1.1. Let \( g^t \) be the flow map associated with the smooth vector field \( V \), so that \( x(t) = g^t x_0 \) is the unique solution to the ODE \( \dot{x} = V(x) \) with \( x(0) = x_0 \). Show that:

\[
g^0 = I, \quad g^{t+s} = g^t g^s, \quad g^{-t} = (g^t)^{-1}.
\]

1.2. Let \( \psi^s_1 \) and \( \psi^s_2 \) be commuting 1-parameter groups of transformations generated by the smooth vector fields \( V_1 \) and \( V_2 \) respectively. Show that \( \psi^s = \psi^s_1 \circ \psi^s_2 \) also defines a 1-parameter group of transformations and show that it is generated by \( V = V_1 + V_2 \).

Conversely, show that if a 1-parameter group of transformations \( \psi^s \) is generated by \( V = V_1 + V_2 \) where \( [V_1, V_2] = 0 \), then \( \psi^s = \psi^s_1 \circ \psi^s_2 \) where the \( \psi^s_1 \) and \( \psi^s_2 \) are generated by \( V_1 \) and \( V_2 \) as before.

1.3. Establish the Leibniz rule (derivation property) and the Jacobi identity of Poisson brackets:

\[
\{f, gh\} = \{f, g\} h + \{f, h\} g, \quad \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0
\]

Deduce that if \( f, g : M \to \mathbb{R} \) are two first integrals of the Hamiltonian system \((M, H)\), then so is \( h = \{f, g\} \).

1.4. We defined linear maps \( U : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \) to be symplectic if \( U^T J U = J \). Show that these form a group, and also that \( J \) is itself symplectic and that \( U \) is symplectic if \( U^T \) is symplectic. (For the first part you may find it convenient to consider the symplectic form \( \omega(X, Y) = X^T J Y \).)

1.5. Let \( x = (q, p) \) and \( y = (Q, P) \). Using the chain rule, show that a smooth coordinate transformation \( x \mapsto y = y(x) \) whose derivative\(^1\) \( D_y(x) \) is symplectic preserves the form of Hamilton’s equations \( \dot{x} = J \nabla H(x) \) for some transformed Hamiltonian, which you should give. Give an example of a linear transformation for which \( D_y \) is not symplectic but which preserves the form of Hamilton’s equations, and give the transformed Hamiltonian. (Hint: scale.) By expressing \( D_y(x) \) in terms of \( q, p, Q, P \), show that \( x \mapsto y(x) \) has symplectic derivative if and only if:

\[
\{Q_i, Q_j\}_{q, p} = \{P_i, P_j\}_{q, p} = 0, \quad \{Q_i, P_j\}_{q, p} = -\{P_j, Q_i\}_{q, p} = \delta_{ij}
\]

1.6. Consider the four dimensional phase space with coordinates \((q, p) = (\phi, r, p_\phi, p_r)\) and take as Hamiltonian:

\[
H(\phi, r, p_\phi, p_r) = \frac{p_\phi^2}{2r^2} + \frac{p_r^2}{2} - \frac{\alpha}{r}
\]

\(^1\)Here \( D_y \) denotes the Jacobian matrix with entries \( (D_y)_{ij} = \partial y_i / \partial x_j \).
where $\alpha$ is a positive constant. Use the fact that $\partial_\phi H = 0$ to show the existence of two first integrals in involution and deduce that the system is integrable in the sense of the Arnol’d-Liouville theorem. Show that on the level set $M_c = \{ H = c_1, p_\phi = c_2 \}$ the coordinate $p_r$ can be written in the form:

$$p_r^2 = 2c_1 + \frac{2\alpha}{r} - \frac{c_2^2}{r^2} \equiv -\frac{2c_1}{r^2} (r_1 - r)(r - r_2),$$

where you should find $r_1, r_2$ explicitly. The coordinates $(\phi, p_\phi) = (\phi, I_\phi)$ already look like an “action-angle” pair. Construct the remaining action-angle coordinates by considering

$$I_r = \frac{1}{2\pi} \oint_{\Gamma_r} p \cdot dq,$$

where $\Gamma_r$ is the cycle on $M_c$ on which $\phi = \text{const}$. Conclude that

$$H(\phi, r, p_\phi, p_r) = \tilde{H}(I_\phi, I_r) = -\frac{\alpha^2}{2(|I_\phi| + I_r)^2}.$$ 

1.7. Let $\xi = x - ct$. By seeking a solution of the form $u(x, t) = f(\xi)$, find the 1-soliton solution to each of the following nonlinear (integrable) PDEs:

- **KdV**: $u_t + u_{xxx} - 6uu_x = 0$,
- **Sine-Gordon**: $u_{tt} - u_{xx} + \sin u = 0$.

