1 Nonlinear Modelling and Integrability

The most well known integrable equations in one space dimension are the following:

1. The Korteweg-deVries equation
$$q_t + q_{xxx} + 6qq_x = 0. \tag{1.1}$$

2. The nonlinear Schrödinger equation
$$iq_t + q_{xx} - 2\lambda|q|^2 q = 0, \qquad \lambda = \pm 1. \tag{1.2}$$

3. The sine-Gordon equation
$$q_{tt} - q_{xx} + \sin q = 0. \tag{1.3}$$

The linearized parts of the above equations are given respectively by

$$u_t + u_{xxx} = 0, \tag{1.1$'$}$$

$$iu_t + u_{xx} = 0, \tag{1.2$'$}$$

$$u_{tt} - u_{xx} + u = 0. \tag{1.3$'$}$$

Substituting in equations $(1.1)'$ - $(1.3)'$ the expression

$$u = e^{ikx - iw(k)t},$$

we find that $w(k)$ is given respectively by

$$w(k) = -k^3, \quad w(k) = k^2, \quad w(k)^2 = k^2 + 1. \tag{1.4}$$

Thus equations (1.1) - (1.3) combine some of the simplest types of dispersion with the simplest types of nonlinearity. In particular suppose that a general dispersion relation can be approximated by a Taylor series

$$w(k - k_0) = w(k_0) + kw'(k_0) + k^2 \frac{w''(k_0)}{2} + k^3 \frac{w'''(k_0)}{3!} + \cdots .$$

Equations (1.1) and (1.2) are the relevant approximations associated with odd and even k. Equation (1.4c) is the simplest canonical form involving w^2.

The KdV equation was derived in 1895 by Korteweg and deVries as an approximation to water waves. In this context $q(x,t)$ is related to the height of the water above the mean level. This work was motivated by the British experimentalist J. Scott Russell who first observed a soliton while riding on horseback beside a narrow barge channel. When the boat he was observing stopped, Russell (1844, p. 311) noted that it set forth "a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed ... Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon." Russell, impressed by this phenomenon challenged the theoreticians of the day to explain this discovery: "It now remained to the mathematicians to predict the discovery after it happened, that is to give an a priori demonstration a posteriori." This work created a controversy which, in fact, lasted almost 50 years. It was resolved by Korteweg-and deVries who were able to show that equation (1.1) supports a particular solution exhibiting the behavior described by Russell. This solution, which was later called the one-soliton solution, is given by

$$q_1(x - p^2 t) = \frac{p^2/2}{\cosh^2[\frac{1}{2}p(x - p^2 t) + c]}, \tag{1.5}$$

where p and c are constants. The velocity of this soliton is given by p^2, and its amplitude by $p^2/2$; thus, faster solitons are higher and narrower. It should be noted that q_1 is a *traveling wave* solution, i.e., q_1 depends only on the variable $X = x - p^2 t$, thus in this case the PDE (1.1) reduces (after integration) to the second-order ODE

$$-p^2 q_1(X) + 3q_1^2(X) + \frac{d^2 q_1}{dX^2}(X) = 0. \tag{1.6}$$

Under the assumption that q and dq/dX tend to zero as $|X| \to \infty$, this ODE yields the solution given by equation (1.4). The problem of finding a solution describing the interaction of two one-soliton solutions is much more difficult and was not addressed by Korteweg and deVries.

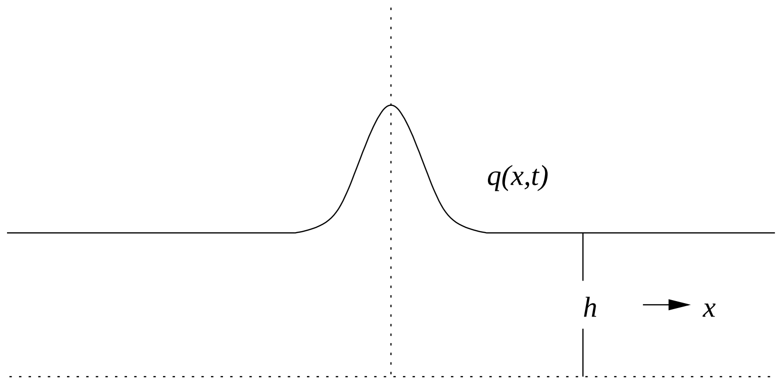


Figure 1 The modelling of water waves by the KdV.

The sine-Gordon equation appears in a variety of geometrical and physical applications. Its first appearance occurred in the study of the geometry of surfaces with constant negative Gaussian curvature. Its physical applications include, see Scott (1970):

- Josephson junction transmission lines, where $\sin\varphi$ is the Josephson current across an insulator between two superconductors (the voltage is proportional to φ_t).
- The propagation of waves in ferromagnetic materials.
- Laser pulses in two state media.
- Elementary particles, see Perring and Skyrme (1962).

The nonlinear Schrödinger equation plays a fundamental role in nonlinear optics. In addition it is a *generic* equation in the following sense. Let $u(x,t)$ satisfy a general nonlinear PDE whose linear part is dispersive; by looking for a solution in the form of an asymptotic expansion in the small parameter ε with the leading term a modulated wave, it can be shown that the amplitude of this wave satisfies equation (1.2).

2 The Burgers Equation

The simplest model combining the effects of weak nonlinearity and diffusion is the Burgers equation

$$u_t + uu_x = \varepsilon u_{xx}. \tag{2.1}$$

This equation appears in various physical applications. For example, it models weak shock waves in compressible fluid dynamics. It is distinguished among other nonlinear equations in that it can be linearized via an explicit transformation. Indeed, Hopf and Cole noted that the Burgers equation can be mapped to the heat equation via the transformation

$$u = -\frac{2\varepsilon v_x}{v} = -2\varepsilon(\ln v)_x : \tag{2.2}$$

$$u_t = \left(\varepsilon u_x - \tfrac{1}{2}u^2\right)_x \quad \text{or} \quad (\log v)_{tx} = \left(\frac{\varepsilon v_{xx}}{v} - \frac{\varepsilon v_x^2}{v^2} + \frac{\varepsilon v_x^2}{v^2}\right)_x$$

or after integrating (assuming for example that $v \to$ const as $x \to \infty$)

$$v_t = \varepsilon v_{xx}. \tag{2.3}$$

Let us consider an initial value problem for the Burgers equation, with

$$u(x,0) = u_0(x). \tag{2.4}$$

The associated initial value problem for the heat equation involves the initial condition

$$v(x,0) = v_0(x) = e^{-\int_0^x \frac{u_0(\eta)d\eta}{2\varepsilon}} \tag{2.5}$$

where we have used the equation $u_0(x) = -2\varepsilon v_{0_x}/v_0$ to express v_0 in terms of u_0. Solving the initial value problem of the heat equation we find

$$v(x,t) = \frac{1}{2\sqrt{\pi\varepsilon}} \int_{-\infty}^{\infty} v_0(\eta) e^{-\frac{(x-\eta)^2}{4\varepsilon t}} d\eta. \tag{2.6}$$

Equation (2.2) implies

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-\eta}{t} e^{-\frac{G}{2\varepsilon}} d\eta}{\int_{-\infty}^{\infty} e^{-\frac{G}{2\varepsilon}} d\eta}, \quad G(\eta; x, t) \equiv \int_0^{\eta} u_0(\eta')d\eta' + \frac{(x-\eta)^2}{2t}. \tag{2.7}$$

In the physical application of shock waves in fluids, ε has the meaning of viscosity. Here we are interested in the inviscid limit, that is, $\varepsilon \to 0$, of the Burgers equation.

As $\varepsilon \to 0$ we use Laplace's method to evaluate the dominant contributions to the integrals appearing in equation (2.7). To achieve this, we need to find the points for which $\partial G/\partial\eta = 0$,

$$\frac{\partial G}{\partial \eta} = u_0(\eta) - \frac{x-\eta}{t}. \tag{2.8}$$

Let $\eta = \xi(x,t)$ be such a point; that is, $\xi(x,t)$ is a solution of

$$x = \xi + u_0(\xi)t. \tag{2.9}$$

Then Laplace's method implies

$$\int_{-\infty}^{\infty} \frac{x-\eta}{t} e^{-\frac{G}{2\varepsilon}} d\eta \sim \frac{x-\xi}{t} \sqrt{\frac{4\pi\varepsilon}{|G''(\xi)|}} e^{-\frac{G(\xi)}{2\varepsilon}},$$

$$\int_{-\infty}^{\infty} e^{-\frac{G}{2\varepsilon}} d\eta \sim \sqrt{\frac{4\pi\varepsilon}{|G''(\xi)|}} e^{-\frac{G(\xi)}{2\varepsilon}}.$$

Indeed, consider the integral

$$I = \int_{-\infty}^{\infty} f(t) e^{-k\phi(t)} dt,$$

containing the *large* parameter k. As $k \to \infty$ we expect that the main contribution comes from the neighborhood of the minimum of $\phi(t)$. Let us denote this minimum by c. Then

$$I \sim \int_{c-R}^{c+R} f(c) e^{-k\left[\phi(c) + \frac{(t-c)^2}{2}\phi''(c)\right]} dt,$$

where R is small but finite. To evaluate this integral, we let

$$\tau = \sqrt{\frac{k}{2}\phi''(c)}(t-c).$$

Thus

$$I \sim \frac{e^{-k\phi(c)} f(c)}{\sqrt{\frac{k}{2}\phi''(c)}} \int_{-R\sqrt{\frac{k}{2}\phi''(c)}}^{R\sqrt{\frac{k}{2}\phi''(c)}} e^{-\tau^2} d\tau.$$

As $k \to \infty$ the above integral becomes $\sqrt{\pi}$, thus

$$\int_{-\infty}^{\infty} f(t) e^{-k\phi(t)} dt \sim e^{-k\phi(c)} f(c) \sqrt{\frac{2\pi}{k\phi''(c)}}, \quad k \to \infty.$$

Hence, if equation (2.9) for a given u_0 has only one solution for ξ, then

$$u(x,t) \sim \frac{x-\xi}{t} = u_0(\xi) \tag{2.10}$$

where ξ is defined by equation (2.9). Equation (2.10) has a simple interpretation: Consider the problem

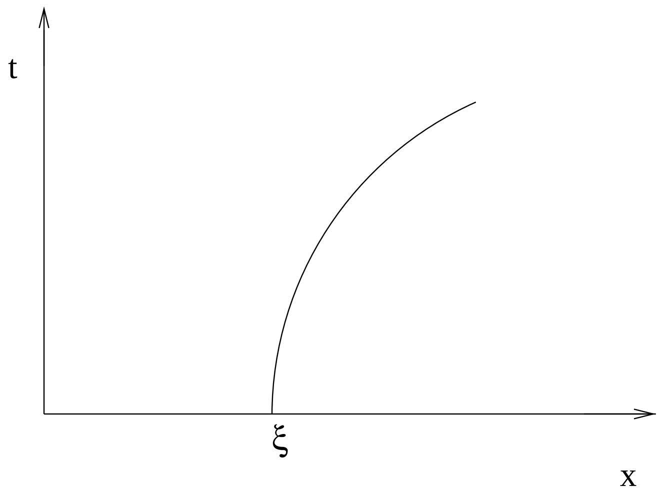
$$\rho_t + \rho\rho_x = 0, \quad \rho(x,0) = u_0(x). \tag{2.11}$$

Equation (2.11) is a first-order hyperbolic equation and can be solved by the method of characteristics

$$\frac{d\rho}{dt} = 0,$$

where on the characteristic $\xi(x,t)$,

$$\frac{dx}{dt} = \rho.$$



On the characteristic ρ is constant and at the point $x = \xi, \quad t = 0$, has the value $u(\xi, 0)$. Hence

$$\rho(x,t) = u_0(\xi) \tag{2.12}$$

where $\xi(x,t)$ is defined by $x = \xi + u_0(\xi)t$, provided that "breaking" does not occur, that is, the characteristics do not cross or, equivalently, provided that equation (2.9) has a single solution.

