0$.

In general consider the ODE

$$\frac{d^2y}{dx^2} + \lambda^2 u(x)y = 0, \tag{7.6}$$

where λ is a large parallel and x is real.

We look for solutions in the form

$$y(x) = e^{\lambda S(x)}(z_0(x) + \frac{1}{\lambda}z_1(x_1) + \cdots).$$

Then (7.6) yields

$$\lambda^2(S'^2 + u)\left(z_0 + \frac{1}{\lambda}z_1 + \frac{1}{\lambda^2}z_2 + \cdots\right)$$

$$+\lambda(2S'z_0' + S''z_0) + O(1) = 0.$$

$$O(\lambda^2) : S'^2 = -u, \quad \text{or} \quad S_\pm = \pm i \int u^{\frac{1}{2}}(x)dx.$$

$$O(\lambda) : 2S'z_0' + S''z_0 = 0, \quad \text{or} \quad \frac{z_0'}{z_0} = -\frac{S''}{2S'}, \quad \text{or} \quad z_{0\pm} = \frac{C_\pm}{u^{\frac{1}{2}}}.$$

Hence, for $u(x) \neq 0$, we find

$$y(x) \sim \frac{1}{u^{\frac{1}{4}}(x)}\left(C_+ e^{i\lambda\int u^{\frac{1}{2}}\,dx} + C_- e^{-i\lambda\int u^{\frac{1}{2}}\,dx}\right). \tag{7.7}$$

If $u(x)$ has a zero, then there exists a *transition* domain. We consider the simplest case where $u(x)$ has a simple zero at $x = 0$. Then, assuming that

$$u(x) \sim -kx, \quad x \to 0, \quad k > 0, \tag{7.8}$$

equation (7.6) can be approximated by the Airy equation. Letting $z = \lambda^{\frac{2}{3}} k^{\frac{1}{3}} x$, (7.6) is approximated by

$$\frac{d^2 y}{dz^2} - zy = 0. \tag{7.9}$$

In addition to the solution $Ai(x)$ which satisfies (7.4) and (7.5), we also introduce $B_i(x)$, where

$$B_i(x) \sim \frac{e^{\frac{2}{3} x^{\frac{3}{2}}}}{\sqrt{\pi} x^{\frac{1}{4}}}, \quad x \to \infty \tag{7.10}$$

and

$$B_i(x) \sim \frac{1}{\sqrt{\pi}(-x)^{\frac{1}{4}}} \cos\left(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{\pi}{4}\right), \quad x \to -\infty. \tag{7.11}$$

Thus,

$$y_L(x) \sim \frac{1}{|u(x)|^{\frac{1}{4}}} \left[C_+^L e^{i\lambda \int_0^x u(t)^{\frac{1}{2}} dx} + C_-^L e^{-i\lambda \int_0^x u(t)^{\frac{1}{2}} dx}\right],$$

$$y_C(x) \sim C_1 Ai\left(k^{\frac{1}{3}} \lambda^{\frac{2}{3}} x\right) + C_2 B_i\left(k^{\frac{1}{3}} \lambda^{\frac{2}{3}} x\right),$$

$$y_R(x) \sim \frac{1}{(-u(x))^{\frac{1}{4}}} \left[C_+^R e^{\lambda \int_0^x (-u(t))^{\frac{1}{2}} dt} + C_-^R e^{-\lambda \int_0^x (-u(t))^{\frac{1}{2}} dt}\right].$$

Letting $\theta = \frac{2}{3} k^{\frac{1}{2}} \lambda |x|^{\frac{3}{2}}$, taking the outer limit of the inner solution and the inner solution of the outer solution, it is straightforward to determine $C_\pm^R$ and $C_\pm^L$ in terms of C_1 and C_2, see A+F, page 503.

Remark Consider the case of two turning points shown in the figure below, namely consider $u(x) < 0$ for $A < x < B$ and $u(x) > 0$ outside $[A, B]$. Furthermore, assume $u(x) = O(x^{-2})$, $x \to \infty$.

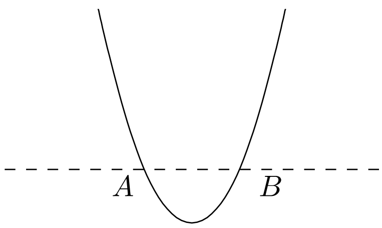

Figure 7:

The one-turning point solution which decays like

$$\frac{C_1}{(u(x))^{\frac{1}{4}}} e^{-\lambda \int_B^x u^{\frac{1}{2}}(t) dt}, \quad x \to \infty,$$

behaves like

$$\frac{2C_1}{(-u(x))^{\frac{1}{4}}} \sin\left[\lambda \int_x^B (-u(t))^{\frac{1}{2}} dt + \frac{\pi}{4}\right], \quad A < x < B.$$

Similarly, the one-turning point solution which decays like

$$\frac{C_2}{(u(x))^{\frac{1}{4}}} e^{-\lambda \int_x^A u^{\frac{1}{2}}(t)dt}, \quad x \to -\infty,$$

behaves like

$$\frac{2C_2}{(-u(x))^{\frac{1}{4}}} \sin\left[\lambda \int_A^x (-u(t))^{\frac{1}{2}} dt + \frac{\pi}{4}\right], \quad A < x < B.$$

