Part II Integrable Systems, Sheet One

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1. **Jacobi identity.** Assume that (p_j, q_j) satisfy the Hamilton equations and show that any function $f = f(p, q, t)$ satisfies
$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, H\},$$
where H is the Hamiltonian.

   Show that the Jacobi identity
$$\{f_1, \{f_2, f_3\}\} + \{f_3, \{f_1, f_2\}\} + \{f_2, \{f_3, f_1\}\} = 0 \qquad (1)$$
holds for Poisson brackets.

   Deduce that if functions f_1 and f_2 which do not explicitly depend on time are first integrals of a Hamiltonian system then so is $f_3 = \{f_1, f_2\}$.

2. **Canonical transformations.**

   - Find the canonical transformation generated by
$$S = \sum_{k=1}^{n} q_k P_k.$$
   - Show that the canonical transformations preserve volume in the two–dimensional phase space, i.e.
$$\frac{\partial(P, Q)}{\partial(p, q)} = 1.$$
[This result also holds in phase spaces of arbitrary dimension.]
   - Show that the transformation
$$Q = \cos(\beta)q - \sin(\beta)p, \quad P = \sin(\beta)q + \cos(\beta)p$$
is canonical for any constant $\beta \in \mathbb{R}$. Find the corresponding generating function. Is it defined for all β?

3. **Action variables for the Kepler problem.** Consider the four–dimensional phase space coordinatised by
$$q_1 = \phi, \quad q_2 = r, \quad p_1 = p_\phi, \quad p_2 = p_r$$
equipped with a Hamiltonian
$$H = \frac{{p_\phi}^2}{2r^2} + \frac{{p_r}^2}{2} - \frac{\alpha}{r}$$
where $\alpha > 0$ is a constant. Use the fact that $\partial_\phi H = 0$ to show the existence of two first integrals in involution and deduce that this system is integrable in a sense of the Arnold–Liouville theorem.

   Construct the action variables. Express the Hamiltonian in terms of the action variables to show that the frequencies associated to the corresponding angles are equal.

   [Hint: ϕ and one function of (r, p_r) parametrise M_f. Varying ϕ and fixing the other coordinate gives one cycle $\Gamma_\phi \subset M_f$. To find the second action coordinate fix ϕ (on top of H and p_ϕ).]

4. **Radial harmonic oscillator.**

   (a) Consider the Hamiltonian system on phase space $\mathbb{R}^4$ defined by $H_1(q_1, q_2, p_1, p_2) = \frac{1}{2}(p_1^2 + \omega_1^2 q_1^2 + p_2^2 + \omega_2^2 q_2^2)$, with ω_1, ω_2 positive real numbers. Find two first integrals which are in involution and action-angle variables. Writing the system in terms of these variables, show that the system is integrable. Find a relation between ω_1 and ω_2 which ensures that all solutions are periodic in t, show this relation holds if $\omega_1 = \omega_2$ and find an additional first integral in this case.

   (b) Consider the Hamiltonian for motion of a particle of unit mass in a radially symmetric harmonic potential on the plane
   $$H_2(\phi, r, p_\phi, p_r) = \frac{p_\phi^2}{2r^2} + \frac{p_r^2}{2} + \frac{1}{2}\omega^2 r^2$$
   in polar coordinates. Working in polar coordinates, and using the integral
   $$\int_b^a \frac{1}{x}\sqrt{(a-x)(x-b)}dx = \pi\Big(\frac{a+b}{2} - \sqrt{ab}\Big), \qquad 0 < b < a < \infty,$$
   find action-angle variables for H_2 and show that all solutions are periodic in t.
   Comment on the relation with part (a) of the question.

5. **Poisson structures.** A Poisson on structure on $\mathbb{R}^{2n}$ is an anti–symmetric matrix ω^{ab} whose components depend on the coordinates $\xi^a \in \mathbb{R}^{2n}, a = 1, \cdots, 2n$ and such that the Poisson bracket
   $$\{f, g\} = \sum_{a,b=1}^{2n} \omega^{ab}(\xi) \frac{\partial f}{\partial \xi^a}\frac{\partial g}{\partial \xi^b}$$
   satisfies the Jacobi identity (1).
   Show that
   $$\{fg, h\} = f\{g, h\} + \{f, h\}g.$$
   Assume that the matrix ω is invertible with $W := (\omega^{-1})$ and show that the antisymmetric matrix $W_{ab}(\xi)$ satisfies
   $$\partial_a W_{bc} + \partial_c W_{ab} + \partial_b W_{ca} = 0. \qquad (2)$$
   [Hint: note that $\omega^{ab} = \{\xi^a, \xi^b\}$.] Deduce that if $n = 1$ then any antisymmetric invertible matrix $\omega(\xi^1, \xi^2)$ gives rise to a Poisson structure (i.e. show that the Jacobi identity holds automatically in this case).

