

2.1. Using the method of characteristics show that the solution to the initial value problem $u_t = 6uu_x$, $u(x, 0) = f(x)$ is given by $u(x, t) = f(\xi)$ where ξ is defined implicitly by $\xi = x + 6tf(\xi)$. Show that the slope u_x first becomes infinite when $t = \min_{\xi} [6f'(\xi)]^{-1}$.

2.2. Let L be a Schrödinger operator with potential u which decays rapidly at infinity. Show that if $L\psi = \lambda\psi$ and $L\psi' = \lambda\psi'$ then the Wronskian $W(\psi, \psi') \equiv \psi\psi'_x - \psi'\psi_x$ is constant. Using this fact establish the following results concerning the discrete and continuous parts of the spectrum of L respectively:

- (i) Show that if ψ and ψ' are bound states corresponding to the same discrete eigenvalue then $\psi \propto \psi'$. Deduce that the discrete eigenvalues are non-degenerate, i.e. each discrete eigenvalue corresponds to exactly one bound state.
- (ii) Show that the reflection and transmission coefficients obey $|R(k)|^2 + |T(k)|^2 = 1$ for all k . [*Hint:* if $L\Phi = k^2\Phi$ then $L\Phi^* = k^2\Phi^*$ also, the star denoting complex conjugation.]

2.3. Let L be the Schrödinger operator associated with the Dirac potential $u(x) = 2\alpha\delta(x)$, $\alpha \neq 0$. Show that the reflection and transmission coefficients associated with the continuous spectrum are

$$R(k) = -\frac{i\alpha}{k + i\alpha}, \quad T(k) = \frac{k}{k + i\alpha}.$$

Verify that $|T|^2 + |R|^2 = 1$. Show that there are no bound states if $\alpha > 0$ and one bound state if $\alpha < 0$.

2.4. Consider the family of self-adjoint operators $L(t)$ on some complex inner product space defined by

$$L(t) = U(t)L(0)U(t)^\dagger$$

where $U(t)$ is a unitary operator, i.e. $U(t)^\dagger U(t) = I$. Show that $L(t)$ and $L(0)$ have the same eigenvalues. Show that there exists a anti-selfadjoint operator $A = -A^\dagger$ such that $U_t = -AU$ and $L_t = [L, A]$.

Conversely, suppose L satisfies $L_t = [L, A]$ for some anti-selfadjoint A . Let $U(t)$ be the solution of $U_t = -AU$, $U(0) = I$. Show that $U(t)$ is unitary. By considering $\frac{d}{dt}U(t)^\dagger L(t)U(t)$ or otherwise, show that $L(t) = U(t)L(0)U(t)^\dagger$.

2.5. For u a real function, define the linear operators

$$L = -\partial_x^2 + u(x, t), \quad A = 4\partial_x^3 - 3u\partial_x - 3\partial_x u.$$

Show that the KdV equation is equivalent to Lax's equation $L_t = [L, A]$.

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Questions marked (*) are optional and should not be attempted at the expense of unstarred questions

Show that L is selfadjoint and A is anti-selfadjoint: $\langle \varphi, L\psi \rangle = \langle L\varphi, \psi \rangle$, $\langle \varphi, A\psi \rangle = -\langle A\varphi, \psi \rangle$ for any smooth, rapidly decaying functions ψ and φ . If $\|\psi(t)\| = 1$ and $\tilde{\psi}(t) = \psi_t(t) + A\psi(t)$, show that ψ and $\tilde{\psi}$ are orthogonal, i.e. $\langle \tilde{\psi}, \psi \rangle = 0$. Conclude that if u satisfies the KdV equation and ψ is a bound state for L then $\psi_t + A\psi = 0$. [Hint: use question 2.2(i).]

2.6. Let $L(t)$ and $A(t)$ be $n \times n$ matrices such that

$$\frac{dL}{dt} = [L, A].$$

Show that $\text{tr}(L^p)$, $p \in \mathbf{Z}$, does not depend on t .

2.7. Show that for any non-singular square matrix $A = A(x)$

$$\frac{1}{\det A} \frac{d}{dx} \det A = \text{tr} \left(A^{-1} \frac{dA}{dx} \right).$$

2.8. Suppose a potential $u = u(x)$ is *reflectionless*, i.e. $R(k) = 0$ in the scattering data for the associated Schrödinger operator L . By writing the GLM equation in the form

$$K(x, y) = -F(x+y) - \int_x^\infty K(x, z)F(z+y) dz, \quad \text{where} \quad F(x) = \sum_{n=1}^N c_n^2 e^{-\chi_n x}$$

show that the unknown function K must have the form $K(x, y) = \sum_{n=1}^N K_n(x) e^{-\chi_n y}$ for some unknown functions $\{K_n\}$. Without looking at your notes, construct an equation of the form $A\mathbf{K} = \mathbf{b}$ where $\mathbf{K} = (K_1, \dots, K_N)^t$ and \mathbf{b} is a vector you should determine. Deduce that $u(x) = -2 \log[\det A]''(x)$.

