1. Let $g^t$ be the flow map generated by the vector field $V$, so that $x(t) = g^t x_0$ is the unique solution to the differential equation $\dot{x} = V(x)$, $x(0) = x_0$. Show that $g^0 = I$, $g^{t+s} = g^t g^s$, $g^{-t} = (g^t)^{-1}$.

2. Let $V_1, V_2$ be vector fields that generate $g_1^t$ and $g_2^t$ respectively. By considering $\Delta(s, t) = g_1^s g_2^t x_0 - g_2^t g_1^s x_0$ and Taylor expanding up to and including second order, show that if $g_1$ and $g_2$ commute then $[V_1, V_2] = 0$.

3. Establish the Leibniz rule and the Jacobi identity for Poisson brackets

\[ \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0, \quad \{f, gh\} = g\{f, h\} + h\{f, g\}. \]

Deduce from the Jacobi identity that if $f = f(q, p)$ and $g = g(q, p)$ are two first integrals of a Hamiltonian system, then so is $h = \{f, g\}$.

4. Let $V_f = J \partial_q f$ and $V_g = J \partial_q g$ be two Hamiltonian vector fields corresponding to the functions $f$ and $g$. By considering $\{V_f, V_g\} = \partial_x h$ for arbitrary $h$, show that $[V_f, V_g] = -V_{\{f, g\}}$. [Hint: Use Jacobi]

5. Let $x = (x_1, x_2, x_3)$ be Cartesian coordinates for $\mathbb{R}^3$. Let $u = u(x, t)$ be the velocity of an incompressible fluid ($\text{div} \, u = 0$) with vorticity $\omega = \nabla \times u$ and pressure $p = p(x, t)$. Starting from Euler’s equation

\[ u_t + (u \cdot \nabla) u + \nabla p = 0 \]

show that $\omega_t = [\omega, u]$. Deduce the flows generated by the velocity field and the vorticity field commute iff the vorticity is time independent. [Hint: use $\partial_x (a \cdot b) = (a \cdot \nabla) b + (b \cdot \nabla) a$]

6. Let $x = (q, p)$ and $y = (Q, P)$. Using Hamilton’s equations in the form $\dot{x} = J \partial_q H(x)$ and the chain rule, show that a coordinate transformation $x \mapsto y = y(x)$ is canonical if and only if the derivative $Dy(x)$ is symplectic, i.e. $(Dy)J(Dy)^t = J$. $(Dy)$ denotes the Jacobian matrix with entries $(Dy)_{ij} = \partial y_i/\partial x_j$.

Hence show the transformation $x \mapsto y(x)$ is canonical iff the inverse transformation $y \mapsto x(y)$ is canonical.

7. Consider the four-dimensional phase space $M$ with coordinates $(q, p) = (\phi, r, p_\phi, p_r)$ and Hamiltonian

\[ H(\phi, r, p_\phi, p_r) = \frac{p_\phi^2}{2r^2} + \frac{p_r^2}{2} - \frac{\alpha}{r} \]

where $\alpha$ is a positive constant. Use the fact that $\partial_\phi H = 0$ to show the existence of two first integrals in involution and deduce that this system is integrable in the sense of the Arnold-Liouville theorem. Show that on the level set $M_\epsilon = \{H = c_1, p_\phi = c_2\}$ the coordinate $p_r$ can be written in the form

\[ p_r^2 = 2c_1 + \frac{2\alpha}{r} - \frac{c_1^2}{r^2} - \frac{2c_1}{r^2}(r_1 - r)(r - r_2) \]

where you should compute the numbers $r_1, r_2$ explicitly.

The coordinates $(\phi, p_\phi) \equiv (\phi, I_\phi)$ look like an “action-angle” pair. Construct the remaining action-angle coordinates by considering

\[ I_r = \frac{1}{2\pi} \oint_{\Gamma_r} p \cdot dq, \]

where $\Gamma_r$ is the cycle on $M_\epsilon$ on which $\phi = \text{const}$. Conclude that

\[ H(\phi, r, p_\phi, p_r) = \hat{H}(I_\phi, I_r) = -\frac{\alpha^2}{2(|I_\phi| + I_r)^2}. \]
8. Let $\xi = x - ct$. By considering solutions of the form $u(x,t) = f(\xi)$ derive the 1-soliton solution to each of the following nonlinear (integrable) PDEs

$$KdV: \quad u_t + u_{xxx} - 6uu_x = 0,$$

$$\text{Sine-Gordon:} \quad u_{tt} - u_{xx} + \sin u = 0.$$ 

In both cases you should assume that $f$ and all its derivatives tend to zero as $\xi \to -\infty$. In each case you will find it useful to multiply an ODE by $f'(\xi)$. Plot the solutions for two differing values of $t$.