In both cases, assume that $f$ and all its derivatives vanish at $\xi \to -\infty$. In each case you will find it useful to multiply an ODE by $f'(\xi)$. Plot the solutions for two differing values of $t$.

1.8. Let $v$ be any solution of the wave equation in double-null coordinates: $v_{xt} = 0$. Show that the two equations:

$$u_x + v_x = \sqrt{2} \exp \left( \frac{u - v}{2} \right), \quad u_t - v_t = \sqrt{2} \exp \left( \frac{u + v}{2} \right),$$

are compatible iff $u$ satisfies Liouville’s equation $u_{xt} = e^u$. These equations constitute a Bäcklund transformation. By considering the most general form of $v = v(x, t)$, show that:

$$u(x, t) = 2 \log \left( \frac{\sqrt{2}}{\int^x \exp[-f(\xi)] d\xi + \int^t \exp[g(\tau)] d\tau} \right) + g(t) - f(x).$$

1.9. Suppose $u = u(x, t)$ satisfies the Hamiltonian evolution equation $u_t = J\delta H$. Show that if $I = I[u]$ then $I_t = \{ I, H \}$, where $\{ F, G \} = \langle \delta F, J \delta G \rangle$ is a Poisson bracket on the space of functionals. Deduce that if $I_1$ and $I_2$ are conserved, then so is $I_3 = \{ I_1, I_2 \}$.

1.10. Show that KdV $u_t + u_{xxx} - 6uu_x = 0$ can be written in Hamiltonian form in two distinct ways

$$H_0[u] = \int \frac{1}{2} u^2 \, dx, \quad J_0 = -\partial_x^3 + 4u\partial_x + 2u_x \quad \text{and} \quad H_1[u] = \int \left( \frac{1}{2} u_x^2 + u^3 \right) \, dx, \quad J_1 = \partial_x.$$ 

In both cases check that the operator $J$ is anti-symmetric.
1.11 (*). Let $g^t$ be the flow associated with the Hamiltonian vector field $X_H = J\nabla H$. If $x(0) = y$, use Taylor’s theorem to show that:

$$g^t y = y + tX_H(y) + o(t).$$

Let $D(t) = g^t D(0)$ be a region in $M$ evolving via the Hamiltonian flow and let $\text{Vol}(t)$ denote the volume of the region. By making the change of variables $x(t) = g^t y$, where $y \in D(0)$, show that:

$$\text{Vol}(t) = \int_{D(t)} d^2x = \int_{D(0)} \det \left( \frac{\partial x_i}{\partial y_j} \right) d^2y.$$

Using $\star$ and $\det(I + \epsilon A + o(\epsilon)) = 1 + \text{tr} A + o(\epsilon)$ for any matrix $A$, deduce that the derivative of $\text{Vol}(t)$ vanishes at $t = 0$. What is the value of the derivative at an arbitrary $t = t_0$? Deduce that the Hamiltonian flow preserves volume. This is known as Liouville’s theorem and is an important result in statistical mechanics and ergodic theory.

1.12 (*). Let $V, W$ be two smooth vector fields on $\mathbb{R}^l$ with corresponding flow maps $g^t, h^s$ (which you may assume to exist for all times, with the maps $(t, x) \mapsto g(t, x) = g^t(x)$ also smooth.) Consider the two curves $s \mapsto g^t(h^s(x))$ and $s \mapsto h^s(g^t(x))$ for fixed (arbitrary) $t, x$. Show that (i) they satisfy the same initial condition at $s = 0$, and (ii) if $[V, W] = 0$ they are integral curves of the same ODE. Deduce that if $[V, W] = 0$ the two flows commute, i.e., $g^t(h^s(x)) = h^s(g^t(x))$. (You may find it useful to consider the $t$ dependence of the two vector fields defined by the ODEs obtained in (ii).)