Mathematical Tripos Part II Michaelmas 2016
Integrable Systems, Example sheet 2 Dr A.C.L. Ashton

Comments and corrections to `acla2@damtp.cam.ac.uk`. *Sheet with commentary available to supervisors.*

1. Using the method of characteristics show that the solution to the initial value problem $u_t = 6uu_x$, $u(x,0) = f(x)$ is given by $u(x,t) = f(\xi)$ where ξ is defined implicitly by $\xi = x + 6tf(\xi)$. Show that the slope u_x first becomes infinite when $t = \min_\xi[6f'(\xi)]^{-1}$.

2. Let L be a Schrödinger operator with potential u which decays rapidly at infinity. Show that if $L\psi = \lambda\psi$ and $L\psi' = \lambda\psi'$ then the Wronskian $W(\psi,\psi') \equiv \psi\psi'_x - \psi'\psi_x$ is constant. Using this fact establish the following results concerning the discrete and continuous parts of the spectrum of L respectively:

(i) Show that if ψ and ψ' are bound states corresponding to the same discrete eigenvalue then $\psi \propto \psi'$. Deduce that the discrete eigenvalues are non-degenerate, i.e. each discrete eigenvalue corresponds to exactly one bound state.

(ii) Show that the reflection and transmission coefficients obey $|R(k)|^2 + |T(k)|^2 = 1$ for all k. [*Hint*: if $L\Phi = k^2\Phi$ then $L\Phi^* = k^2\Phi^*$ also, the star denoting complex conjugation.]

3. Let L be the Schrödinger operator associated with the Dirac potential $u(x) = 2\alpha\delta(x)$, $\alpha \neq 0$. Show that the reflection and transmission coefficients associated with the continuous spectrum are

$$R(k) = -\frac{\mathrm{i}\alpha}{k+\mathrm{i}\alpha}, \quad T(k) = \frac{k}{k+\mathrm{i}\alpha}.$$

Verify that $|T|^2 + |R|^2 = 1$. Show that there are no bound states if $\alpha > 0$ and one bound state if $\alpha < 0$.

4. Consider the family of self-adjoint operators $L(t)$ on some complex inner product space defined by

$$L(t) = U(t)L(0)U(t)^*$$

where $U(t)$ is a unitary operator, i.e. $U(t)U(t)^* = I$. Show that $L(t)$ and $L(0)$ have the same eigenvalues. Show that there exists a anti-symmetric operator A such that $U_t = -AU$ and $L_t = [L, A]$.

5. Define the linear operators

$$L = -\partial_x^2 + u(x,t), \quad A = 4\partial_x^3 - 3u\partial_x - 3\partial_x u.$$

Show that the KdV equation is equivalent to Lax's equation $L_t = [L, A]$.

Show that A is anti-symmetric $\langle\varphi, A\psi\rangle = -\langle A\varphi, \psi\rangle$ for any smooth, rapidly decaying functions ψ and φ. If $\|\psi\| = 1$ and $\psi' = \psi_t + A\psi$, show that ψ and ψ' are orthogonal, i.e. $\langle\psi', \psi\rangle = 0$. Conclude that if u satisfies the KdV equation and ψ is a bound state for L then $\psi_t + A\psi = 0$. [*Hint*: use question 2(i).]

6. Let $L(t)$ and $A(t)$ be $n \times n$ matrices such that

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [L, A].$$

Show that $\mathrm{tr}(L^p)$, $p \in \mathbf{Z}$, does not depend on t.

7. Show that for any non-singular square matrix $A = A(x)$

$$\frac{1}{\det A}\frac{\mathrm{d}}{\mathrm{d}x}\det A = \mathrm{tr}\left(A^{-1}\frac{\mathrm{d}A}{\mathrm{d}x}\right).$$

8. Suppose a potential $u = u(x)$ is *reflectionless*, i.e. $R(k) = 0$ in the scattering data for the associated Schrödinger operator L. By writing the GLM equation in the form

$$K(x,y) = -F(x+y) - \int_x^\infty K(x,z)F(z+y)\,\mathrm{d}z, \quad \text{where} \quad F(x) = \sum_{n=1}^N c_n^2 e^{-\chi_n x}$$

show that the unknown function K *must* have the form $K(x,y) = \sum_{n=1}^N K_n(x)e^{-\chi_n y}$ for some unknown functions $\{K_n\}$. Without looking at your notes, construct an equation of the form $A\mathbf{K} = \mathbf{b}$ where $\mathbf{K} = (K_1, \ldots, K_N)^t$ and $\mathbf{b}$ is a vector you should determine. Deduce that $u(x) = -2\log[\det A]''(x)$.