The above analysis shows that for appropriate u_0, the limit of the solution of the Burgers equation is given by the solution of the limit equation (2.11). However, the relationship between the Burgers equation and equation (2.11) must be further clarified. Indeed, for some $u_0(x)$, equation (2.11) gives multivalued solutions (after the characteristics cross), while the solution (2.7) is always single valued. This means that, out of all the possible ("weak") solutions that equation (2.11) can support, there exists a unique solution that is the correct limit of the Burgers equation as $\varepsilon \to 0$. It is interesting that Laplace's method provides us with a way of picking this correct solution: When the characteristics of equation (2.11) cross, equation (2.9) admits two solutions, which we denote by ξ_1 and ξ_2 with $\xi_1 > \xi_2$ (both ξ_1 and ξ_2 yield the same values of x and t from equation (2.9). Laplace's method shows that for the sum of these contributions, using equation (2.7)

$$u(x,t) \sim \frac{u_0(\xi_1)|G''(\xi_1)|^{-\frac{1}{2}} e^{-\frac{G(\xi_1)}{2\varepsilon}} + u_0(\xi_2)|G''(\xi_2)|^{-\frac{1}{2}} e^{-\frac{G(\xi_2)}{2\varepsilon}}}{|G''(\xi_1)|^{-\frac{1}{2}} e^{-\frac{G(\xi_1)}{2\varepsilon}} + |G''(\xi_2)|^{-\frac{1}{2}} e^{-\frac{G(\xi_2)}{2\varepsilon}}}. \tag{2.13}$$

Hence, owing to the dominance of exponentials

$$\begin{aligned} u(x,t) &\sim u_0(\xi_1) \quad \text{for} \quad G(\xi_1) < G(\xi_2); \\ u(x,t) &\sim u_0(\xi_2) \quad \text{for} \quad G(\xi_1) > G(\xi_2); \end{aligned} \tag{2.14}$$

The changeover will occur at those (x,t) for which $G(\xi_1) = G(\xi_2)$, or using the definition of G we find

$$\int_0^{\xi} u_0(\eta')d\eta' + \frac{(x-\xi_1)^2}{2t} = \int_0^{\xi} u_0(\eta')d\eta' + \frac{(x-\xi_2)^2}{2t},$$

or

$$\frac{(x-\xi_1)^2}{2t} - \frac{(x-\xi_2)^2}{2t} = -\int_{\xi_2}^{\xi_1} u_0(\eta)d\eta,$$

or

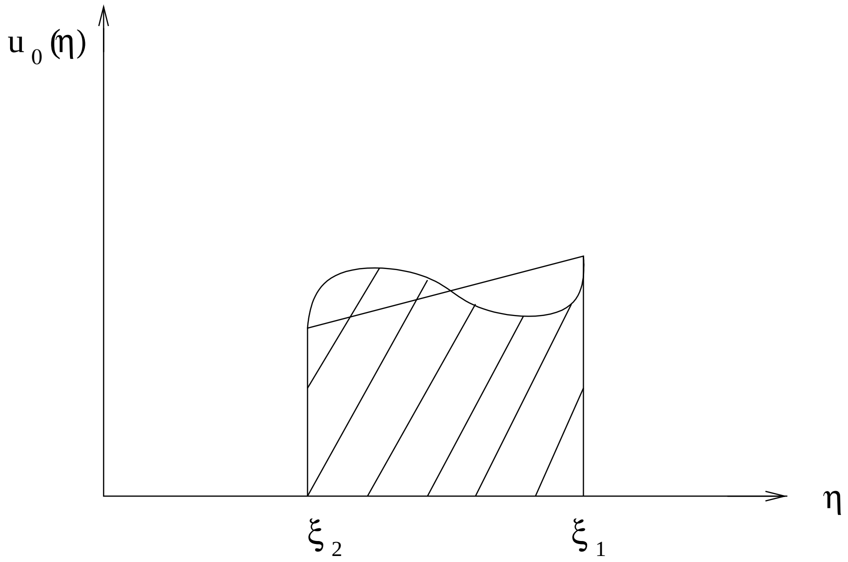
$$(\xi_1 - \xi_2)\left(-\frac{x}{t} + \frac{(\xi_1+\xi_2)}{2t}\right) = -\int_{\xi_2}^{\xi_1} u_0(\eta)d\eta.$$

Using equation (2.9) for ξ_1 and ξ_2, and summing, we find that

$$-\frac{x}{t} + \frac{\xi_1 + \xi_2}{2t} = -\tfrac{1}{2}(u_0(\xi_1) + u_0(\xi_2)),$$

hence

$$\tfrac{1}{2}(u_0(\xi_1) + u_0(\xi_2))(\xi_1 - \xi_2) = \int_{\xi_2}^{\xi_1} u_0(\eta)d\eta. \tag{2.15}$$



Equations (2.14) with (2.9) show that at $\varepsilon \to 0$ the changeover in the behavior of $u(x,t)$ leads to a discontinuity. In this way the solution of the Burgers equation tends to a *shock wave* as $\varepsilon \to 0$. This solution is that particular solution of the limiting equation (2.11) that satisfies the *shock condition* (2.15).

3 Riemann-Hilbert Problems and Singular Integral Equations

It is remarkable that a large number of diverse problems of physical and mathematical significance involve the solution of the so-called Riemann-Hilbert (RH) problem. Let us mention a few such problems.

(1) Find a function $w(z) = u(x,y) + iv(x,y)$, u, v real, which is analytic inside a region enclosed by a contour C, such that

$$\alpha(t)u(t) + \beta(t)v(t) = \gamma(t), \quad t \text{ on } C \tag{3.1}$$

where α, β, and γ are given, real functions. In the special case of $\alpha = 1$, $\beta = 0$, C a circle, this problem reduces to deriving the well-known Poisson formula.

(2) Solve the linear singular integral equation,

$$f(t) + \oint_a^b \frac{\alpha(t')}{t' - t} f(t')dt' = \beta(t) \tag{3.2}$$

where $\alpha(t)$ and $\beta(t)$ are given functions and $\oint$ denotes a principal value integral. Such equations arise in many applications. For example, the equation $\oint_a^b \frac{f(t')dt'}{t'-t} = \beta$, plays an important role in airfoil theory.

(3) Solve the linear integral equation

$$f(t) + \int_0^\infty \alpha(t-t')f(t')dt' = \beta(t), \quad t > 0 \tag{3.3}$$

where α and β are given, integrable functions.

(4) Solve the time independent wave equation (Helmholtz equation)

$$\varphi_{xx} + \varphi_{yy} + k^2\varphi = 0, \quad -\infty < x < \infty, \quad y \geq 0, \quad k \ \text{ real } \ \text{constant}, \tag{3.4}$$

where $\varphi(x,0) = f(x)$ for $-\infty < x \leq 0$, $(\partial\varphi/\partial y)(x,0) = g(x)$ for $0 < x < \infty$, and φ satisfies an appropriate boundary ("radiation") condition at infinity.

(5) Derive the inverse Radon transform and the inverse attenuated Radon transform. These transforms play a fundamental role in the mathematical foundation of computerized tomography.

(6) Solve an inverse scattering problem associated with the time-independent Schrödinger equation

$$\psi_{xx} + (q(x) + k^2)\psi = 0, \quad -\infty < x < \infty \tag{3.5}$$

that is, reconstruct the potential $q(x)$ from appropriate scattering data. Inverse problems arise in many areas of applications, for example, geophysics, image reconstruction, quantum mechanics, etc. In many cases, they can be solved using Riemann-Hilbert problems.

(7) Solve the following initial value problem for the Korteweg-deVries (KdV) equation

$$\begin{gathered} q_t + q_{xxx} + 6qq_x = 0, \ -\infty < x < \infty, \quad t > 0 \\ q(x,0) = q_0(x); \ q \to 0 \quad \text{as} \quad |x| \to \infty. \end{gathered} \tag{3.6}$$

Many other nonlinear PDEs as well as many nonlinear ODEs can also be related to RH problems.

The above list is by no means exhaustive. Several aspects of RH theory were motivated and developed owing to the relation of RH problems with problems arising in physical application, for example, elasticity and hydrodynamics (Freund, 1990; Gakhov, 1966; Muskhelishvili, 1977).

Problems 1-5 above are associated with *scalar* RH problems. The simplest such problem involves finding two analytic functions $\mathbf{\Phi}^+(z)$ and $\mathbf{\Phi}^-(z)$, defined inside and outside a closed contour C of the complex z plane such that

$$\mathbf{\Phi}^+(t) - g(t)\mathbf{\Phi}^-(t) = f(t), \quad t \ \text{ on } \ C \tag{3.7}$$

for given functions $g(t)$ and $f(t)$. This problem can be solved in closed form. Its solution is intimately related to the Cauchy type integral

$$\mathbf{\Phi}(z) = \frac{1}{2\pi}\int_C \frac{\varphi(t)dt}{z-t} \tag{3.8}$$

where φ is a certain function related to f and g. A generalization of the above problem allows C to be an open contour; this problem can also be solved in closed form.

Problem 1 above was first formulated by Riemann in 1851. In 1904, Hilbert reduced this problem to a RH problem of the form (3.7), which he also expressed in terms of a singular integral equation of the form (3.2). In 1908, Plemelj gave the first closed form solution of a simple RH problem (an RH problem of "zero index," see below). The closed form solution of a general scalar RH problem was given by Gakhov (1938). Integral equations of the form (3.3) were studied by Carleman, who solved such an equation in 1932 using a method similar to the so-called Wiener-Hopf method. This method, introduced originally in 1931, was also in connection with the solution of a particular integral equation of the type (3.3). The Wiener-Hopf method, which also can be used for the solution of Problem 4, actually reduces to solving a certain RH problem. (The interested reader can find relevant references in the books of Gakhov (1966) and Muskhelishvili (1977).) The derivation of transforms, such as the Radon transform, via RH techniques appears to be rather recent (Fokas and Novikov, 1991).

Problems 6 and 7 are associated with *vector* RH problems. The formulation of such problems is similar to that for scalar ones, but $\mathbf{\Phi}^+$ and $\mathbf{\Phi}^-$ are now vectors instead of scalars. Unfortunately, in general, vector RH problems cannot be solved in closed form; their solution can be given in terms of linear integral equations of Fredholm type.

There exists a significant generalization of the RH problem that is called a $\bar{\partial}$ (DBAR) problem. This problem involves solving the equation

$$\frac{\partial \mathbf{\Phi}(x,y)}{\partial \bar{z}} = f(x,y), \quad z \in D, \quad z = x + iy \tag{3.9}$$

for $\mathbf{\Phi}(x,y)$, where g is given, D is some domain in the complex z plane, and $\bar{z}$ is the complex conjugate of z. To appreciate the relationship between $\bar{\partial}$ and RH problems, it is convenient to consider the particular RH problem

$$\mathbf{\Phi}^+(x) - \mathbf{\Phi}^-(x) = f(x), \quad -\infty < x < \infty \tag{3.10}$$

where $\mathbf{\Phi}^+(z)$ and $\mathbf{\Phi}^-(z)$ are analytic in the upper and lower half complex z plane. Let $\mathbf{\Phi}(z) = \mathbf{\Phi}^+(z)$ for $y > 0$ and $\mathbf{\Phi}(z) = \mathbf{\Phi}^-(z)$ for $y < 0$. Solving the RH problem (3.10) means finding a function $\mathbf{\Phi}(z)$ that is analytic in the entire z complex plane except on the real axis, where it has a prescribed "jump." The quantity $\partial\mathbf{\Phi}/\partial\bar{z}$ measures the departure of $\mathbf{\Phi}$ from analyticity; if $\mathbf{\Phi}(z)$ is analytic, then $\partial\mathbf{\Phi}/\partial\bar{z} = 0$. But for equation (3.10), $\partial\mathbf{\Phi}/\partial\bar{z}$ vanishes everywhere in the z complex plane *except* on the real axis, where it is given by $f(x)\delta(y)$ and where $\delta(y)$ denotes the Dirac delta function. Thus the RH problem (3.10) can be viewed as a special case of a DBAR problem where $f(x,y) = f(x)\delta(y)$.

