1.1. Let $g^t$ be the flow map associated with the vector field $V$, so that $x(t) = g^t x_0$ is the unique solution to the ODE $\dot{x} = V(x)$ with $x(0) = x_0$. Show that:

$$g^0 = I, \quad g^{t+s} = g^t g^s, \quad g^{-t} = (g^t)^{-1}.$$ 

1.2. Let $V, W$ be vector fields that generate flows $g^t, h^s$ respectively. By considering \( \Delta(s, t) = g^t h^s x_0 - h^s g^t x_0 \) and Taylor expanding up to and including second order, show that if $g, h$ commute then $[V, W] = 0$.

1.3. Establish the Leibniz rule and the Jacobi identity of Poisson brackets:

$$\{ f, gh \} = \{ f, g \} h + \{ f, h \} g, \quad \{ f, \{ g, h \} \} + \{ g, \{ f, h \} \} + \{ h, \{ f, g \} \} = 0$$

Deduce that if $f, g : M \to \mathbb{R}$ are two first integrals of the Hamiltonian system $(M, H)$, then so is $h = \{ f, g \}$.

1.4. Let $X_f = \Omega \partial f$ and $X_g = \Omega \partial g$ be two Hamiltonian vector fields corresponding to the functions $f, g$. By considering $[X_f, X_g] \cdot \partial h$ for an arbitrary function $h : M \to \mathbb{R}$ and using the Jacobi identity, show:

$$[X_f, X_g] = -X_{\{ f, g \}}.$$ 

1.5. Suppose $M = \mathbb{R}^6 = \{(q, p)\}$ with the usual $\Omega$ and define the angular momentum by $L = q \times p$. Show that if the Hamiltonian takes the form

$$H = \frac{1}{2} |p|^2 + U(|q|)$$

then each $L_i, \, i = 1, 2, 3$ is a first integral of the motion. Show that $\{L_i, L_j\} = \epsilon_{ijk} L_k$ and establish that $L_3$ and $|L|^2 = L_1^2 + L_2^2 + L_3^2$ are constants of the motion in involution. Deduce that $(M, H)$ is integrable.

1.6. Let $x = (q, p)$ and $y = (Q, P)$. Using Hamilton’s equations in the form $\dot{x} = \Omega \partial H(x)$ and using the chain rule, show that a coordinate transformation $x \mapsto y = y(x)$ is canonical if and only if the derivative\(^1\) $Dy(x)$ is symplectic. Hence show that the transformation $x \mapsto y(x)$ is canonical if and only if the inverse transformation $y \mapsto x(y)$ is canonical. By expressing $Dy(x)$ in terms of $q, p, Q, P$, show that $x \mapsto y(x)$ is canonical if and only if:

$$\{Q_i, Q_j\}_{q,p} = \{P_i, P_j\}_{q,p} = 0, \quad \{Q_i, P_j\}_{q,p} = - \{P_j, Q_i\}_{q,p} = \delta_{ij}$$

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Please send any corrections to cmw50@cam.ac.uk

Questions marked (*) are optional and should not be attempted at the expense of unstared questions

\(^1\)Here $Dy$ denotes the Jacobian matrix with entries $(Dy)_{ij} = \partial y_i / \partial x_j$.\)
1.7. Consider the four dimensional phase space with coordinates \((q, p) = (\phi, r, p\phi, p_r)\) and take as Hamiltonian:

\[
H(\phi, r, p\phi, p_r) = \frac{p^2}{2r^2} + \frac{p^2_r}{2} - \frac{\alpha}{r}
\]

where \(\alpha\) is a positive constant. Use the fact that \(\partial_\phi H = 0\) to show the existence of two first integrals in involution and deduce that the system is integrable in the sense of the Arnol’d-Liouville theorem. Show that on the level set \(M_c = \{H = c_1, p\phi = c_2\}\) the coordinate \(p_r\) can be written in the form:

\[
p_r^2 = 2c_1 + \frac{2\alpha}{r} - \frac{c_2^2}{r^2} \equiv -\frac{2c_1}{r^2} (r_1 - r)(r - r_2),
\]

where you should find \(r_1, r_2\) explicitly. The coordinates \((\phi, p\phi) = (\phi, I_\phi)\) already look like an “action-angle” pair. Construct the remaining action-angle coordinates by considering

\[
I_r = \frac{1}{2\pi} \oint_{\Gamma_r} p \cdot dq,
\]

where \(\Gamma_r\) is the cycle on \(M_c\) on which \(\phi = \text{const.}\) Conclude that

\[
H(\phi, r, p\phi, p_r) = \tilde{H}(I_\phi, I_r) = -\frac{\alpha^2}{2(|I_\phi| + I_r)^2}.
\]

1.8. Let \(\xi = x - ct\). By seeking a solution of the form \(u(x, t) = f(\xi)\), find the 1–soliton solution to each of the following nonlinear (integrable) PDEs:

- KdV: \(u_t + uu_{xxx} - 6u u_x = 0\),
- Sine-Gordon: \(u_{tt} - uu_{xx} + \sin u = 0\).