In order to match the above solutions in $A < x < B$ we note that

$$\sin\left[\lambda \int_x^B a dt + \frac{\pi}{4}\right] = -\sin\left[\lambda \int_B^x a dt - \frac{\pi}{4}\right]$$
$$= -\sin\left\{\lambda \int_A^x a dt + \frac{\pi}{4} - \left[\lambda \int_A^B a dt + \frac{\pi}{2}\right]\right\}.$$

Thus, we require the constraint

$$\lambda \int_A^B (-u(x))^{\frac{1}{2}} dx + \frac{\pi}{2} = \text{integer multiple of } \pi.$$

If $u(x) = V(x) - E$, then

$$\lambda \int_A^B \sqrt{E - V(x)} dx = (n + \frac{1}{2})\pi, \quad n = 0, 1, \cdots.$$

8 Stokes Phenomenon

Let us consider the asymptotic behavior of $\sinh(z^{-1})$, as $z \to 0$, in the complex plane. Letting $z = \rho e^{i\theta}$, we find

$$\sinh\left(\frac{e^{-i\theta}}{\rho}\right) = \frac{1}{2} e^{\frac{1}{\rho}(\cos\theta - i\sin\theta)} - \frac{1}{2} e^{-\frac{1}{\rho}(\cos\theta - i\sin\theta)}.$$

Thus,

$$\sinh\left(z^{-1}\right) \sim \left[\begin{array}{ll} \frac{1}{2} e^{z^{-1}}, & \cos\theta > 0 \\ & \\ -\frac{1}{2} e^{-z^{-1}}, & \cos\theta < 0 \end{array}\right., \quad z \to 0.$$

Hence, the asymptotic expansion changes discontinuously across the rays $\theta = \frac{\pi}{2}$, $\frac{3\pi}{2}$. This phenomenon, called Stoke's phenomenon, occurs often in applications. In particular, it occurs in connection to the asymptotic expansion of linear ODEs in the vicinity of an irregular singular point.

Example

$$I(z) = \int_0^\infty \frac{e^{-zt}}{1+t^4} dt, \quad z \in \mathbb{C}, \quad |z| \to \infty.$$

Letting $z = \rho e^{i\theta}$, it follows that if $\cos\theta > 0$, i.e. $|\theta| < \frac{\pi}{2}$, then the integral decays exponentially for large $|z|$ unless $t \to 0$. Thus, we expect the main contribution comes from the neighborhood of the origin:

$$I(z) \sim \int_0^\infty e^{-zt} dt - \int_0^\infty e^{-zt} t^4 dt, \quad z \to \infty$$

and

$$I(z) \sim \frac{1}{z} - \frac{4!}{z^5}, \quad z \to \infty, \quad |\arg z| < \frac{\pi}{2}.$$

The constraint $|\arg z| < \frac{\pi}{2}$ is a consequence of the requirement that Re $(zt) > 0$. Thus, by complexifying t we will obtain a different constraint on z. For example, deforming the contour of integration to the lower imaginary axis and using Cauchy's theorem in the fourth quadrant of the complex z-plane we find

$$I(z) = \int_0^{\infty e^{-\frac{i\pi}{2}}} \frac{e^{-z\zeta}}{1+\zeta^4} d\zeta + \frac{i\pi}{2} \frac{e^{-ze^{-\frac{i\pi}{4}}}}{e^{-\frac{3i\pi}{4}}}. \tag{7.11}$$

Now convergence requires Re $(z\zeta) > 0$, or $\cos\left(\theta - \frac{\pi}{2}\right) > 0$, or $0 < \theta < \pi$. The LHS of (7.11) is well defined for $\arg z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, while the rhs for $\arg z \in (0, \pi)$ (for Cauchy's theorem we need $0 < z < \frac{\pi}{2}$). Thus, we *can* compute $I(z)$ for $\arg z \in (-\frac{\pi}{2}, \pi)$.

The pole contribution is exponentially small in $\arg z \in (-\frac{\pi}{4}, \frac{3\pi}{4})$ and exponentially large in $(\frac{3\pi}{4}, \pi)$. Thus,

$$I(z) \sim \left[\begin{array}{ll} \frac{1}{z} - \frac{4!}{z^5}, & \arg z \in \left(\frac{-\pi}{4}, \frac{3\pi}{4}\right), \\ & \\ \frac{i\pi}{2} e^{\frac{3i\pi}{4}} e^{-ze^{\frac{-i\pi}{4}}}, & \arg z \in \left(\frac{3\pi}{4}, \pi\right) \cup \left(-\frac{\pi}{2}, \frac{-\pi}{4}\right) \end{array}\right. , \quad |z| \to \infty.$$

9 Review of Useful Definitions and Theorems

9.1 Partition

Consider the interval $[a, b]$, where $a < b$, a partition of $[a, b]$ is a finite set $P = \{a_0, ..., a_n : a = a_0 < a_1, ... < a_n = b\}$.