The solution of the DBAR problem is based on the following formula

$$\Phi(z,\bar{z}) = \frac{1}{2i\pi}\int_{\partial D} \frac{\Phi(\zeta,\bar{\zeta})}{\zeta - z} d\zeta - \frac{1}{\pi}\int\int_D \frac{\frac{\partial\Phi}{\partial\bar{\zeta}} d\xi d\eta}{\zeta - z}, \quad \zeta = \xi + i\eta. \tag{3.11}$$

This formula appears with several different names in the literature which include the "Cauchy-Green formula," and the "Borel formula". The first name refers to the occurrence in equation (3.11) of the Cauchy integral and to the fact that equation (3.11) *can* be derived from Green's theorem. The second name is probably related to the fact that Borel presented this formula (giving full credit to Pompeiu) at the fifth international congress of mathematicians, at Cambridge in 1913. It appears that a more appropriate name is the "Pompeiu formula," since it was first derived in 1909 (without the boundary term) and fully in 1912 by Pompeiu in connection with a question posed by Painlevé in 1897.

3.1 Cauchy Type Integrals

Consider the integral

$$\mathbf{\Phi}(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - z} d\tau \tag{3.1.1}$$

where L is a smooth curve (L may be an arc or a closed contour) and $\varphi(\tau)$ is a function satisfying the *Hölder condition* on L, that is for any two points τ and τ_1 on L

$$|\varphi(\tau) - \varphi(\tau_1)| \leq \wedge|\tau - \tau_1|^\lambda, \quad \wedge > 0, \quad 0 < \lambda \leq 1. \tag{3.1.2}$$

If $\lambda = 1$, the Hölder condition becomes the so-called Lipschitz condition. For example, a differentiable function $\varphi(\tau)$ satisfies the Hölder (Lipschitz) condition with $\lambda = 1$. (This follows from the definition of a derivative.) If $\lambda > 1$ on L, it follows, from the definition of the derivative, that $d\varphi/d\tau = 0$ and hence $\varphi =$ const on L, which is a trivial case. The integral (3.1.1) is well defined and $\mathbf{\Phi}(z)$ is analytic provided that z is not on L. We also note that from the series expansion, as $|z| \to \infty$ off L, we have $\mathbf{\Phi}(z) \sim c/z$ where $c = -\frac{1}{2\pi i}\int_L \varphi(\tau)d\tau$. However, if z is on L, this integral becomes ambiguous; to give it a unique meaning we must know how z approaches L. We denote by $+$ the region that is on the left of the positive direction of L and by $-$ the region

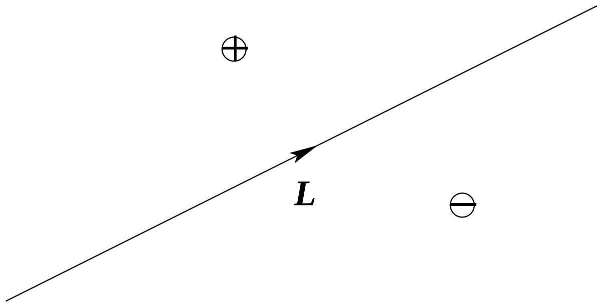


Figure 3.1.1. Regions on either side of L

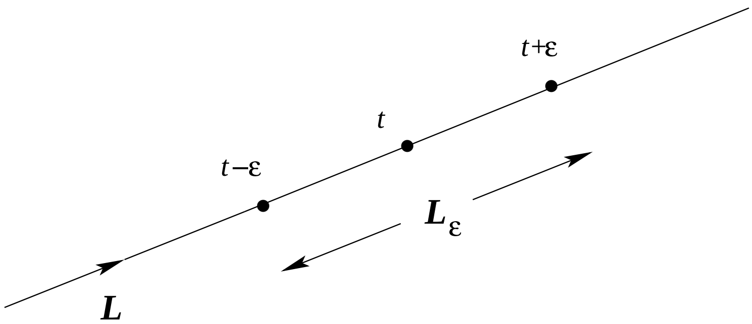


Figure 3.1.2. Definition of L_ε

on the right (see Figure (3.1.1). It turns out that $\mathbf{\Phi}(z)$ has a limit $\mathbf{\Phi}^+(t)$, t on L, when z approaches L along a curve entirely in the $+$ region. Similarly, $\mathbf{\Phi}(z)$ has a limit $\mathbf{\Phi}^-(t)$, when z approaches L along a curve entirely in the $-$ region. These limits, which play a fundamental role in the theory of RH problems, are given by the so-called Plemelj formulae.

Plemelj Formulae Let L be a smooth contour (closed or open) and let $\varphi(\tau)$ satisfy a Hölder condition on L. Then the Cauchy type integral $\mathbf{\Phi}(z)$, defined in equation (3.1.1), has the limiting values $\mathbf{\Phi}^+(t)$ and $\mathbf{\Phi}^-(t)$ as z approaches L from the left and the right, respectively, and t is not an endpoint of L. These limits are given by

$$\mathbf{\Phi}^\pm(t) = \pm\tfrac{1}{2}\varphi(t) + \frac{1}{2\pi i}\fint_L \frac{\varphi(\tau)}{\tau - t}d\tau. \tag{3.1.3$^\pm$}$$

In these equations, $\fint$ denotes the principal value integral defined by

$$\fint_L \frac{\varphi(\tau)d\tau}{\tau - t} = \lim_{\varepsilon\to 0}\int_{L-L_\varepsilon} \frac{\varphi(\tau)d\tau}{\tau - t} \tag{3.1.4}$$

where L_ε is the part of L that has length 2ε and is centered around t, as depicted in Figure 3.1.2.

In the above formulation we have assumed that L is a finite contour, otherwise $\varphi(\tau)$ must satisfy an additional condition. Suppose, for example, that L is the real axis; then we assume that $\varphi(\tau)$ satisfies a Hölder condition for all finite τ, and that as $t \to \pm\infty$, $\varphi(\tau) \to \varphi(\infty)$, where

$$|\varphi(\tau) - \varphi(\infty)| < \frac{M}{|\tau|^\mu}, \quad M > 0, \quad \mu > 0. \tag{3.1.5}$$

Equations (3.1.3) are equivalent to

$$\mathbf{\Phi}^+(t) - \mathbf{\Phi}^-(t) = \varphi(t), \quad \mathbf{\Phi}^+(t) + \mathbf{\Phi}^-(t) = \frac{1}{\pi i} \oint \frac{\varphi(\tau)}{\tau - t} d\tau. \tag{3.1.6}$$

The function $\mathbf{\Phi}(z)$ is said to be *sectionally analytic*; functions that are the boundary values of $\mathbf{\Phi}(z)$ as $z \to L$ from the left and the right will sometimes be referred to as $\oplus$ and $\ominus$ functions.

Example 3.1.1 Find $\mathbf{\Phi}^\pm(t)$ corresponding to $\mathbf{\Phi}(z) = \frac{1}{2\pi i} \oint_C (\tau + \frac{1}{\tau}) \frac{d\tau}{\tau - z}$, where C is the unit circle (see Figure 3.1.3).

To compute $\mathbf{\Phi}^+(z)$, we consider $\mathbf{\Phi}(z)$ with z inside the circle, thus using contour integration $\mathbf{\Phi}^+(z) = z + \frac{1}{z} - \frac{1}{z} = z$. Similarly, to compute $\mathbf{\Phi}^-(z)$ we consider $\mathbf{\Phi}(z)$ with z outside the circle and use contour integration to find $\mathbf{\Phi}^-(z) = -\frac{1}{z}$. Therefore on the contour $z = t$

$$\mathbf{\Phi}^+(t) = t, \quad \mathbf{\Phi}^-(t) = -\frac{1}{t}.$$

Also, using contour integration, it follows that $\frac{1}{i\pi} \oint (\tau + \frac{1}{\tau}) \frac{d\tau}{\tau - t} = (t - \frac{1}{t})$; therefore equations (3.1.6) are verified. We note that $\mathbf{\Phi}^+(z)$ is indeed analytic inside the unit circle, while $\mathbf{\Phi}^-(z)$ is analytic outside the unit circle. Taking this into consideration, as well as that $\mathbf{\Phi}^+(t) - \mathbf{\Phi}^-(t) = \varphi(t)$, it follows that in this simple example, $\mathbf{\Phi}^+(z) = z$ and $\mathbf{\Phi}^-(z) = -1/z$ could have been found by inspection.

Example 3.1.2 Find $\mathbf{\Phi}^\pm(t)$ corresponding to $\mathbf{\Phi}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{2}{\tau^2 + 1} \frac{d\tau}{\tau - z}$.

We split $2/(t^2 + 1)$ as follows:

$$\frac{2}{t^2 + 1} = \frac{i}{t + i} - \frac{i}{t - i}.$$

Furthermore, $\frac{i}{(z-i)}$ is analytic in the lower half plane, while $\frac{i}{(z+i)}$ is analytic in the upper half plane. Hence this suggests that

$$\mathbf{\Phi}^+(t) = \frac{i}{t + i} \quad \mathbf{\Phi}^-(t) = \frac{i}{t - i}.$$

These formulae can be verified by contour integration. For example, computing $\mathbf{\Phi}(z)$ with z in the upper half plane (we can consider a large semicircular contour in the lower half plane), we find $\mathbf{\Phi}^+(z) = \frac{i}{(z+i)}$.

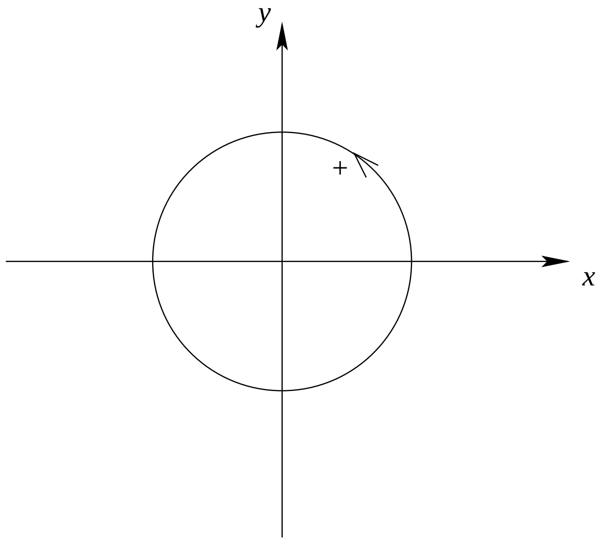


Figure 3.1.3 Inside and outside unit circle C

3.2 Scalar Riemann-Hilbert Problems

The machinery introduced in §3.1, namely the formulae which express the behavior of a Cauchy integral as z approaches any point on the contour, will now be used to solve any scalar RH problem. We first introduce some definitions.

(1) Let C be a simple smooth, closed contour dividing the complex z-plane into two regions D^+ and D^-, where the positive direction of C will be taken as that for which D^+ is on the left (see Figure 3.2.1). A scalar function $\mathbf{\Phi}(z)$ defined in the entire plane, except for points on C, will be called *sectionally analytic* if:

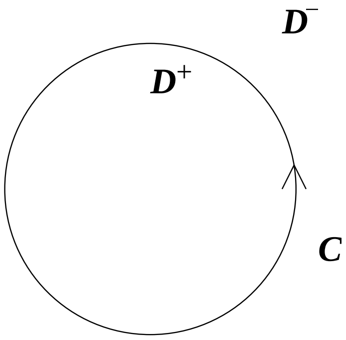


Figure 3.2.1. Simple closed contour C and the "+", "−" regions

(a) the function $\mathbf{\Phi}(z)$ is analytic in each of the regions D^+ and D^- except, perhaps, at $z = \infty$, and (b) the function $\mathbf{\Phi}(z)$ is sectionally continuous with respect to C, i.e., as z approaches any point t on C along any path which lies wholly in either D^+ or D^-, the function $\mathbf{\Phi}(z)$ approaches a definite limiting value $\mathbf{\Phi}^+(t)$ or $\mathbf{\Phi}^-(t)$ respectively.