9. Assuming $u = u(x,t)$ is small, obtain the dispersion relation for the linearised Sine-Gordon equation. Deduce that small amplitude solutions to the Sine-Gordon equation are dispersive.

10. Let $v$ be any solution of the equation $v_{xt} = 0$. Show that the two equations

$$u_x + v_x = \sqrt{2} \exp\left(\frac{u-v}{2}\right), \quad u_t - v_t = \sqrt{2} \exp\left(\frac{u+v}{2}\right)$$

are compatible if and only if $u$ satisfies Liouville's equation: $u_{xt} = e^u$. These equations constitute a Backlund transformation. By considering the most general form of $v = v(x,t)$, show that

$$u(x,t) = 2 \log \left(\frac{\sqrt{2}}{\int_s^x \exp[-f(\xi)] d\xi + \int_s^y \exp[g(\tau)] d\tau}\right) + g(t) - f(x)$$

solves Liouville's equation for arbitrary functions $f$ and $g$.

**Additional problems**

These questions should not be attempted at the expense of earlier ones.

11. Let $g^i$ be the flow associated with the Hamiltonian vector field $V_H = J\partial_\xi H$. If $x(0) = y$, use Taylor’s theorem to show that

$$g^i y = y + tV_H(y) + o(t). \quad (\star)$$

Let $D(t) = g^i D(0)$ be a region in $M \in M$ evolving via the Hamiltonian flow and let $\text{Vol}(t)$ denote the volume of this region. By making the change of variables $x(t) = g^i y$, where $y \in D(0)$, show that

$$\text{Vol}(t) \equiv \int_{D(t)} d^2n x = \int_{D(0)} \det \left(\frac{\partial x_i}{\partial y_j}\right) d^2n y.$$ 

Using $(\star)$ and $\det(I + \epsilon A + o(\epsilon)) = 1 + \epsilon \text{tr}(A) + o(\epsilon)$ for any matrix $A$, deduce that the derivative of $\text{Vol}(t)$ vanishes at $t = 0$. What is the value of the derivative at arbitrary $t = t_0$? Deduce that the Hamiltonian flow preserves volume (this is known as Liouville’s theorem – it’s awesome).

12. Recall in the sketch proof of the Arnold-Liouville theorem we used the map $\varphi : \mathbb{R}^n/\text{Stab}(x) \to M_c$ defined by $\varphi(t) = g^t x$, where $g^t = g_1^t \cdots g_n^t$ was the flow map associated with the independent, commutative Hamiltonian vector fields $V_{f_i}$, $i = 1, \ldots, n$. In this exercise we will show that $\varphi$ is surjective.

(a) Show that $\varphi(t + \epsilon e_i) \equiv g_1^t \cdots g_i^{t+\epsilon} \cdots g_n^t x = \varphi(t) + \epsilon V_{f_i}(\varphi(t)) + o(\epsilon)$.

(b) Deduce that the Jacobian matrix $D\varphi(t)$ is of maximal rank for all $t \in \mathbb{R}^n$.

(c) Using the inverse function theorem, conclude that $\varphi$ is a local diffeomorphism.

(d) Let $\gamma$ be a curve in $M_c$ connecting $x$ to an arbitrary point $y \in M_c$. By applying part (c) to small open sets covering the curve $\gamma$, show that $\varphi$ is surjective.

13. Define $y(s) = g_1^{-1} g_2^s g_1^t x$ with $x$ arbitrary, $t$ fixed and $g_1^t$, $g_2^s$ are the flow maps associated with the vector fields $V_1$ and $V_2$. Compute $\dot{y}(s)$ and show that it is independent of $t$ if $[V_1, V_2] = 0$. Deduce that $\dot{y}(s) = V_2(y)$. Conclude that if two vector fields commute, then their corresponding flows commute.