Mathematical Tripos Part II Michaelmas 2016
Integrable Systems, Example sheet 1 Dr A.C.L. Ashton

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1. Let g^t be the flow map generated by the vector field V, so that $\mathbf{x}(t) = g^t\mathbf{x}_0$ is the unique solution to the differential equation $\dot{\mathbf{x}} = V(\mathbf{x})$, $\mathbf{x}(0) = \mathbf{x}_0$. Show that

$$g^0 = I, \quad g^{t+s} = g^t g^s, \quad g^{-t} = (g^t)^{-1}.$$

2. Let V_1, V_2 be vector fields that generate g_1^t and g_2^t respectively. By considering $\Delta(s,t) = g_1^t g_2^s \mathbf{x}_0 - g_2^s g_1^t \mathbf{x}_0$ and Taylor expanding upto and including second order, show that if g_1 and g_2 commute then $[V_1, V_2] = 0$.

3. Establish the Leibniz rule and the Jacobi identity for Poisson brackets

$$\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0, \quad \{f, gh\} = g\{f,h\} + h\{f,g\}.$$

Deduce from the Jacobi identity that if $f = f(\mathbf{q},\mathbf{p})$ and $g = g(\mathbf{q},\mathbf{p})$ are two first integrals of a Hamiltonian system, then so is $h = \{f,g\}$.

4. Let $V_f = J\partial_{\mathbf{x}} f$ and $V_g = J\partial_{\mathbf{x}} g$ be two Hamiltonian vector fields corresponding to the functions f and g. By considering $[V_f, V_g]\cdot\partial_{\mathbf{x}} h$ for arbitrary h, show that $[V_f, V_g] = -V_{\{f,g\}}$. [*Hint*: Use Jacobi]

5. Let $\mathbf{x} = (x_1, x_2, x_3)$ be Cartesian coordinates for $\mathbf{R}^3$. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ be the velocity of an incompressible fluid ($\operatorname{div}\mathbf{u} = 0$) with vorticity $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$ and pressure $p = p(\mathbf{x}, t)$. Starting from Euler's equation

$$\mathbf{u}_t + (\mathbf{u}\cdot\partial_{\mathbf{x}})\mathbf{u} + \partial_{\mathbf{x}} p = 0$$

show that $\boldsymbol{\omega}_t = [\boldsymbol{\omega}, \mathbf{u}]$. Deduce the flows generated by the velocity field and the vorticity field commute iff the vorticity is time independent. [*Hint*: use $\partial_{\mathbf{x}}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\partial_{\mathbf{x}})\mathbf{b} + (\mathbf{b}\cdot\partial_{\mathbf{x}})\mathbf{a} + \mathbf{a}\times\operatorname{curl}\mathbf{b} + \mathbf{b}\times\operatorname{curl}\mathbf{a}$]

6. Let $\mathbf{x} = (\mathbf{q},\mathbf{p})$ and $\mathbf{y} = (\mathbf{Q},\mathbf{P})$. Using Hamilton's equations in the form $\dot{\mathbf{x}} = J\partial_{\mathbf{x}} H(\mathbf{x})$ and the chain rule, show that a coordinate transformation $\mathbf{x} \mapsto \mathbf{y} = \mathbf{y}(\mathbf{x})$ is canonical if and only if the derivative $D\mathbf{y}(\mathbf{x})$ is symplectic, i.e. $(D\mathbf{y})J(D\mathbf{y})^t = J$. ($D\mathbf{y}$ denotes the Jacobian matrix with entries $(D\mathbf{y})_{ij} = \partial y_i/\partial x_j$).
Hence show the transformation $\mathbf{x} \mapsto \mathbf{y}(\mathbf{x})$ is canonical iff the inverse transformation $\mathbf{y} \mapsto \mathbf{x}(\mathbf{y})$ is canonical.

7. Consider the four-dimensional phase space M with coordinates $(\mathbf{q},\mathbf{p}) = (\phi, r, p_\phi, p_r)$ and Hamiltonian

$$H(\phi, r, p_\phi, p_r) = \frac{p_\phi^2}{2r^2} + \frac{p_r^2}{2} - \frac{\alpha}{r}$$

where α is a positive constant. Use the fact that $\partial_\phi H = 0$ to show the existence of two first integrals in involution and deduce that this system is integrable in the sense of the Arnold-Liouville theorem. Show that on the level set $M_{\mathbf{c}} = \{H = c_1, p_\phi = c_2\}$ the coordinate p_r can be written in the form

$$p_r^2 = 2c_1 + \frac{2\alpha}{r} - \frac{c_2^2}{r^2} \equiv -\frac{2c_1}{r^2}(r_1 - r)(r - r_2)$$

where you should compute the numbers r_1, r_2 explicitly.
The coordinates $(\phi, p_\phi) \equiv (\phi, I_\phi)$ look like an "action-angle" pair. Construct the remaining action-angle coordinates by considering

$$I_r = \frac{1}{2\pi}\oint_{\Gamma_r} \mathbf{p}\cdot \mathrm{d}\mathbf{q},$$

where Γ_r is the cycle on $M_{\mathbf{c}}$ on which $\phi = \text{const}$. Conclude that

$$H(\phi, r, p_\phi, p_r) = \tilde{H}(I_\phi, I_r) = -\frac{\alpha^2}{2(|I_\phi| + I_r)^2}.$$