It then follows that $\mathbf{\Phi}(z)$ is continuous in the closed region $D^+ + C$ if it is assigned the value $\mathbf{\Phi}^+(t)$ on C. A similar statement applies for the region $D^- + C$.

(2) The sectionally analytic function $\mathbf{\Phi}(z)$ has a finite degree at infinity if for some finite integer m, $\lim_{z\to\infty} \mathbf{\Phi}(z)/|z|^m = 0$. The function $\mathbf{\Phi}(z)$ is said to have degree k at infinity if

$$\mathbf{\Phi}(z) \sim c_k z^k + O(z^{k-1}) \quad \text{as} \quad z \to \infty, \quad c_k \quad \text{a nonzero constant.} \tag{3.2.1}$$

(3) The index of a function $\varphi(t)$ with respect to C is the increment of its argument in transversing a curve C in the positive direction, divided by 2π, that is

$$ind\varphi(t) := \frac{1}{2\pi}\left[arg\varphi(t)\right]_C = \frac{1}{2\pi i}\left[\ln\varphi(t)\right]_C = \frac{1}{2\pi i}\int_C d\left(\ln\varphi(t)\right). \tag{3.2.2}$$

Example 3.2.1 Show that if the function $\varphi(t)$ is analytic inside the contour C except at a finite number of points where it may have poles, then its index equals to the difference between the number of zeros and the number of poles inside the contour.

If $\varphi(t)$ is differentiable, then equation (3.2.2) yields

$$ind\varphi(t) = \frac{1}{2\pi i}\int_C \frac{\varphi'(t)}{\varphi(t)}dt = N - P \tag{3.2.3}$$

where N and P are the number of zeroes and poles of $\varphi(t)$, respectively, and where a multiple zero or pole is counted according to its multiplicity.

To illustrate this result, let us compute the index of $\varphi(t) = t^n$ where C is an arbitrary contour enclosing the origin. Because t^n has a zero of order n inside the contour, it follows that $indt^n = n$.

The scalar *homogeneous* RH problem for closed contours is formulated as follows: Given a closed contour C and a function $g(t)$ which is Hölder on C with $g(t) \neq 0$ on C, find a sectionally analytic function $\mathbf{\Phi}(z)$, with finite degree at infinity, such that

$$\mathbf{\Phi}^+(t) = g(t)\mathbf{\Phi}^-(t) \quad \text{on} \quad C, \tag{3.2.4}$$

where $\mathbf{\Phi}^{\pm}(t)$ are the boundary values of $\mathbf{\Phi}(z)$ on C. We assume that the index of $g(t)$ is k. Here we only consider RH problems in a simply connected region. The extension to a multiply connected region can be dealt with by simple methods, cf. Gakhov (1966).

The solution of this RH problem with degree m at infinity is given by

$$\mathbf{\Phi}(z) = X(z)P_{m+k}(z), \tag{3.2.5}$$

where $P_{m+k}(z)$ is a polynomial of degree $m+k$ and $X(z)$, called the fundamental solution of (3.2.4), is given by

$$X(z) \equiv \begin{bmatrix} e^{\Gamma(z)}, & z \text{ in } D^+ \\ z^{-k}e^{\Gamma(z)}, & z \text{ in } D^- \end{bmatrix} \tag{3.2.6}$$

$$\Gamma(z) \equiv \frac{1}{2\pi i}\int_C \frac{d\tau \ln(\tau^{-k}g(\tau))}{\tau - z}, \tag{3.2.7}$$

with $k = indg(t)$ on C. We assume C encloses the origin so that $ind(\tau^{-k}) = -k$.

Note that this definition of $X(z)$ implies the normalization $X^-(z) \sim z^{-k}$, $z \to \infty$. One can modify this normalization as needed in a given problem; for example, $X(z)$ can be multiplied by a constant.

These results follow from a straightforward application of the Plemelj formulae. However, care must be taken to ensure that the function appearing in the integrand of the relevant Cauchy integral used in the Plemelj formulae satisfies a Hölder condition. The index of $g(t)$ is k, and as such, $\log g(t)$ does not satisfy the Hölder condition and the arguments leading to the Plemelj formulae fail. To remedy this, we note that the index of $t^{-k}g(t)$ is zero, and hence $\log(t^{-k}g(t))$ satisfies a Hölder condition. This suggests rewriting equation (3.2.4) in the form

$$\mathbf{\Phi}^+(t) = \left(t^{-k}g(t)\right) t^k \mathbf{\Phi}^-(t),$$

or

$$\ln \mathbf{\Phi}^+(t) - \ln\left(t^k \mathbf{\Phi}^-(t)\right) = \ln\left(t^{-k}g(t)\right).$$

The representation (3.2.7) for $\Gamma(z)$ follows from the Plemelj formulae. Equation (3.2.6) follows from the fact that $X(z) \sim z^{-k}$ as $z \to \infty$ (recall that $\Gamma(z) \sim O(1/z)$ as $z \to \infty$). Note that $X^+(z)$ is nonvanishing in D^+ and $X^-(z)$ is nonvanishing in D^- except perhaps as $z \to \infty$.

Thus the solution of equation (3.2.4) that satisfies the requisite condition at infinity is therefore given by $\mathbf{\Phi}(z) = X(z)P_{m+k}(z)$. Note that multiplying the fundamental solution $X(z)$ by a polynomial that is analytic for all z has no effect on equation (3.2.4).

RH problems are closely related to singular integral equations. For this and other reasons one is interested in finding all solutions of (3.2.4) which vanish at infinity. Equation (3.2.5) implies that:

(a) If $k > 0$ then there exist k linearly independent solutions of (3.2.4) vanishing at infinity. This follows from the fact that as $z \to \infty$, $\Phi(z) \sim z^{-k}P_{m+k}(z)$, and for decaying solutions we require $m = -1$. The polynomial $P_{k-1}(z) = A_0 + A_1 z + A_2 z^2 + \cdots + A_{k-1}z^{k-1}$ has k arbitrary constants.

(b) If $k \leq 0$ then there exists no solution of (3.2.4) vanishing at infinity. Stated differently, the fundamental solution $X(z)$ grows algebraically at infinity for $k < 0$ or is bounded at infinity for $k = 0$. Consequently, to have a vanishing solution we must take $P_{m+k}(z) = 0$ and we only have the trivial solution.

The so-called *inhomogeneous* RH problem differs from the homogeneous one in that equation (3.2.4) is replaced by

$$\mathbf{\Phi}^+(t) = g(t)\mathbf{\Phi}^-(t) + f(t), \quad t \text{ on } C, \tag{3.2.8}$$

where $f(t)$ is also Hölder on C. The solution of this problem is given by

$$\mathbf{\Phi}(z) = X(z)\left[P_{m+k}(z) + \mathbf{\Psi}(z)\right], \quad \mathbf{\Psi}(z) \equiv \frac{1}{2\pi i}\int_C \frac{f(\tau)d\tau}{X^+(\tau)(\tau - z)}, \tag{3.2.9}$$

where $X(z)$ is given by equation (3.2.6).

To derive equation (3.2.9) we rewrite $g(t)$ as $X^+(t)/X^-(t)$, hence equation (3.2.8) becomes

$$\frac{\mathbf{\Phi}^+(t)}{X^+(t)} - \frac{\mathbf{\Phi}^-(t)}{X^-(t)} = \frac{f(t)}{X^+(t)}.$$

Then a special solution $\frac{\mathbf{\Phi}}{X}(z) = \mathbf{\Psi}(z)$ is obtained from the Plemelj formula, and the equation for $\mathbf{\Phi}(z)$ given by equation (3.2.9) follows.

Again (thinking ahead to applications) it is useful to find the solutions vanishing at infinity. Because for large z, $X(z) = O(z^{-k})$ and $\mathbf{\Psi}(z) = O(z^{-1})$, it follows that:

(a) If $k > 0$ then there exist k linearly independent solutions given by (3.2.9) with $m = -1$. (This follows by the same argument as discussed in part (a) of the homogeneous problem, above.)

(b) If $k = 0$ then there exists a unique solution $X(z)\mathbf{\Psi}(z)$; here we need to take $P_{m+k}(z) = 0$.

(c) If $k < 0$ then there exists a unique solution $X(z)\mathbf{\Psi}(z)$ if and only if the orthogonality conditions

$$\int_C \frac{f(\tau)\tau^{n-1}}{X^+(\tau)}d\tau, \quad n = 1, 2, \cdots, -k, \tag{3.2.10}$$

hold. As in (b), we need to take $P_{m+k}(z) = 0$. These orthogonality conditions follow from the asymptotic expansion of $\mathbf{\Psi}(z)$ for large z and the requirement that $\mathbf{\Psi}(z) \sim O(z^{-|k|-1})$ as $z \to \infty$, because $X(z) \sim z^{|k|}$.

Note that

$$\frac{1}{2\pi i} \int_C \frac{f(\tau)}{X+(\tau)} \frac{1}{\tau - z} d\tau$$

$$= \frac{1}{2\pi i} \frac{(-1)}{z} \int_C \frac{f(\tau)}{X+(\tau)} \frac{d\tau}{(1-\tau/z)}$$

$$\sim \frac{-1}{2\pi i} \int_C \frac{f(\tau)}{X+(\tau)} \left(\frac{1}{z} + \frac{\tau}{z^2} + \frac{\tau^2}{z^3} + \cdots + \frac{\tau^{n-1}}{z^n} + \cdots \right) d\tau$$

The behavior of $\mathbf{\Psi}(z)$ and $X(z)$ is such that all coefficients of z^{-n}, $n = 1, 2, \cdots, |k|$ must vanish in order for $\mathbf{\Phi}(z) \to 0$ as $z \to \infty$.

Example 3.2.2 Solve the RH problem (3.2.8) with $g(t) = t/(t^2 - 1)$, $f(t) = (t^3 - t^2 + 1)/(t^2 - t)$, C encloses the points $0, 1, -1$, and $\Phi(z)$ vanishes at infinity.

