**1.1.** Let $g^t$ be the flow map associated with the vector field $V$, so that $\boldsymbol{x}(t) = g^t\boldsymbol{x}_0$ is the unique solution to the ODE $\dot{\boldsymbol{x}} = V(\boldsymbol{x})$ with $\boldsymbol{x}(0) = \boldsymbol{x}_0$. Show that:

$$g^0 = I, \quad g^{t+s} = g^t g^s, \quad g^{-t} = (g^t)^{-1}.$$

**1.2.** Let $V, W$ be vector fields that generate flows $g^t, h^s$ respectively. By considering $\Delta(s,t) = g^t h^s \boldsymbol{x}_0 - h^s g^t \boldsymbol{x}_0$ and Taylor expanding up to and including second order, show that if $g, h$ commute then $[V, W] = 0$.

**1.3.** Establish the Leibniz rule and the Jacobi identity of Poisson brackets:

$$\{f, gh\} = \{f, g\}\, h + \{f, h\}\, g, \qquad \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

Deduce that if $f, g : M \to \mathbb{R}$ are two first integrals of the Hamiltonian system $(M, H)$, then so is $h = \{f, g\}$.

**1.4.** Let $X_f = \Omega\partial f$ and $X_g = \Omega\partial g$ be two Hamiltonian vector fields corresponding to the functions $f, g$. By considering $[X_f, X_g] \cdot \partial h$ for an arbitrary function $h : M \to \mathbb{R}$ and using the Jacobi identity, show:

$$[X_f, X_g] = -X_{\{f,g\}}.$$

**1.5.** Suppose $M = \mathbb{R}^6 = \{(\boldsymbol{q}, \boldsymbol{p})\}$ with the usual $\Omega$ and define the angular momentum by $\boldsymbol{L} = \boldsymbol{q} \times \boldsymbol{p}$. Show that if the Hamiltonian takes the form

$$H = \frac{1}{2}|\boldsymbol{p}|^2 + U(|\boldsymbol{q}|)$$

then each $L_i$, $i = 1, 2, 3$ is a first integral of the motion. Show that $\{L_i, L_j\} = \epsilon_{ijk}L_k$ and establish that $L_3$ and $|\boldsymbol{L}|^2 = L_1^2 + L_2^2 + L_3^2$ are constants of the motion in involution. Deduce that $(M, H)$ is integrable.

**1.6.** Let $\boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{p})$ and $\boldsymbol{y} = (\boldsymbol{Q}, \boldsymbol{P})$. Using Hamilton's equations in the form $\dot{\boldsymbol{x}} = \Omega\partial H(\boldsymbol{x})$ and using the chain rule, show that a coordinate transformation $\boldsymbol{x} \mapsto \boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x})$ is canonical if and only if the derivative[1] $D\boldsymbol{y}(\boldsymbol{x})$ is symplectic. Hence show that the transformation $\boldsymbol{x} \mapsto \boldsymbol{y}(\boldsymbol{x})$ is canonical if and only if the inverse transformation $\boldsymbol{y} \mapsto \boldsymbol{x}(\boldsymbol{y})$ is canonical. By expressing $D\boldsymbol{y}(\boldsymbol{x})$ in terms of $\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{Q}, \boldsymbol{P}$, show that $\boldsymbol{x} \mapsto \boldsymbol{y}(\boldsymbol{x})$ is canonical if and only if:

$$\{Q_i, Q_j\}_{q,p} = \{P_i, P_j\}_{q,p} = 0, \qquad \{Q_i, P_j\}_{q,p} = -\{P_j, Q_i\}_{q,p} = \delta_{ij}$$

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Please send any corrections to cmw50@cam.ac.uk

Questions marked $(*)$ are optional and should not be attempted at the expense of unstarred questions

[1]Here $D\boldsymbol{y}$ denotes the Jacobian matrix with entries $(D\boldsymbol{y})_{ij} = \partial y_i / \partial x_j$.

**1.7.** Consider the four dimensional phase space with coordinates $(\boldsymbol{q}, \boldsymbol{p}) = (\phi, r, p_\phi, p_r)$ and take as Hamiltonian:

$$H(\phi, r, p_\phi, p_r) = \frac{p_\phi^2}{2r^2} + \frac{p_r^2}{2} - \frac{\alpha}{r}$$

where $\alpha$ is a positive constant. Use the fact that $\partial_\phi H = 0$ to show the existence of two first integrals in involution and deduce that the system is integrable in the sense of the Arnol'd-Liouville theorem. Show that on the level set $M_{\boldsymbol{c}} = \{H = c_1, p_\phi = c_2\}$ the coordinate $p_r$ can be written in the form:

$$p_r^2 = 2c_1 + \frac{2\alpha}{r} - \frac{c_2^2}{r^2} \equiv -\frac{2c_1}{r^2}(r_1 - r)(r - r_2),$$

where you should find $r_1, r_2$ explicitly. The coordinates $(\phi.p_\phi) = (\phi, I_\phi)$ already look like an "action-angle" pair. Construct the remaining action-angle coordinates by considering

$$I_r = \frac{1}{2\pi} \oint_{\Gamma_r} \boldsymbol{p} \cdot d\boldsymbol{q},$$

where $\Gamma_r$ is the cycle on $M_{\boldsymbol{c}}$ on which $\phi = \text{const}$. Conclude that

$$H(\phi, r, p_\phi, p_r) = \tilde{H}(I_\phi, I_r) = -\frac{\alpha^2}{2(|I_\phi| + I_r)^2}.$$

**1.8.** Let $\xi = x = ct$. By seeking a solution of the form $u(x,t) = f(\xi)$, find the 1−soliton solution to each of the following nonlinear (integrable) PDEs:

$$\begin{aligned} \text{KdV:} \quad & u_t + u_{xxx} - 6uu_x = 0, \\ \text{Sine-Gordon:} \quad & u_{tt} - u_{xx} + \sin u = 0. \end{aligned}$$

In both cases, assume that $f$ and all its derivatives vanish at $\xi \to -\infty$. In each case you will find it useful to multiply an ODE by $f'(\xi)$. Plot the solutions for two differing values of $t$.