In both cases, assume that \(f\) and all its derivatives vanish at \(\xi \to -\infty\). In each case you will find it useful to multiply an ODE by \(f'(\xi)\). Plot the solutions for two differing values of \(t\).

1.9. Assuming \(u = u(x, t)\) is small, obtain the dispersion relation for the linearised Sine-Gordon equation. Deduce that small amplitude solutions to the Sine-Gordon equation are dispersive.

1.10. Let \(v\) be any solution of the wave equation in double-null coordinates: \(v_{xt} = 0\). Show that the two equations:

\[
u_x + v_x = \sqrt{2} \exp \left(\frac{u - v}{2}\right), \quad u_t - v_t = \sqrt{2} \exp \left(\frac{u + v}{2}\right),
\]

are compatible iff \(u\) satisfies Liouville’s equation \(u_{xt} = e^u\). These equations constitute a Bäcklund transformation. By considering the most general form of \(v = v(x, t)\), show that:

\[
u(x, t) = 2 \log \left(\frac{\sqrt{2}}{\int_x^\infty \exp[-f(\xi)]d\xi - \int_x^t \exp[g(\tau)]d\tau}\right) + g(t) - f(x).
\]
1.11 (*). Let $g^t$ be the flow associated with the Hamiltonian vector field $X_H = \Omega \partial H$. If $x(0) = y$, use Taylor’s theorem to show that:

$$g^t y = y + tX_H(y + o(t)).$$

(*)

Let $D(t) = g^tD(0)$ be a region in $M$ evolving via the Hamiltonian flow and let $\text{Vol}(t)$ denote the volume of the region. By making the change of variables $x(t) = g^t y$, where $y \in D(0)$, show that:

$$\text{Vol}(t) \equiv \int_{D(t)} d^2m = \int_{D(0)} \det \left( \frac{\partial x_i}{\partial y_j} \right) d^2y.$$

Using (*) and $\det(I + \epsilon A + o(\epsilon)) = 1 + \epsilon \text{tr} A + o(\epsilon)$ for any matrix $A$, deduce that the derivative of $\text{Vol}(t)$ vanishes at $t = 0$. What is the value of the derivative at an arbitrary $t = t_0$? Deduce that the Hamiltonian flow preserves volume. This is known as Liouville’s theorem and is an important result in statistical mechanics and ergodic theory.

1.12 (*). In sketching the proof of the Arnol’d-Liouville theorem we asserted the surjectivity of the map $\varphi : \mathbb{R}^n / \text{Stab}(x) \to M_c$ given by $\varphi(t) = g^t x$, where $g^t = g^{t_1} \cdot g^{t_2} \cdot \cdots \cdot g^{t_n}$ was the flow map associated with the independent, commutative, Hamiltonian flows $X_{f_i}$, $i = 1, \ldots, n$. In this exercise we establish this surjectivity.

(a) Show that $\varphi(t + \epsilon e_i) \equiv g^{t_1} \cdot \cdots \cdot g^{t_i+\epsilon} \cdot \cdots \cdot g^{t_n} x = \varphi(t) + \epsilon X_{f_i}(\varphi(t)) + o(\epsilon)$.

(b) Deduce that the Jacobian matrix $D\varphi(t)$ is of maximal rank for all $t \in \mathbb{R}^n$.

(c) Using the inverse function theorem, deduce that $\varphi$ is a local diffeomorphism.

(d) Let $\gamma$ be a curve in $M_c$ connecting $x$ to an arbitrary point $y \in M_c$. By applying part (c) to small open sets covering the curve $\gamma$, show that $\varphi$ is surjective.

1.13 (*). Let $V, W$ be two vector fields with corresponding flow maps $g^t, h^s$. Define $y(s) = g^{-t} h^s g^t x$ with $x$ arbitrary and $t$ fixed. Compute $\dot{y}(s)$ and show that it is independent of $t$ if $[V, W] = 0$. Deduce that $\dot{y}(s) = W(y)$. Conclude that if two vector fields commute, then their corresponding flows commute.