Since $g(t)$ is analytic inside C and it has one zero and two poles, it follows that $k = ind g = -1$. The fundamental solution $X(z)$ satisfies (recall $X^-(z) \to z^{-k}$ as $z \to \infty$),

$$X^+(t) = \frac{t}{(t-1)(t+1)} X^-(t), \quad X^-(z) \to z \ \text{ as } \ z \to \infty. \tag{3.2.11}$$

The solution of (3.2.11) can be found by inspection: since $t/(t-1)(t+1)$ is analytic outside C, it follows that the RHS of (3.2.11) is a $\ominus$ function; thus using the boundary condition at infinity it follows that

$$X^+(t) = \frac{t}{(t-1)(t+1)} X^-(t) = 1,$$

or

$$X^+(t) = 1, \quad X^-(t) = \frac{t^2 - 1}{t}.$$

To compute $\mathbf{\Psi}(z)$ of equation (3.2.9), we need to evaluate the Cauchy integral associated with $f(t)/X^+(t) = t + 1/[t(t-1)]$. This can also be done by inspection: t is analytic in D^+, while $1/[t(t-1)]$ is analytic in D^-, hence the Plemelj formulae yield

$$\mathbf{\Psi}^+(z) = z, \quad \mathbf{\Psi}^-(z) = -\frac{1}{z(z-1)}.$$

Of course these formulae can be verified by contour integration; for example, if $z \epsilon D^+$

$$\mathbf{\Psi}^+(z) = \frac{1}{2\pi i} \int_C \left(\tau + \frac{1}{\tau(\tau - 1)} \right) \frac{d\tau}{\tau - z} = z.$$

A solution of this RH problem exists only if the orthogonality condition (3.2.10), is satisfied. Evaluating (3.2.10) with $n = 1$, $X^+ = 1$, $f(t) = t + \frac{1}{t(t-1)}$ we find

$$\int_C \left(\tau + \frac{1}{\tau(\tau - 1)} \right) d\tau = 0.$$

Thus the above inhomogeneous RH problem has the unique solution

$$\mathbf{\Phi}^+(z) = z, \quad \mathbf{\Phi}^-(z) = -\frac{z+1}{z^2}.$$

3.3 Singular Integral Equations

We will now show that the solution of scalar RH problems provides an effective way of solving certain singular integral equations. We note the following equivalence: Solving the singular integral equation

$$a(t)\varphi(t) + \frac{b(t)}{\pi i} \oint_L \frac{\varphi(\tau)}{\tau - t} d\tau = c(t) \tag{3.3.1}$$

where $a(t)$, $b(t)$, $c(t)$ satisfy the Hölder condition on L with $a \pm b \neq 0$ on L, is equivalent to finding the sectionally analytic function

$$\mathbf{\Phi}(z) = \frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - z} d\tau \tag{3.3.2}$$

associated with the RH problem

$$\mathbf{\Phi}^+(t) = g(t)\mathbf{\Phi}^-(t) + f(t) \quad \text{for} \quad t \quad \text{on} \quad L; \quad \mathbf{\Phi}^-(\infty) = 0,$$

$$g(t) \equiv \frac{a(t) - b(t)}{a(t) + b(t)}, \quad f(t) \equiv \frac{c(t)}{a(t) + b(t)}. \tag{3.3.3}$$

To show that equation (3.3.1) reduces to the RH problem (3.3.3), we use the Plemelj formula for $\mathbf{\Phi}(z)$, that is,

$$\varphi(t) = \mathbf{\Phi}^+(t) - \mathbf{\Phi}^-(t), \quad \frac{1}{\pi i} \oint \frac{\varphi(\tau)}{\tau - t} = \mathbf{\Phi}^+(t) + \mathbf{\Phi}^-(t)$$

Substituting these equations in equation (3.3.1), we immediately find equation (3.3.3). The converse is also true: If $\mathbf{\Phi}(z)$ solves the RH problem (3.3.3) *with the boundary condition* $\mathbf{\Phi}^-(\infty) = 0$, then $\varphi = \mathbf{\Phi}^+ - \mathbf{\Phi}^-$ solves equation (3.3.1) (see Muskhelishvili, 1977).

More general singular integral equations can occasionally be solved in closed form:

Example 3.3.1 Solve the singular integral equation

$$(t + t^{-1})\varphi(t) + \frac{t - t^{-1}}{\pi i} \oint_C \frac{\varphi(\tau)}{\tau - t} d\tau - \frac{1}{2\pi i} \int_C (t + t^{-1})(\tau + \tau^{-1})\varphi(\tau) d\tau = 2t^2, \tag{3.3.4}$$

where C is the unit circle.

This equation can be solved in closed form because the Fredholm kernel $(t+t^{-1})(\tau+\tau^{-1})$ is degenerate: Let

$$A \equiv \frac{1}{2\pi i} \int_C (\tau + \tau^{-1})\varphi(\tau) d\tau. \tag{3.3.5}$$

Then (3.3.4) yields the singular integral equation

$$(t + t^{-1})\varphi(t) + \frac{t - t^{-1}}{\pi i} \oint_C \frac{\varphi(\tau)}{\tau - t} d\tau = 2t^2 + A(t + t^{-1}).$$

This equation is equivalent to the scalar RH problem

$$(t + t^{-1})(\mathbf{\Phi}^+ - \mathbf{\Phi}^-) + (t - t^{-1})(\mathbf{\Phi}^+ + \mathbf{\Phi}^-) = 2t^2 + A(t + t^{-1}).$$

Thus

$$\mathbf{\Phi}^+(t) = t^{-2}\mathbf{\Phi}^-(t) + t + \frac{A}{2}(1 + t^{-2}), \mathbf{\Phi}^-(\infty) = 0. \tag{3.3.6}$$

The analytic function $g(t) = t^{-2}$ has a second order pole inside C, thus $k = \text{index}\, g = -2$. The fundamental solution $X(z)$ of the homogeneous RH problem satisfies $X^-(z) = O(z^2)$ as $z \to \infty$. It can be found by inspection:

$$t^2 X^+(t) = X^-(t) = t^2,$$

thus

$$X^+(t) = 1, \quad X^-(t) = t^2.$$

Since the index is -2, the solution of (3.3.6) exists if and only if

$$\int_C \left[\tau + \frac{A}{2}(1+\tau^{-2})\right] d\tau = 0, \quad \int_C \left[\tau + \frac{A}{2}(1+\tau^{-2})\right] \tau d\tau = 0.$$

The second equation above implies that $A = 0$. Thus

$$\mathbf{\Phi}(z) = \frac{X(z)}{2\pi i} \int_C \frac{\tau d\tau}{\tau - z} = \begin{bmatrix} z, & z \ \text{ inside circle} \\ 0, & z \ \text{ outside circle.} \end{bmatrix}$$

Therefore $\mathbf{\Phi}^+(t) = t$, $\mathbf{\Phi}^-(t) = 0$ and $\varphi(t) = \mathbf{\Phi}^+(t) - \mathbf{\Phi}^-(t) = t$. Hence equation (3.3.4) has the unique solution $\varphi(t) = t$ provided that $A = 0$ which is indeed the case since

$$\frac{1}{2\pi i} \int_C (\tau + \tau^{-1}) \tau d\tau = 0.$$

4 Direct Methods of Solitons

4.1 Bäcklund Transformations

Using the characteristic coordinates

$$\xi = \frac{x-t}{2}, \quad \eta = \frac{x+t}{2}, \tag{4.1.1}$$

we can write the sine-Gordon equation (1.3) in the canonical form

$$\frac{\partial^2 q}{\partial \xi \partial \eta} = \sin q. \tag{4.1.2}$$

In general a Bäcklund transformation for a second order equation is of the form

$$\tilde{q}_\xi = A(\tilde{q}, q, q_\xi, q_\eta, \xi, \eta),$$

$$\tilde{q}_\eta = B(\tilde{q}, q, q_\xi, q_\eta, \xi, \eta), \tag{4.1.3}$$

where the smooth functions A, B are chosen in such a way that $\tilde{q}$ also solves some equation of second order. If $\tilde{q}$ satisfies the same equation as q then this transformation is sometimes called an auto-Bäcklund transformation. Such a transformation for equation (4.1.2) (which was first derived in a geometrical context) is given by

$$\tilde{q}_\xi = q_\xi + 2\lambda \sin \frac{\tilde{q} + q}{2}$$

$$\tilde{q}_\eta = -q_\eta + \frac{2}{\lambda} \sin \frac{\tilde{q} - q}{2}, \tag{4.1.4}$$

where λ is a constant parameter. It can be verified that equations (4.1.4), which are *two* equations for *one* function $\tilde{q}$, are compatible iff q and $\tilde{q}$ solve (4.1.2).

Since $q_0 = 0$ is a solution of (4.1.2), substituting $q = 0$ in (4.1.4), it follows that another solution $\tilde{q} = q_1$ is given by

$$q_{1_\xi} = 2\lambda \sin\frac{q_1}{2}, \quad q_{1_\eta} = \frac{2}{\lambda}\sin\frac{q_1}{2}. \tag{4.1.5}$$

This yields the one-soliton solution

$$\tan\frac{q_1}{4} = ce^{\lambda\xi+\frac{\eta}{\lambda}} = ce^{\frac{x-Ut}{\sqrt{1-U^2}}}, \quad \lambda = \sqrt{\frac{1+U}{1-U}}, \tag{4.1.6}$$

where x, t are related to ξ, η by equations (4.1.1) and c is a constant.

4.2 The Direct Linearizing Method

Let $\varphi(k; x, t)$ be the unique solution of the linear integral equation

$$\varphi(k; x, t) + ie^{i(kx+k^3t)}\int_L \frac{\varphi(l; x, t)}{l+k} d\lambda(l) = e^{i(kx+k^3t)}, \tag{4.2.1}$$

where $d\lambda(k)$ and L are an arbitrary measure and contour respectively. Let $q(x, t)$ be defined by

$$q(x, t) = -\frac{\partial}{\partial x}\int_L \varphi(k; x, t) d\lambda(k). \tag{4.2.2}$$

Then $q(x, t)$ solves the KdV equation (1.1).

Indeed, it can be verified that the KdV equation is the compatibility condition of the linear equations

$$M\varphi = 0, \quad N\varphi = 0, \tag{4.2.3}$$

where

$$\begin{aligned} M\varphi &\doteqdot \varphi_{xx} - ik\varphi_x + q\varphi, \\ N\varphi &\doteqdot \varphi_t + \varphi_{xxx} + 3q\varphi_x. \end{aligned} \tag{4.2.4}$$

Equations (4.2.4) are called the *Lax pair* of the KdV equation (1.1). Instead of verifying that q solves the KdV equation we will verify that (φ, q) satisfy equations (4.2.3). Denoting $\exp[i(kx + k^3t)]$ by e and $\int_L \frac{d\lambda(l)}{l+k}$ by $\mathbf{I}$, equation (4.2.1) implies the following equations:

$$\varphi_{xx} + ie\mathbf{I}\varphi_{xx} - ik^2e\mathbf{I}\varphi - 2ke\mathbf{I}\varphi_x = -k^2e,$$

$$-ik\varphi_x + ke\mathbf{I}\varphi_x + ik^2e\mathbf{I}\varphi = k^2e,$$

$$(q\varphi) + ie\mathbf{I}(q\varphi) = qe.$$

After adding these equations we note that the terms involving $\mathbf{I}\varphi_x$ simplify as follows,

$$-ke\mathbf{I}\varphi_x = ie\mathbf{I}(-il\varphi_x) - e\int_L \varphi_x d\lambda(l).$$

The second term of the rhs of this equation cancels with qe (using the definition (4.2.2)), thus

$$(M\varphi) + ie\mathbf{I}(M\varphi) = 0. \tag{4.2.5}$$

Since by assumption the homogeneous version of equation (4.2.1) has only the zero solution, the above equation implies $M\varphi = 0$.

A similar calculation yields

$$(N\varphi) + ie\mathbf{I}(N\varphi) = 3ke\mathbf{I}M\varphi \tag{4.2.6}$$

which implies $N\varphi = 0$.

It is straightforward to calculate soliton solutions using (4.2.1), (4.2.2): Substituting

$$d\lambda(k) = c_1\delta(k - ip)dk, \tag{4.2.7}$$

in these equations we find

$$\varphi(k; x, t) + ic_1 e^{i(kx+k^3t)}\frac{\varphi(ip; x, t)}{ip + k} = e^{i(kx+k^3t)}, \tag{4.2.8}$$

$$q(x, t) = -\frac{\partial}{\partial x}c_1\varphi(ip; x, t). \tag{4.2.9}$$

Evaluating equation (4.2.8) at $k = ip$, solving the resulting equation for $\varphi(ip; x, t)$, and then using (4.2.9) we find

$$q(x, t) = -\frac{\partial}{\partial x}2p\frac{c_1 e^{-px+p^3t}}{2p + c_1 e^{-px+p^3t}}. \tag{4.2.10}$$

This is the one-soliton solution. Indeed, letting $c_1/2p = e^{-2c}$ in equation (4.2.10) and denoting $e = \exp[-px + p^3t - 2c]$, we find

$$q(x, t) = -2p\frac{\partial}{\partial x}\left(\frac{e}{1 + e}\right) = 2p^2\frac{e}{(1 + e)^2} = \frac{2p^2}{(e^{-\frac{1}{2}} + e^{\frac{1}{2}})^2} = \frac{p^2/2}{\cosh^2[\frac{p}{2}x - \frac{p^3t}{2} + c]},$$

which is equation (1.5).