2.9. The $N = 2$ soliton solution to the KdV is given by ($\chi_1 > \chi_2$)

$$u(x, t) = -8 \left[\frac{(\chi_1^2 e^{\eta_1} + \chi_2^2 e^{\eta_2}) + 2(\chi_1 - \chi_2)^2 e^{\eta_1 + \eta_2} + \alpha_{12}(\chi_1^2 e^{\eta_1 + 2\eta_2} + \chi_2^2 e^{2\eta_1 + \eta_2})}{(1 + e^{\eta_1} + e^{\eta_2} + \alpha_{12} e^{\eta_1 + \eta_2})^2} \right]$$

where $\eta_i(x, t) = 2\chi_i x - 8\chi_i^3 t + \beta_i$ for $i = 1, 2$ and $\alpha_{12} = (\chi_1 - \chi_2)^2 (\chi_1 + \chi_2)^{-2}$. By setting $\eta_1 = \text{const}$ and taking the limit $t \rightarrow \infty$ show that in a frame of reference travelling at speed $4\chi_1^2$ the 2-soliton reduces to a one soliton solution

$$u(x, t) = -2\chi_1^2 \text{sech}^2[\chi_1(x - 4\chi_1^2 t) + \phi_\infty]$$

where you should determine the constant ϕ_∞ . By instead taking the limit $t \rightarrow -\infty$, calculate the phase shift $\Delta\phi = \phi_\infty - \phi_{-\infty}$ induced by the soliton interaction.

2.10. Suppose $u = u(x, t)$ satisfies the Hamiltonian evolution equation $u_t = \mathcal{J}\delta H$. Show that if $I = I[u]$ then $I_t = \{I, H\}$, where $\{F, G\} = \langle \delta F, \mathcal{J}\delta G \rangle$ is a Poisson bracket on the space of functionals. Deduce that if I_1 and I_2 are conserved, then so is $I_3 = \{I_1, I_2\}$.

2.11. Show that KdV $u_t + u_{xxx} - 6uu_x = 0$ can be written in Hamiltonian form in two distinct ways

$$H_0[u] = \int \frac{1}{2}u^2 dx, \quad \mathcal{J}_0 = -\partial_x^3 + 4u\partial_x + 2u_x \quad \text{and} \quad H_1[u] = \int \left(\frac{1}{2}u_x^2 + u^3\right) dx, \quad \mathcal{J}_1 = \partial_x.$$

In both cases check that the operator \mathcal{J} is anti-symmetric.

2.12 (*). Let $u : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and satisfy $\int_{\mathbb{R}} |u(x)| dx < \infty$.

- i) Suppose $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a uniformly continuous, bounded, function which solves the integral equation:

$$\varphi(x) = e^{-ikx} + \frac{1}{k} \int_{-\infty}^x \sin[k(x-y)]u(y)\varphi(y) dy, \quad (*)$$

Show that in fact φ is twice differentiable and satisfies:

$$L\varphi := -\frac{\partial^2 \varphi}{\partial x^2} + u\varphi = k^2\varphi, \\ \left| \varphi(x) - e^{-ikx} \right| \rightarrow 0, \quad \text{as } x \rightarrow -\infty.$$

- ii) Let $f_0(x) = e^{-ikx}$ and define the sequence $\{f_n\}_{n \geq 0}$ by $f_{n+1} = Kf_n$, where

$$(Kf_n)(x) \equiv \frac{1}{k} \int_{-\infty}^x \sin[k(x-y)]u(y)f_n(y) dy.$$

Prove by induction that f_n is uniformly continuous and satisfies the bound:

$$|f_n(x)| \leq \frac{\mathcal{E}(x)^n}{k^n n!}, \quad \text{where } \mathcal{E}(x) = \int_{-\infty}^x |u(y)| dy.$$

Deduce that the series $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly to a continuous, bounded function.

- iii) By rewriting (*) as $(1-K)\varphi = e^{-ikx}$, conclude that the function $\varphi = \sum_{n=0}^{\infty} K^n(e^{-ikx})$ solves (*).

- iv) Show that (*) may be re-written as:

$$\varphi(x) - a(k)e^{-ikx} - b(k)e^{ikx} = -\frac{1}{k} \int_x^{\infty} \sin[k(x-y)]u(y)\varphi(y) dy$$

where

$$a(k) = 1 - \frac{1}{2ik} \int_{-\infty}^{\infty} e^{iky}u(y)\varphi(y, k) dy, \quad b(k) = \frac{1}{2ik} \int_{-\infty}^{\infty} e^{-iky}u(y)\varphi(y, k) dy.$$

Deduce that

$$\left| \varphi(x) - a(k)e^{-ikx} - b(k)e^{ikx} \right| \rightarrow 0, \quad \text{as } x \rightarrow \infty.$$