8. Let $\xi = x - ct$. By considering solutions of the form $u(x,t) = f(\xi)$ derive the 1-soliton solution to each of the following nonlinear (integrable) PDEs

$$\begin{aligned} \text{KdV:} \quad & u_t + u_{xxx} - 6uu_x = 0, \\ \text{Sine-Gordon:} \quad & u_{tt} - u_{xx} + \sin u = 0. \end{aligned}$$

In both cases you should assume that f and all its derivatives tend to zero as $\xi \to -\infty$. In each case you will find it useful to multiply an ODE by $f'(\xi)$. Plot the solutions for two differing values of t.

9. Assuming $u = u(x,t)$ is small, obtain the dispersion relation for the linearised Sine-Gordon equation. Deduce that small amplitude solutions to the Sine-Gordon equation are dispersive.

10. Let v be any solution of the equation $v_{xt} = 0$. Show that the two equations

$$u_x + v_x = \sqrt{2}\exp\left(\frac{u-v}{2}\right), \quad u_t - v_t = \sqrt{2}\exp\left(\frac{u+v}{2}\right)$$

are compatible only if u satisfies Liouville's equation: $u_{xt} = e^u$. These equations constitute a Bäcklund transformation. By considering the most general form of $v = v(x,t)$, show that

$$u(x,t) = 2\log\left(\frac{\sqrt{2}}{\text{const} - \int^x \exp[-f(\xi)]\,\mathrm{d}\xi - \int^t \exp[g(\tau)]\,\mathrm{d}\tau}\right) + g(t) - f(x)$$

solves Liouville's equation for arbitrary functions f and g.

Additional problems

These questions should not be attempted at the expense of earlier ones.

11. Let g^t be the flow associated with the Hamiltonian vector field $V_H = J\partial_{\mathbf{x}}H$. If $\mathbf{x}(0) = \mathbf{y}$, use Taylor's theorem to show that

$$g^t\mathbf{y} = \mathbf{y} + tV_H(\mathbf{y}) + o(t). \qquad (\star)$$

Let $D(t) = g^t D(0)$ be a region in M evolving via the Hamiltonian flow and let $\mathrm{Vol}(t)$ denote the volume of this region. By making the change of variables $\mathbf{x}(t) = g^t\mathbf{y}$, where $\mathbf{y} \in D(0)$, show that

$$\mathrm{Vol}(t) \equiv \int_{D(t)} \mathrm{d}^{2n}\mathbf{x} = \int_{D(0)} \det\left(\frac{\partial x_i}{\partial y_j}\right) \mathrm{d}^{2n}\mathbf{y}.$$

Using $(\star)$ and $\det(I + \epsilon A + o(\epsilon)) = 1 + \epsilon\,\mathrm{tr}(A) + o(\epsilon)$ for any matrix A, deduce that the derivative of $\mathrm{Vol}(t)$ vanishes at $t = 0$. What is the value of the derivative at arbitrary $t = t_0$? Deduce that the Hamiltonian flow preserves volume (this is known as Liouville's theorem – it's awesome).

12. Recall in the sketch proof of the Arnold-Liouville theorem we used the map $\varphi : \mathbf{R}^n/\mathrm{Stab}(\mathbf{x}) \to M_{\mathbf{c}}$ defined by $\varphi(\mathbf{t}) = g^{\mathbf{t}}\mathbf{x}$, where $g^{\mathbf{t}} = g_1^{t_1} \cdots g_n^{t_n}$ was the flow map associated with the independent, commutative Hamiltonian vector fields V_{f_i}, $i = 1, \ldots, n$. In this exercise we will show that φ is surjective.

(a) Show that $\varphi(\mathbf{t} + \epsilon\mathbf{e}_i) \equiv g_1^{t_1} \cdots g_i^{t_i+\epsilon} \cdots g_n^{t_n}\mathbf{x} = \varphi(\mathbf{t}) + \epsilon V_{f_i}(\varphi(\mathbf{t})) + o(\epsilon)$.

(b) Deduce that the Jacobian matrix $D\varphi(\mathbf{t})$ is of maximal rank for all $\mathbf{t} \in \mathbf{R}^n$.

(c) Using the inverse function theorem, conclude that φ is a local diffeomorphism.

(d) Let γ be a curve in $M_{\mathbf{c}}$ connecting $\mathbf{x}$ to an arbitrary point $\mathbf{y} \in M_{\mathbf{c}}$. By applying part (c) to small open sets covering the curve γ, show that φ is surjective.

13. Define $\mathbf{y}(s) = g_1^{-t} g_2^s g_1^t \mathbf{x}$ with $\mathbf{x}$ arbitrary, t fixed and g_1^t, g_2^s are the flow maps associated with the vector fields V_1 and V_2. Compute $\dot{\mathbf{y}}(s)$ and show that it is independent of t if $[V_1, V_2] = 0$. Deduce that $\dot{\mathbf{y}}(s) = V_2(\mathbf{y})$. Conclude that if two vector fields commute, then their corresponding flows commute.