The N-soliton solution can be obtained by substituting

$$d\lambda(k) = \sum_1^N c_j\delta(k - ip_j)dk, \tag{4.2.11}$$

in equations (4.2.1), (4.2.2) and then evaluating the equation resulting from (4.2.1) at $k = ip_j$, $j = 1, \cdots, N$.

The two-soliton solution is given by

$$q_2(x, t) = \frac{2(p_1^2e^{\eta_1} + p_2^2e^{\eta_2}) + 4(p_1 - p_2)^2e^{\eta_1+\eta_2} + 2A_{12}(p_2^2e^{2\eta_1+\eta_2} + p_1^2e^{2\eta_2+\eta_1})}{(1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2})^2}, \tag{4.2.12}$$

where

$$\eta_j = p_jx - p_j^3t + c_j, \quad j = 1, 2. \tag{4.2.13}$$

Let $p_1 > p_2$; then the soliton 1 is taller and moves faster than the soliton 2. Thus as $t \to \infty$, soliton 1 will overtake soliton 2:

Comoving with soliton 1. Let $A_{12} = e^{\phi_{12}}$. Replace x by the new variable $\xi = x - p_1^2 t$. Then η_1 is constant and

$$\eta_2 = p_2\xi + p_2(p_1^2 - p_2^2)t + c_2.$$

As $t \to \infty, \quad \eta_2 \to -\infty, \quad e^{\eta_2} \to 0$ and

$$q \to \frac{2p_1^2 e^{\eta_1}}{(1+e^{\eta_1})^2}.$$

Thus

$$\eta_2 \to \pm\ \infty \quad \text{as} \quad t \to \pm\infty,$$

$$q_2 \to \begin{cases} \frac{p_1^2/2}{\cosh^2(\frac{1}{2}\eta_1)}, & t \to -\infty \\ \\ \frac{p_1^2/2}{\cosh^2(\frac{1}{2}\eta_1 + \frac{1}{2}\varphi_{12})}, & t \to +\infty. \end{cases} \tag{4.2.14}$$

Similarly, as $t \to \infty, \quad \eta_2 \to \infty, \quad e^{2\eta_2}$ dominates and

$$q \to \frac{2A_{12}p_1^2 e^{2\eta_2} e^{\eta_1}}{e^{2\eta_2}(1 + A_{12}e^{\eta_1})^2}.$$

Comoving with soliton 2. Replace x by the new variable $\xi = x - p_2^2 t$. Then η_2 is constant and

$$\eta_1 = p_1\xi + p_1(p_2^2 - p_1^2)t + c_1.$$

Thus

$$\eta_1 \to \pm\infty \quad \text{as} \quad t \to \mp\infty.$$

Hence

$$q_2 \to \begin{cases} \frac{p_2^2/2}{\cosh^2(\frac{1}{2}\eta_2)}, & t \to +\infty \\ \\ \frac{p_2^2/2}{\cosh^2(\frac{1}{2}\eta_2 + \frac{1}{2}\varphi_{12})}, & t \to -\infty. \end{cases} \tag{4.2.15}$$

The maximum positions of the soliton 1 at ∞ and at $-\infty$ are given respectively by $\eta_1 = -\varphi_{12}$ and $\eta_1 = 0$, therefore they occur at

$$x = -\frac{c_1}{p_1} + p_1^2 t - \frac{\varphi_{12}}{p_1}, \quad x = -\frac{c_1}{p_1} + p_1^2 t.$$

Thus there exists a "phase shift" of φ_{12}/p_1. Similarly for soliton 2.

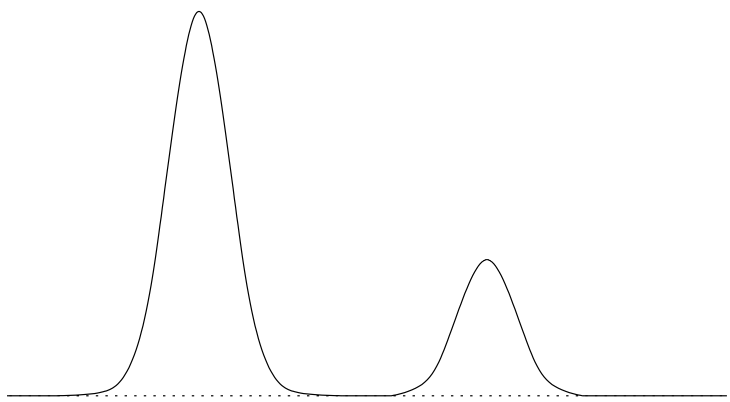


Figure 4.2 A Two-Soliton Solution for the KdV.

4.3 The Dressing Method

Let $M^{\pm}(x,t,k)$ satisfy the following 2×2 matrix RH problem

$$M^{+}(x,t,k) = M^{-}(x,t,k)e^{-i(kx+2k^2t)\sigma_3}S(k)e^{i(kx+2k^2t)\sigma_3}, \quad k\in\mathbb{C}, \tag{4.3.1}$$

$$M(x,t,k) = I_2 + O\left(\frac{1}{k}\right), \quad k\to\infty, \tag{4.3.2}$$

where $S(k)$ is an arbitrary 2×2 unimodular matrix with $(S)_{11}=1$, and

$$I_2 = diag(1,1) \quad \sigma_3 = diag(1,-1). \tag{4.3.3}$$

The main idea of the dressing method is to construct two linear operators L and N such that: (i) LM and NM satisfy the same jump condition as M. (ii) LM and NM are of $O(1/k)$ as $k\to\infty$. Then, under the *assumption* that the RH problem defined by equations (4.3.1) and (4.3.2) as a unique solution, it follows that

$$LM = 0, \quad NM = 0. \tag{4.3.4}$$

These equations constitute the Lax pair associated with the above RH problem.

In order to construct L we introduce the operator $\hat{\sigma}_3$ defined by

$$\hat{\sigma}_3 M = [\sigma_3, M]. \tag{4.3.5}$$

Equation (4.3.1) can be rewritten in the form

$$M^{+} = M^{-}e^{-i(kx+2k^2t)\hat{\sigma}_3}S(k). \tag{4.3.6}$$

This equation immediately implies that M satisfies the equation

$$\left\{(\partial_x + ik\hat{\sigma}_3)M^{+}\right\} = \left\{(\partial_x + ik\hat{\sigma}_3)M^{-}\right\}e^{-i(kx+2k^2t)\hat{\sigma}_3}S(k), \tag{4.3.7}$$

as well as a similar equation with the operator $\partial_x + ik\hat{\sigma}_3$ replaced by the operator $\partial_t + 2ik^2\hat{\sigma}_3$.

Since M satisfies the boundary condition (4.3.2), it follows that $(\partial_x + ik\hat{\sigma}_3)M$ is of $O(1)$. Thus in order to construct an operator L such that LM is of $O(1/k)$ we must subtract $Q(x,t)M$ (note that QM satisfies the same jump condition as M). Thus, we let

$$LM \doteqdot M_x + ik[\sigma_3, M] - QM. \tag{4.3.8}$$

Substituting the asymptotic expansion

$$M(x,t,k) = I_2 + \frac{M_1(x,t)}{k} + \frac{M_2(x,t)}{k^2} + O\left(\frac{1}{k^3}\right), \tag{4.3.9}$$

into equation (4.3.8) we find

$$LM = \{i[\sigma_3, M_1] - Q\} + O\left(\frac{1}{k}\right).$$

Thus, if Q is defined by the equation

$$Q(x,t) = i[\sigma_3, M_1(x,t)], \tag{4.3.10}$$

then LM satisfies the following homogeneous RH problem,

$$(LM)^+ = (LM)^- e^{-i(kx+2k^2t)\sigma_3} S(k) e^{i(kx+2k^2t)\sigma_3},$$

$$LM = O\left(\frac{1}{k}\right), \quad k \to \infty.$$

Hence, $LM = 0$, i.e. M and Q satisfy the following equation

$$M_x + ik[\sigma_3, \mu] = QM, \tag{4.3.11}$$

where [,] denotes the usual matrix commutator and

$$Q = \begin{pmatrix} 0 & q(x,t) \\ r(x,t) & 0 \end{pmatrix}. \tag{4.3.12}$$

The operator $(\partial_t + 2ik^2\hat{\sigma}_3)M$ is of $O(k)$, thus we define N by

$$NM \doteqdot M_t + 2ik^2[\sigma_3, M] - kA(x,t)M - B(x,t)M. \tag{4.3.13}$$

Substituting the asymptotic expansion (4.3.9) into equation (4.3.13) we find

$$NM = k\{2i[\sigma_3, M_2] - A\} + \{2i[\sigma_3, M_2] - AM_1 - B\} + O\left(\frac{1}{k}\right).$$

Thus, we define A and B by the equations

$$A = 2i[\sigma_3, M_1], \tag{4.3.14}$$

$$B = 2i[\sigma_3, M_2] - AM_1. \tag{4.3.15}$$

Comparing equations (4.3.10) and (4.3.14) it follows that

$$A = 2Q. \tag{4.3.16}$$

Then equation (4.3.13) becomes

$$B = -2\left(QM_1 - i[\sigma_3, M_2]\right). \tag{4.3.17}$$

The $O(1/k)$ term in the asymptotic expansion of equation $LM = 0$ yields

$$M_{1_x} + i[\sigma_3, M_2] = QM_1. \tag{4.3.18}$$

Comparing this equation with equation (4.3.17) it follows that $B = -2M_{1_x}$, i.e.

$$B = -2\partial_x \left[M_1^{(0)} + M_1^{(D)}\right], \tag{4.3.19}$$

where the superscripts refer to the off-diagonal and the diagonal parts of the matrix M_1. Equation (4.3.10) implies that

$$M_1^{(0)} = -\frac{i}{2}Q\sigma_3. \tag{4.3.20}$$

The diagonal part of equation (4.3.18) yields

$$M_{1_x}^{(D)} = QM_1^{(0)} = -\frac{i}{2}Q^2\sigma_3. \tag{4.3.21}$$

Hence, equations (4.3.19)-(4.3.21) imply

$$B = i(Q\sigma_3 + Q^2\sigma_3). \tag{4.3.22}$$

Equation $NM = 0$, where N is defined by (4.3.13) and A, B are defined by (4.3.16), (4.3.22), is

$$M_t + 2ik^2[\sigma_3, \mu] = (2kQ - iQ_x\sigma_3 - iQ^2\sigma_3)M, \tag{4.3.23}$$

The compatibility condition of equations (4.3.11) and (4.3.23) yields the following pair of nonlinear evolution PDEs for $q(x,t)$ and $r(x,t)$,

$$\begin{aligned} iq_t + q_{xx} - 2rq^2 &= 0 \\ -ir_t + r_{xx} - 2r^2q &= 0. \end{aligned} \tag{4.3.24}$$

The reduction $r = \sigma\bar{q}$, $\sigma = \pm 1$, yields the celebrated NLS equation (1.2).

5 The Inverse Spectral (Scattering) Method

Let $q(x,t)$ satisfy a linear evolution equation whose highest order x-derivative is n. We denote such an equation by

$$(\partial_t + iw(-i\partial_x))q(x,t) = 0, \tag{5.1}$$

where $w(k)$ is a polynomial of degree n,

$$w(k) = \alpha_0 + \alpha_1 k + \cdots + \alpha_n k^n. \tag{5.2}$$

Equation (5.1) can be written in the form

$$\left(e^{-ikx+iw(k)t}q\right)_t + \left(e^{-ikx+iw(k)t}X\right)_x = 0, \tag{5.3}$$

where

$$X(x,t,k) = \left(\frac{w(k) - w(l)}{k - l}\right)q, \quad l \doteqdot -i\partial_x. \tag{5.4}$$

Indeed, replacing in equation (5.3) q_t by $-iw(-i\partial_x)q$ we find

$$i(w(k) - w(-i\partial_x))q + (\partial_x - ik)X = 0.$$

Thus if X is defined by (5.4), this equation is satisfied identically.