   [In differential geometry the invertible antisymmetric matrix W which satisfies (2) is called a symplectic structure. We have therefore deduced that symplectic structures are special cases of Poisson structures.]

6. **KdV and its 1–soliton solution** Verify that the equation
   $$\frac{1}{v}\Psi_t + \Psi_x + \beta\Psi_{xxx} + \alpha\Psi\Psi_x = 0.$$
   where $\Psi = \Psi(x, t)$ and (v, β, α) are non–zero constants is equivalent to the KdV equation
   $$u_t - 6uu_x + u_{xxx} = 0, \qquad u = u(x, t) \qquad (3)$$
   after a suitable change of dependent and independent variables.
   Assume that a solution of the KdV equation (3) is of the form
   $$u(x, t) = f(\xi), \qquad \text{where} \quad \xi = x - ct$$
   for some constant c. Show that the function $f(\xi)$ satisfies the ODE
   $$\frac{1}{2}(f')^2 = f^3 + \frac{1}{2}cf^2 + \alpha f + \beta$$
   where (α, β) are arbitrary constants. Assume that f and its first two derivatives tend to zero as $|\xi| \to \infty$ and solve the ODE to construct the one–soliton solution to the KdV equation.

7. **Sine–Gordon soliton from Backlund transformations.** The Sine–Gordon equation is

$$\phi_{xx} - \phi_{tt} = \sin(\phi), \qquad \phi = \phi(x,t).$$

Set $\tau = (x+t)/2, \rho = (x-t)/2$ and consider the Bäcklund transformations

$$\partial_\rho(\phi_1 - \phi_0) = 2b \sin\left(\frac{\phi_1 + \phi_0}{2}\right), \qquad \partial_\tau(\phi_1 + \phi_0) = 2b^{-1} \sin\left(\frac{\phi_1 - \phi_0}{2}\right),$$

where $b = \text{const}$ and ϕ_0, ϕ_1 are functions of (τ, ρ). Take $\phi_0 = 0$ and construct the 1-soliton (kink) solution ϕ_1. Draw the graph of $\phi_1(x,t)$ for a fixed value of t. What happens when t varies?

8. **Backlünd for Liouville equation.** Let v be any solution of the wave equation in double-null coordinates: $v_{xt} = 0$. Show that the two equations:

$$u_x + v_x = \sqrt{2} \exp\left(\frac{u-v}{2}\right), \qquad u_t - v_t = \sqrt{2} \exp\left(\frac{u+v}{2}\right), \tag{4}$$

are compatible iff u satisfies Liouville's equation $u_{xt} = e^u$. These equations constitute a Bäcklund transformation. By considering the most general form of $v = v(x,t)$, show that:

$$u(x,t) = 2 \log\left(-\frac{\sqrt{2}}{\int^x \exp[-f(\xi)]d\xi + \int^t \exp[g(\tau)]d\tau}\right) + g(t) - f(x). \tag{5}$$

9. **Miura transformation**. Let $v = v(x,t)$ satisfy the modified KdV equation

$$v_t - 6v^2 v_x + v_{xxx} = 0.$$

Show that the function $u(x,t)$ given by

$$u = v^2 + v_x \tag{6}$$

satisfy the KdV equation. Is it true that any solution u to the KdV equation gives rise, via (6), to a solution of the modified KdV equation?

Books. The course follows the first four chapters of
Dunajski, M. (2024) *Solitons, Instantons and Twistors*, 2nd edition, OUP.
Other interesting books are

- Hamiltonian Systems.

  Arnold, V. I. *Mathematical Methods of Classical Mechanics.* (This uses a language of differential forms but has the best possible exposition of the Arnold–Liouville theorem. Chapter 10 is most relevant).

  Schuster, H. G. *Deterministic Chaos: An Introduction.* (A popular introduction to KAM theorem and ergodicity with some mention of integrable systems).

- Solitons and Inverse Scattering.

  Novikov S., Manakov S. V., Pitaevskii L. P., Zaharov V. E., *Theory of Solitons.* (The lectures follow Chapter 1 of this book in the treatment of the KdV equation and solitons).

  Drazin, P. G., Johnson, R.S. *Solitons: an introduction.* (A very readable text. Chapters 3, 4, 5 are most relevant).

- Lie symmetries, Painleve equations.

  Hydon P. E. *Symmetry Methods for Differential Equations: A Beginner's Guide.* (Elementary and very easy to follow)

  Olver, P. J. *Applications of Lie groups to differential equations.*

  Fokas, A.S. et. al. *Painleve transcendents. The Riemann-Hilbert approach.*