**1.9.** Assuming $u = u(x,t)$ is small, obtain the dispersion relation for the linearised Sine-Gordon equation. Deduce that small amplitude solutions to the Sine-Gordon equation are dispersive.

**1.10.** Let $v$ be any solution of the wave equation in double-null coordinates: $v_{xt} = 0$. Show that the two equations:

$$u_x + v_x = \sqrt{2}\exp\left(\frac{u-v}{2}\right), \qquad u_t - v_t = \sqrt{2}\exp\left(\frac{u+v}{2}\right),$$

are compatible iff $u$ satisfies Liouville's equation $u_{xt} = e^u$. These equations constitute a Bäcklund transformation. By considering the most general form of $v = v(x,t)$, show that:

$$u(x,t) = 2\log\left(-\frac{\sqrt{2}}{\int^x \exp[-f(\xi)]d\xi - \int^t \exp[g(\tau)]d\tau}\right) + g(t) - f(x).$$

**1.11** (*). Let $g^t$ be the flow associated with the Hamiltonian vector field $X_H = \Omega \partial H$. If $\boldsymbol{x}(0) = \boldsymbol{y}$, use Taylor's theorem to show that:

$$g^t \boldsymbol{y} = \boldsymbol{y} + t X_H(\boldsymbol{y} + o(t). \tag{$\star$}$$

Let $D(t) = g^t D(0)$ be a region in $M$ evolving via the Hamiltonian flow and let $\mathrm{Vol}(t)$ denote the volume of the region. By making the change of variables $\boldsymbol{x}(t) = g^t \boldsymbol{y}$, where $\boldsymbol{y} \in D(0)$, show that:

$$\mathrm{Vol}(t) \equiv \int_{D(t)} d^{2m}\boldsymbol{x} = \int_{D(0)} \det\left(\frac{\partial x_i}{\partial y_j}\right) d^{2m}\boldsymbol{y}.$$

Using ($\star$) and $\det(I + \epsilon A + o(\epsilon)) = 1 + \epsilon \mathrm{tr} A + o(\epsilon)$ for any matrix $A$, deduce that the derivative of $\mathrm{Vol}(t)$ vanishes at $t = 0$. What is the value of the derivative at an arbitrary $t = t_0$? Deduce that the Hamiltonian flow preserves volume. This is known as Liouville's theorem and is an important result in statistical mechanics and ergodic theory.

**1.12** (*). In sketching the proof of the Arnol'd-Liouville theorem we asserted the surjectivity of the map $\varphi : \mathbb{R}^n/\mathrm{Stab}(\boldsymbol{x}) \to M_{\boldsymbol{c}}$ given by $\varphi(\boldsymbol{t}) = g^{\boldsymbol{t}}\boldsymbol{x}$, where $g^{\boldsymbol{t}} = g_1^{t_1} g_2^{t_2} \cdots g_n^{t_n}$ was the flow map associated with the independent, commutative, Hamiltonian flows $X_{f_i}$, $i = 1, \ldots, n$. In this exercise we establish this surjectivity.

(a) Show that $\varphi(\boldsymbol{t} + \epsilon \boldsymbol{e}_i) \equiv g_1^{t_1} \cdots g_i^{t_i+\epsilon} \cdots g_n^{t_n}\boldsymbol{x} = \varphi(\boldsymbol{t}) + \epsilon X_{f_i}(\varphi(\boldsymbol{t})) + o(\epsilon)$.

(b) Deduce that the Jacobian matrix $D\varphi(\boldsymbol{t})$ is of maximal rank for all $\boldsymbol{t} \in \mathbb{R}^n$.

(c) Using the inverse function theorem, deduce that $\varphi$ is a local diffeomorphism.

(d) Let $\gamma$ be a curve in $M_{\boldsymbol{c}}$ connecting $\boldsymbol{x}$ to an arbitrary point $\boldsymbol{y} \in M_{\boldsymbol{c}}$. By applying part ($c$) to small open sets covering the curve $\gamma$, show that $\varphi$ is surjective.

**1.13** (*). Let $V, W$ be two vector fields with corresponding flow maps $g^t, h^s$. Define $\boldsymbol{y}(s) = g^{-t} h^s g^t \boldsymbol{x}$ with $\boldsymbol{x}$ arbitrary and $t$ fixed. Compute $\dot{\boldsymbol{y}}(s)$ and show that it is independent of $t$ if $[V, W] = 0$. Deduce that $\dot{\boldsymbol{y}}(s) = W(\boldsymbol{y})$. Conclude that if two vector fields commute, then their corresponding flows commute.