Example 5.1. For equation $(1.1)'$, $w(k) = -k^3$. Thus

$$X = \frac{l^3 - k^3}{k - l} q = -(k^2 + l^2 + kl)q = q_{xx} + ikq_x - k^2 q. \tag{5.5}$$

For equation $(1.2)'$, $w(k) = k^2$. Thus

$$X = \frac{k^2 - l^2}{k - l} q = (k + l)q = -iq_x + kq. \tag{5.6}$$

A Lax Pair for Linear PDEs. Equation (5.1) is the compatibility condition of the following pair (Lax pair) of linear eigenvalue equations for the scalar function $\mu(x, t, k)$:

$$\begin{aligned} \mu_x - ik\mu &= q(x, t) \\ \mu_t + iw(k)\mu &= -X(x, t, k). \end{aligned} \tag{5.7}$$

Indeed, equations (5.3) imply that there exists a function $M(x, t, k)$ such that

$$\begin{aligned} M_x &= e^{-ikx+iw(k)t} q, \\ M_t &= -e^{-ikx+iw(k)t} X. \end{aligned}$$

Letting $M = \mu e^{-ikx+iw(k)t}$, the above equations become (5.7).

Example 5.2. The linearized KdV equation

$$q_t + q_{xxx} = 0, \tag{5.8}$$

admits the Lax pair

$$\begin{aligned} \mu_x - ik\mu &= q, \\ \mu_t - ik^3\mu &= -q_{xx} - ikq_x + k^2 q. \end{aligned} \tag{5.9}$$

Similarly the linearized NLS equation

$$iq_t + q_{xx} = 0, \tag{5.10}$$

admits the Lax pair

$$\begin{aligned} \mu_x - ik\mu &= q, \\ \mu_t - ik^2\mu &= iq_x - kq. \end{aligned} \tag{5.11}$$

5.1 The Fourier Transform in One Dimension

Let $\mu(x, k)$ satisfy the equation

$$\mu_x - ik\mu = q(x), \quad -\infty < x < \infty, \quad k \in \mathbb{C}, \tag{5.1.1}$$

where $q(x)$ is an arbitrary function with appropriate smoothness and decay. We will derive the Fourier transform by performing the *spectral analysis* of equation (5.1.1). The most important step in this analysis is the construction of a solution $\mu(x, k)$ which for all $-\infty < x < \infty$, is *sectionally analytic* with respect to

$k \in \mathbb{C}$. This means that we must find a function μ which solves equation (5.1.1) and which is well defined for all $k \in \mathbb{C}$. Such a function is given by

$$\mu(x,k) = \begin{cases} \mu^+(x,k), \ Imk \geq 0 \\ \\ \mu^-(x,k), \ Imk \leq 0, \end{cases} \tag{5.1.2}$$

where $\mu^\pm$ are defined by

$$\begin{aligned} \mu^+(x,k) &= \int_{-\infty}^{x} e^{ik(x-\xi)} q(\xi) d\xi, \quad Imk \geq 0 \\ \mu^-(x,k) &= -\int_{x}^{\infty} e^{ik(x-\xi)} q(\xi) d\xi, \quad Imk \leq 0. \end{aligned} \tag{5.1.3}$$

Indeed, both functions μ^+ and μ^- satisfy equation (5.1.1). Furthermore, since $x-\xi \geq 0$, the rhs of equation (5.1.3a) is analytic for $Imk > 0$, similarly the rhs of (5.1.3b) is analytic for $Imk < 0$.

Since both μ^+ and μ^- satisfy the same equation (5.1.1), it follows that in the domain of k where both μ^+ and μ^- are well defined, these functions are simply related. Indeed subtracting equations (5.1.3) we find

$$\mu^+(x,k) - \mu^-(x,k) = e^{ikx}\hat{q}(k), \quad k \in \mathbb{R}, \tag{5.1.4}$$

where $\hat{q}(k)$ is defined by

$$\hat{q}(k) = \int_{-\infty}^{\infty} e^{-ik\xi} q(\xi) d\xi, \quad k \in \mathbb{R}. \tag{5.1.5}$$

Equations (5.1.3), using integration by parts, imply

$$\mu(x,k) = O(\frac{1}{k}), \quad k \to \infty. \tag{5.1.6}$$

Equations (5.1.4) and (5.1.6) define an elementary Riemann-Hilbert (RH) problem. Its unique solution is

$$\mu(x,k) = \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{e^{ilx}\hat{q}(l)}{l-k} dl, \quad k \in \mathbb{C}. \tag{5.1.7}$$

Substituting equation (5.1.7) into equation (5.1.1) we find

$$q(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\hat{q}(k) dk. \tag{5.1.8}$$

Equations (5.1.5) and (5.1.8) define the classical Fourier transform pair.

5.2 The Inverse Spectral Method for Linear PDEs

Let $q(x,t)$ satisfy equation (5.1) on the infinite line, $-\infty < x < \infty$, $t > 0$, with decaying initial conditions $q(x,0) = q_0(x)$, $-\infty < x < \infty$.

It was shown in §5.1 that associated with equation (5.7a) there exist the pair (5.1.5), (5.1.8). Allowing for q to depend on t we find

$$q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\hat{q}(k,t) dk, \tag{5.2.1}$$

where

$$\hat{q}(k,t) = \int_{-\infty}^{\infty} e^{-ikx} q(x,t) dx. \tag{5.2.2}$$

Equation (5.1.3a) implies

$$\hat{q}(k,t) = \lim_{x\to\infty} e^{-ikx}\mu^+(x,t,k).$$

This equation, the assumption that $q \to 0$ as $x \to \infty$, and equation (5.7b) imply

$$\hat{q}_t + iw(k)\hat{q} = 0.$$

Thus

$$\hat{q}(k,t) = \hat{q}(k,0)e^{-iw(k)t}, \tag{5.2.3}$$

where

$$\hat{q}(k,0) = \int_{-\infty}^{\infty} e^{-ikx}q_0(x)dx. \tag{5.2.4}$$

In summary, the *spectral analysis* of the x-part of the Lax pair yields the solution $q(x,t)$ in terms of the *spectral function* $\hat{q}(k,t)$. The t-evolution of this function can be easily obtained using the t-part of the Lax pair.

In what follows we will use precisely the same conceptual steps to solve the initial value problem for the KdV equation.

5.3 Inverse Scattering Transform for the Korteweg-deVries equation

The KdV equation can be written as the compatibility condition of the following pair of linear eigenvalue equations for the eigenfunction $\psi(x,t,k)$,

$$\psi_{xx} + (q+k^2)\psi = 0, \tag{5.3a}$$

$$\psi_t + (2q-4k^2)\psi_x - (q_x+\nu)\psi = 0, \quad k \in \mathbb{C}, \tag{5.3b}$$

where ν is an arbitrary constant.
The Direct Problem

As $|x| \to \infty$, $q \to 0$, thus there exist solutions of equation (5.3a) which tend to $\exp[\pm ikx]$ as $|x| \to \infty$. Let $\psi(k,x)$ and $\hat{\psi}(k,x)$ denote solutions of equation (5.3a) with the following asymptotic property:

$$\psi \to e^{ikx}, \quad \hat{\psi} \to e^{-ikx}, \text{ as } x \to \infty, \quad k \in \mathbb{R}. \tag{5.3.1}$$

Under the transformation $k \to -k$, equation (5.3a) remains invariant and the boundary condition for ψ is mapped to the boundary condition for $\hat{\psi}$. Hence

$$\hat{\psi}(k,x) = \psi(-k,x). \tag{5.3.2}$$

We denote by $\phi(k,x)$ the solutionof equation (5.3a) which tends to $\exp[-ikx]$ as $x \to -\infty$,

$$\phi \to e^{-ikx}, \text{ as } x \to -\infty, \quad k \in \mathbb{R}. \tag{5.3.3}$$

It is more convenient to work with eigenfunctions (i.e. solutions of (5.3a)) normalised to unity as $x \to \infty$, thus we introduce $M(k,x)$ and $N(k,x)$ as follows

$$M = \phi e^{ikx}, \quad N = \psi e^{-ikx}. \tag{5.3.4}$$

The functions M and N can be expressed in terms of q through the solution of linear Volterra integral equations. Indeed, M satisfies

$$M_{xx} - 2ikM_x = -qM, \quad k \in \mathbb{R}, \tag{5.3.5}$$

$$M \to 1, \quad x \to -\infty.$$

The homogeneous version of (5.3.5a) has solutions 1 and e^{2ikx}. Thus

$$M = c_1 + c_2 e^{2ikx} + M_p, \tag{5.3.6}$$

where c_1, c_2 are constants and M_p is given by

$$M_p = u_1(x) + u_2(x)e^{2ikx}. \tag{5.3.7}$$

The functions u_1, u_2 satisfy

$$u_1' + e^{2ikx}u_2' = 0, \quad 2ike^{2ikx}u_2' = -qM.$$

Thus

$$u_1(x) = \frac{1}{2ik}\int_{-\infty}^{x} d\xi q(\xi)M(k,\xi), \quad u_2(x) = -\frac{1}{2ik}\int_{-\infty}^{x} d\xi e^{-2ik\xi}q(\xi)M(k,\xi). \tag{5.3.8}$$

Substituting (5.3.7), (5.3.8) into (5.3.6) and using the boundary condition (5.3.5b) we find

$$M(k,x) = 1 + \frac{i}{2k}\int_{-\infty}^{x} d\xi(-1 + e^{2ik(x-\xi)})q(\xi)M(k,\xi). \tag{5.3.9}$$

Similarly, one may establish that N satisfied

$$N(k,x) = 1 + \frac{i}{2k}\int_{x}^{\infty} d\xi(-1 + e^{-2ik(x-\xi)})q(\xi)N(k,\xi). \tag{5.3.10}$$

The kernel of equation (5.3.9), as a function of k is bounded and analytic for $Imk > 0$. Thus if $q \in L_1$, $M(k,x)$ as a function of k is holomorphic for $Imk > 0$. Similarly, $N(k,x)$ as a function of k is holomorphic for $Imk > 0$.

Thus, we have found particular solutions of equation (5.3a) which are holomorphic for $Imk > 0$. Furthermore, these solutions are simply related for k real. Indeed, the linear independence of solutions of the second order ODE (5.3a) implies

$$\phi(k,x) = a(k)\hat{\psi}(k,x) + b(k)\psi(k,x), \quad k \in \mathbb{R}.$$

Using (5.3.2) and replacing ϕ and ψ in terms of M and N, we find

$$\frac{M(k,x)}{a(k)} = N(-k,x) + \rho(k)e^{2ikx}N(k,x), \rho(k) = \frac{b(k)}{a(k)}, \quad k \in \mathbb{R}. \tag{5.3.11}$$

The functions $a(k)$ and $b(k)$ are given by

$$a(k) = 1 - \frac{i}{2k}\int_{-\infty}^{\infty} d\xi q(\xi)M(k,\xi), \quad k \in \mathbb{R},$$

$$b(k) = \frac{i}{2k}\int_{-\infty}^{\infty} d\xi q(\xi)M(k,\xi)e^{-2ik\xi}, \quad k \in \mathbb{R}. \tag{5.3.12}$$

Indeed as $x \to \infty$, $N \to 1$, thus equation (5.3.11) implies

$$M \to a(k) + b(k)e^{2ikx} \text{ as } x \to \infty. \tag{5.3.13}$$

On the other hand, equation (5.3.9) implies that

$$M \to 1 + \frac{i}{2k}\int_{-\infty}^{\infty} d\xi(-1 + e^{2ik(x-\xi)}q(\xi)M(k,\xi)), \quad x \to \infty. \tag{5.3.14}$$

Comparing equations (5.3.13) and (5.3.14) we find equations (5.3.12).