9. The $N = 2$ soliton solution to the KdV is given by ($\chi_1 > \chi_2$)

$$u(x,t) = -8\left[\frac{(\chi_1^2 e^{\eta_1} + \chi_2^2 e^{\eta_2}) + 2(\chi_1 - \chi_2)^2 e^{\eta_1+\eta_2} + \alpha_{12}(\chi_1^2 e^{\eta_1+2\eta_2} + \chi_2^2 e^{2\eta_1+\eta_2})}{(1 + e^{\eta_1} + e^{\eta_2} + \alpha_{12}e^{\eta_1+\eta_2})^2}\right]$$

where $\eta_i(x,t) = 2\chi_i x - 8\chi_i^3 t + \beta_i$ for $i = 1, 2$ and $\alpha_{12} = (\chi_1 - \chi_2)^2(\chi_1 + \chi_2)^{-2}$. By setting $\eta_1 = \text{const}$ and taking the limit $t \to \infty$ show that in a frame of reference travelling at speed $4\chi_1^2$ the 2-soliton reduces to a one soliton solution

$$u(x,t) = -2\chi_1^2 \operatorname{sech}^2[\chi_1(x - 4\chi_1^2 t) + \phi_\infty]$$

where you should determine the constant ϕ_∞. By instead taking the limit $t \to -\infty$, calculate the phase shift $\Delta\phi = \phi_\infty - \phi_{-\infty}$ induced by the soliton interaction.

10. Suppose $u = u(x,t)$ satisfies the Hamiltonian evolution equation $u_t = \mathcal{J}\delta H$. Show that if $I = I[u]$ then $I_t = \{I, H\}$, where $\{F, G\} = \langle \delta F, \mathcal{J}\delta G\rangle$ is a Poisson bracket on the space of functionals. Deduce that if I_1 and I_2 are conserved, then so is $I_3 = \{I_1, I_2\}$.

11. Show that KdV $u_t + u_{xxx} - 6uu_x = 0$ can be written in Hamiltonian form in two distinct ways

$$H_0[u] = \int \tfrac{1}{2}u^2\,\mathrm{d}x, \quad \mathcal{J}_0 = -\partial_x^3 + 4u\partial_x + 2u_x \quad \text{and} \quad H_1[u] = \int \left(\tfrac{1}{2}u_x^2 + u^3\right)\mathrm{d}x, \quad \mathcal{J}_1 = \partial_x.$$

In both cases check that the operator $\mathcal{J}$ is anti-symmetric.

Additional problems

These questions should not be attempted at the expense of earlier ones.

12. Let $f_0(x) = e^{-\mathrm{i}kx}$ where $k \in \mathbf{R} \setminus \{0\}$ is fixed. Define the sequence $\{f_n\}_{n\geq 0}$ by $f_{n+1} = Kf_n$, where

$$(Kf_n)(x) \equiv \frac{1}{k}\int_{-\infty}^x \sin[k(x-y)]u(y)f_n(y)\,\mathrm{d}y,$$

and u, the potential associated with a Schrödinger operator L, has compact support. Prove by induction

$$|f_n(x)| \leq \frac{\mathcal{E}(x)^n}{k^n n!}, \quad \text{where} \quad \mathcal{E}(x) = \int_{-\infty}^x |u(y)|\,\mathrm{d}y.$$

Deduce that the series $\sum_{n=0}^\infty f_n(x)$ converges uniformly. Conclude that the function $\varphi = \sum_{n=0}^\infty K^n(e^{-\mathrm{i}kx})$ satisfies $L\varphi = k^2\varphi$ and $\varphi = e^{-\mathrm{i}kx}$ for sufficiently negative x. What if u is only assumed to be integrable?

13. Let φ be as in the previous question, so that $L\varphi = k^2\varphi$ and

$$\varphi(x,k) = \begin{cases} e^{-\mathrm{i}kx} & \text{as } x \to -\infty, \\ a(k)e^{-\mathrm{i}kx} + b(k)e^{\mathrm{i}kx} & \text{as } x \to +\infty. \end{cases}$$

By considering the equation $(I - K)\varphi(x,k) = e^{-\mathrm{i}kx}$, with K defined in the previous question, show that

$$a(k) = 1 - \frac{1}{2\mathrm{i}k}\int_{-\infty}^\infty e^{\mathrm{i}ky}u(y)\varphi(y,k)\,\mathrm{d}y, \quad b(k) = \frac{1}{2\mathrm{i}k}\int_{-\infty}^\infty e^{-\mathrm{i}ky}u(y)\varphi(y,k)\,\mathrm{d}y.$$