9.2 Refinement of a Partition

We say that $P*$ is a refinement of P if $P \subset P*$ (that is every point of P is a point of $P*$). Given, two partitions, P_1 and P_2, we say that $P* = P_1 \cup P_2$ is their common refinement.

9.3 Upper and Lower Riemann Sums

Let $f : [a, b] \to \mathbb{R}$, be a bounded real function. The partition P gives rise to the following **upper** and **lower** Riemann sums:

$$U(P, f) := \sum_{i=1}^{n} M_i \Delta a_i, \qquad L(P, f) = \sum_{i=1}^{n} m_i \Delta a_i$$

where $\Delta a_i = a_i - a_{i-1}$, and for every $i \leq n$,

$$M_i := \sup\{f(x) : a_{i-1} \leq x \leq a_i\} \qquad m_i := \inf\{f(x) : a_{i-1} \leq x \leq a_i\},$$

and also

$$\int_a^{b-} f(x)dx := \inf\{U(P, f) : P \text{ is a partition of } [a, b]\} \tag{1}$$

$$\int_{a-}^{b} f(x)dx := \sup\{U(P, f) : P \text{ is a partition of } [a, b]\}. \tag{2}$$

Here, inf **and** sup are taken over all partitions of $[a, b]$. The quantity $\int_a^{b-} f(x)dx$ is called the *upper* and $\int_{a-}^{b} f(x)dx$ is called the *lower* Riemann integrals of f over $[a, b]$. Now, if

$$\int_a^{b-} f(x)dx = \int_{a-}^{b} f(x)dx \tag{3}$$

then, we say f *is Riemann integrable on* [a,b]. We denote this common value by

$$\int_a^{b} f(x)dx = \int_a^{b-} f(x)dx = \int_{a-}^{b} f(x)dx \tag{4}$$

9.4 Riemann Criterion

The function $f : [a, b] \to \mathbb{R}$ is Riemann-integrable **if and only if** ($\Longleftrightarrow$), if for every $\epsilon > 0$, $\exists$ a partition P_ϵ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.

10 Riemann Lebesgue Lemma

10.1 A brief statement on the proof of Riemann Lebesgue Lemma

The proof of the Riemann Lebesgue lemma is straightforward, using integration by parts in the case that $f(t)$ is a continously differentiable. In the general case, the proof is a consequence of the fact that a Riemann integrable function can be approximated by a piecewise constant function.

10.2 Riemann Lebesgue Lemma

- Let the function $f : [a, b] \to \mathbb{R}$ be **Riemann-integrable**, and $k \in \mathbb{R}$.
- The function $I(k)$ is defined by

$$I(k) = \int_a^b f(t)e^{ikt}dt. \tag{5}$$

Then

$$\lim_{|k|\to\infty} I(k) = 0. \tag{6}$$

Proof

- Since, $f(x)$ is **Riemann Integrable** $\Rightarrow$ for every $\epsilon > 0$, $\exists$ a partition P_ϵ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$, (For P and every refinement $P*$). Moreover , $\sup\{U(P, f) : P\} = \inf\{U(P, f) : P\} = \int_a^b f(x)dx$.
- e^{ikt} is a smooth bounded function and hence $f(t)e^{ikt}$ is also **Riemann Integrable**. So $I(k)$ exists, and can be estimated using an appropriate partition.
- Given a $\epsilon > 0, \exists$ a partition P_ϵ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.
- Rewrite $U(P_\epsilon, f) = \int_a^b M(t)dt$, where $M_i := \sup\{f(x) : a_{i-1} \leq x \leq a_i\}$.
- Rewrite $L(P_\epsilon, f) = \int_a^b m(t)dt$, where $m_i := \inf\{f(x) : a_{i-1} \leq x \leq a_i\}$.
- $M(t) \geq f(t) \geq m(t)$.
- $U(P_\epsilon, f) - L(P_\epsilon, f) = \int_a^b \Big[M(t) - m(t)\Big]dt < \epsilon$.
- $|\int_a^b e^{ikt}\Big[f(t) - m(t)\Big]dt| \leq \int_a^b |[M(t) - m(t)]|dt < \epsilon$, since $|e^{ikt}| = 1$.
- The function $m(t)$ is a piecewise continous function. Thus, for this function, integration by parts immediately implies Riemann Lebesgue Lemma. Hence given the above $\epsilon > 0$, we can find a K such that $\forall|k| > K$, $|\int_a^b m(t)e^{ikt}dt| < \epsilon$.
- Combining the two statements ,$\forall|k| > K$, $|\int_a^b m(t)e^{ikt}dt| < \epsilon$, **and** $|\int_a^b e^{ikt}\Big[f(t) - m(t)\Big]dt| \leq \int_a^b |[M(t) - m(t)]|dt < \epsilon$, we find that $\forall|k| > K$, $\int_a^b f(t)e^{ikt} < 2\epsilon$. This is equivalent to $\lim_{|k|\to\infty} I(k) = 0$.