The expression for $a(k)$ implies that this function is also holomorphic for $Imk > 0$.

In summary, in the "direct problem", we have found particular solutions of equation (5.3a) which are *sectionally holomorphic:*

$$\begin{pmatrix} M(k,x) \\ N(k,x) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} M(-k,x) \\ N(-k,x) \end{pmatrix}$$

are holomorphic for $Imk > 0$ and $Imk < 0$ respectively. These solutions, which are characterized in terms of q by equations (5.3.9) and (5.3.10), are simply related by equation (5.3.11).

The Inverse Problem

Equation (5.3.10) expresses N in terms of q. Is it possible to find an alternative expression for N in terms of some appropriate "spectral data"? The answer is positive and is a direct consequence of the fact that equation (5.3.11) defines the "jump condition" of a Riemann-Hilbert problem. Indeed, it can be shown that $a(k)$ may have simple zeros $k_1, \cdots, k_n$ in the positive imaginary axis of the k-complex plane. Hence in general, M/a can be expressed in the form

$$\frac{M(k,x)}{a(k)} = \mathcal{M}(k,x) + \sum_{j=1}^{n} \frac{A_j(x)}{k - ip_j}, \quad p_j > 0,$$

where $\mathcal{M}(k,x)$ as a function of k is holomorphic for $Imk > 0$. It can also be shown that $A_j(x) = C_j \exp[-2p_j, x] N(k_j, x)$. Hence equation (5.3.11) becomes

$$\mathcal{M}(k,x) - N(-k,x) = -\sum_{j=1}^{n} \frac{C_j e^{-2p_j x} N(ip_j, x)}{k - ip_j} + \rho(k) e^{2ikx} N(k,x), k \in \mathbb{R}.$$

Taking the $(-)$ projection of this equation, and using, the fact that both $\mathcal{M}$ and N tend to 1 as $k \to \infty$, we find

$$N(k,x) - \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{dl \rho(l) e^{2ilx} N(l,x)}{l + k + i0} = 1 + \sum_{j=1}^{n} \frac{C_j e^{-2p_j x}}{k + ip_j} N(ip_j, x). \tag{5.3.15}$$

In summary, this equation expressed $N(k,x)$ in terms of the scattering data $(\rho(k), \{C_j, p_j\}_1^n)$.

Since both equations (5.3.10) and (5.3.15) are associated with the same q, these equations can be used to obtain the following expression for q

$$q = -2 \frac{\partial}{\partial x} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} dl \rho(l) e^{2ilx} N(l,x) + i \sum_{j=1}^{n} C_j e^{-2p_j x} N(ip_j, x) \right]. \tag{5.3.16}$$

Indeed, equation (5.3.10) implies

$$\lim_{k \to \infty} N(k,x) = 1 - \frac{i}{2k} \int_{x}^{\infty} d\xi q(\xi).$$

Comparing this expression with the large k behavior of equation (5.3.15) we find (5.3.16).

Time Dependence of the Scattering Data

We now use equation (5.3b) to compute the time dependence of the scattering data: Evaluating equation (5.3b) as $x \to -\infty$, where $\phi \sim e^{-ikx}$ we find $\nu = 4ik^3$. Then evaluating it as $x \to \infty$ and using

$$\phi \sim a e^{-ikx} + b e^{ikx}, \quad x \to +\infty,$$

we find

$$a_t = 0, \quad b_t = 8ik^3 b.$$

Hence

$$a(t,k) = a(0,k), \quad \rho(t,k) = \rho(0,k)e^{8ik^3 t}. \tag{5.3.16}$$

Thus

$$p_j(t) = p_j(0), \quad C_j(t) = C_j(0)e^{8p_j^3 t}. \tag{5.3.17}$$

The above formal results motivate the following definitions (for simplicity we assume that $a(k)$ has no zeros). Given a decaying real function $q_0(x)$, $x \in \mathbb{R}$, define $M_0(k,x)$ as the solution of the linear Volterra integral equation

$$M_0(k,x) = 1 + \frac{i}{2k}\int_{-\infty}^{x} d\xi(-1 + e^{2ik(x-\xi)}q(\xi)M_0(k,\xi), \quad Imk \geq 0. \tag{5.3.18}$$

Given $M_0(k,x)$, defined $a_0(k)$ and $b_0(k)$ by

$$M_0(k,x) \to a_0(k) + b_0(k)e^{2ikx}, \quad x \to \infty, \quad k \in \mathbb{R}.$$

Given a_0 and b_0, define $N(k,x,t)$ by the solution of the linear integral equation

$$N(k,x,t) - \frac{1}{2\pi i}\int_{-\infty}^{\infty} dl \frac{b_0(l)}{a_0(l)} e^{8il^3 t + 2ilx} \frac{N(l,x,t)}{l+k+i0} = 1. \tag{5.3.19}$$

A theorem of Gohberg and Krein, implies that this equation has a unique global solution. Given a_0, b_0, N, define $q(x,t)$ by

$$q(x,t) = -\frac{1}{\pi}\frac{\partial}{\partial x}\int_{-\infty}^{\infty} dk \frac{b_0(k)}{a_0(k)} e^{8ik^3 t + 2ikx} N(k,x,t). \tag{5.3.20}$$

Then it can be shown that $q(x,t)$ satisfies the KdV equation and $q(x,0) = q_0(x)$.

The linear limit

Let

$$q_0(x) = \varepsilon u_0(x) + O(\varepsilon^2).$$

Then the equation (5.3.18) for M_0 implies

$$M_0(k,x) = 1 + O(\varepsilon),$$

and equations (5.3.12), with M replaced by M_0, imply

$$a_0(k) = 1 + O(\varepsilon), \quad b_0(k) = \frac{i\varepsilon}{2k}\int_{-\infty}^{\infty} d\xi u_0(\xi)e^{-2ik\xi} = \frac{i\varepsilon}{2k}\widehat{u}_0(k).$$

Since $a_0 \sim 1$ there exist no solitons and equation (5.3.19) for N implies

$$N(k,x,t) = 1 + O(\varepsilon).$$

Finally, equation (5.3.20) for q implies

$$q(x,t) = -\frac{1}{\pi}\frac{\partial}{\partial x}\int_{-\infty}^{\infty} dk \frac{i\varepsilon}{2k}\widehat{u}_0(k)e^{2ikx + 8ik^3 t},$$

or

$$q(x,t) = \frac{\varepsilon}{\pi}\int_{-\infty}^{\infty} dk \widehat{u}_0(k)e^{2ikx + 8ik^3 t}, \quad \widehat{u}_0(k) = \int_{-\infty}^{\infty} d\xi u_0(\xi)e^{-2ik\xi}.$$

Letting $q(x,t) = \varepsilon u(x,t)$, it follows that the above formulae solve the inital value problem of

$$u_t + u_{xxx} = 0.$$

6 Painlevé Equation

The modified KdV equation

$$q_t + q_{xxx} + 6q^2 q_x = 0, \tag{6.1}$$

admits two obvious ODE reductions.

(i) Using the fact that the mKdV equation is invariant under x and t translations it follows that we can look for a solution in the form

$$q = f(\xi), \quad \xi = x - vt, \tag{6.2}$$

where v is constant. Then equation (6.1) yields

$$-vf' + f''' + 6f^2 f' = 0,$$

or

$$f'' + 2f^3 - vf = c, \tag{6.3}$$

c constant. Equation (6.3) can be solved in terms of elliptic functions.

(ii) Equation (6.1) in invariant under the scaling transformation

$$x \to \lambda x, \quad t \to \lambda^3 t, q \to \lambda^{-1} q.$$

This implies that we can look for a solution in the form

$$q = t^{-1/3} f(\xi), \quad \xi = xt^{-1/3}.$$

Then equation (6.1) yields

$$f'' + 2f^3 - \frac{1}{3}\xi f = c, \tag{6.4}$$

c constant. Equation (6.4) is the so-called Pailevé II equation.

Equations (6.3) and (6.4) share the so-called Painlevé property: A first or a second order ODE has the Painlevé property if the branch points and the essential singularities of its solution do *not* depend on the initial conditions. The only ODE of first order, of the type

$$\frac{df}{dz} = F(z, f),$$

F meromorphic in z and rational in f, is the Ricatti equation. Painlevé and his school classified all equations of the type

$$\frac{d^2 f}{dz^2} = F\left(z, f, \frac{df}{dz}\right),$$

F meromorphic in z and rational in f, f', with the above property and found 50 ODEs. Among these there exist 6 ODEs which could not be solved by known functions. These equations are called the 6 Painlevé equations. The first two are

$$f'' = 3f^2 + z, \tag{6.5}$$

$$f'' = 2f^3 + zf + \alpha, \quad \alpha \quad \text{constant}. \tag{6.6}$$

6.1 The Analogue of Bäcklund Transformations

Let $v(z,\alpha)$ be a solution of Painlevé II equation (6.6) and let $u(z,\nu)$ be a solution of

$$u'' + 2u^2 - zu + \frac{\nu + u' - (u')^2}{2u - z} = 0, \quad \nu = \alpha^2 + \alpha. \tag{6.7}$$

There exist the following one-to-one correspondence between solutions of (6.6) and (6.7):

$$u = -v' - v^2, \quad v = \frac{u' + \alpha}{2u - z}. \tag{6.8}$$

Indeed,

$$\begin{aligned} v' &= \frac{u''}{2u - z} - \frac{(u' + \alpha)(2u' - 1)}{(2u - z)^2} \\ &= \left(-2u^2 + zu - \frac{\nu + u' - (u')^2}{2u - z}\right)\frac{1}{2u - z} - \frac{(u' + \alpha)(2u' - 1)}{(2u - z)^2} \\ &= -u - \frac{(u' + \alpha)^2}{(2u - z)^2} = -u - v^2. \end{aligned}$$

Similarly, differentiating (6.8a) and using Painlevé II we find (6.8b).

The above results imply the following auto-Bäcklund transformation for Painlevé II: Let $v(z,\alpha)$ and $\tilde{v}(z,\alpha+1)$ be solutions of the Painleé II equation. Then

$$\tilde{v}(z,\alpha+1) = -v(z,\alpha) - \frac{1 + 2\alpha}{2v^2 + 2v' + z}, \quad \alpha \neq -\frac{1}{2}. \tag{6.9}$$

Indeed, equation (6.7) implies that $u(z,\alpha) = u(z,\tilde{\alpha})$, where $\tilde{\alpha} = -(\alpha + 1)$. Thus

$$\begin{aligned} \tilde{v}(\tilde{\alpha}) &= \frac{u'(\tilde{\alpha}) - (\alpha + 1)}{2u(\tilde{\alpha}) - z} = \frac{u'(\alpha) - (\alpha + 1)}{2u(\alpha) - z} \\ &= \frac{[v(\alpha)(2u(\alpha) - z) - \alpha] - (\alpha + 1)}{2u(\alpha) - z} = v(\alpha) - \frac{2\alpha + 1}{2u(\alpha) - z} \\ &= v(\alpha) + \frac{2\alpha + 1}{-2v'(\alpha) - 2v^2(\alpha) - z}. \end{aligned}$$

Using $\tilde{v}(\tilde{\alpha}) = \tilde{v}(-\alpha - 1) = -\tilde{v}(\alpha + 1)$, we find equation (6.9).

The case of $\alpha = -1/2$ is covered by the following result: A particular solution of the Painlevé II equation with $\alpha = -1/2$ satisfies

$$v' + v^2 + \frac{z}{2} = 0. \tag{6.10}$$

Indeed, differentiating this equation and using equation (6.10) to replace v' we find

$$v'' - 2v\left(v^2 + \frac{z}{2}\right) + \frac{1}{2} = 0.$$