Example Sheet 2.  
David Stuart  
Part II: Integrable systems  
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2.1. Let \( L(t) \) and \( A(t) \) be \( n \times n \) matrices depending differentiably on \( t \in \mathbb{R} \), and such that 

\[
\frac{dL}{dt} = [L, A].
\]  

Show, without considering the eigenvalues/vectors of \( L \), that \( \text{tr}(L^p) \), \( p \in \mathbb{N} \), does not depend on \( t \).

2.2. Show that if in (1) the matrix \( A \) is skew-symmetric \((A^T = -A)\) and \( L \) is symmetric then both sides of the equation are symmetric.

2.3. Write down the Hamiltonian equations for the Toda Hamiltonian for \( N \) particles moving in one dimension, \( H = \frac{1}{2} \sum_{j=1}^{N} p_j^2 + \sum_{j=1}^{N-1} \exp(q_j - q_{j+1}) \) and show that with the definitions \( a_j = \frac{1}{2} \exp((q_j - q_{j+1})/2) \) and \( b_j = -\frac{1}{2} p_j \) they are equivalent to

\[
\dot{a}_j = a_j(b_{j+1} - b_j), \quad \dot{b}_j = 2(a_j^2 - a_j^2). 
\]  

(Use the convention that \( q_0 = -\infty, e^{q_0} = 0, q_{N+1} = +\infty, e^{-q_{N+1}} = 0. \))

2.4. Recall the Toda problem with \( N = 2 \) can be written as the Lax pair \( \dot{L} = [B, L] \) with

\[
L = \begin{pmatrix} b_1 & a_1 \\ a_1 & b_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}. 
\]

Express the eigenvalues of \( L \) in terms of the total momentum \( p_1 + p_2 \) and the energy \( H \), check they are in involution. Obtain the general solution to the system.

2.5. Extend the Lax pair formulation of the Toda problem to general \( N \), by considering the tri-diagonal\(^1\) \( N \times N \) matrices whose diagonal elements are \( L_{jj} = b_j \) and \( B_{jj} = 0 \) for \( j = 1, \ldots, N \), and whose near diagonal elements are \( L_{j,j+1} = L_{j+1,j} = a_j \) and \( B_{j,j+1} = -B_{j+1,j} = a_j \) for \( j = 1, \ldots, N-1 \). Show that the equations (2) are equivalent to \( \dot{L} = [B, L] \). For the case \( N = 3 \) deduce that \( F_1 = \lambda_1 + \lambda_2 + \lambda_3 \), \( F_2 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3 \) and \( F_3 = \lambda_1 \lambda_2 \lambda_3 \) are all 1st integrals (where \( \lambda_j \) are the eigenvalues of \( L \), which you may assume to be real and distinct). Calculate \( F_1, F_2, F_3 \) in terms of \( a_1, a_2, b_1, b_2, b_3 \) and hence show that \( F_1, F_2, F_3 \) are in involution. * Prove the eigenvalues of \( L \) are real and distinct.

2.6. Consider a family of Hermitian \( N \times N \) matrices \( L(t) \) obeying

\[
L(t) = U(t)L(0)U(t)^\dagger
\]

where \( U(t) \) is unitary. Show that \( L(t) \) and \( L(0) \) have the same eigenvalues. Assuming differentiability, show that there exists a skew-Hermitian matrix \( A \) (i.e., a matrix obeying \( A = -A^\dagger \)) such that \( U_t = -AU \) and \( L_t = [L, A] \).

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\(^1\)A tri-diagonal matrix is one whose only nonzero elements are either on the diagonal or nearest neighbour to the diagonal.

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Conversely, suppose $L$ satisfies $L_t = [L, A]$ for some skew-Hermitian $A$. Let $U(t)$ be the solution of $U_t = -AU$, $U(0) = I$. Show that $U(t)$ is unitary. By considering $\frac{d}{dt} U(t)^\dagger L(t) U(t)$ or otherwise, show that $L(t) = U(t) L(0) U(t)^\dagger$.

Extend the preceding results to the following situation. A complex inner product on a vector space $X$ is a map $u, v \mapsto \langle u, v \rangle \in \mathbb{C}$ which is linear in $v$ and obeys $\langle u, v \rangle = \langle v, u \rangle$ for all vectors $u, v$ in $X$, and $|\langle v, v \rangle| = \langle v, v \rangle > 0$ for nonzero vectors $v$, making $X$ into a normed space. An operator is a continuous linear map $X \to X$. An operator is a continuous linear map $O : X \to X$, and its adjoint $O^\dagger$ is defined by $\langle O^\dagger u, v \rangle = \langle u, Ov \rangle$. If $O^\dagger = O$ (resp. $O^\dagger = -O$) it is called self-adjoint (resp. skew-adjoint). Now replace "(skew)-Hermitian matrix" by "(skew)-adjoint operator" in the above question, and show that the conclusions still hold. You may assume all operators depend smoothly on $t$ and that the equation $U_t = -AU$ does indeed have a smooth solution.

2.7. Let $L = -\frac{d^2}{dx^2} + u(x)$ be the one dimensional Schrödinger operator with potential $u$, assumed to decay rapidly at infinity. Show that if $L\psi = \lambda\psi$ and $L\psi' = \lambda\psi''$ then the Wronskian $W(\psi, \psi') \equiv \psi\psi_x' - \psi'\psi_x$ is constant. Using this fact establish the following results concerning the discrete and continuous parts of the spectrum of $L$ respectively:

(i) Show that if $\psi$ and $\psi'$ are bound states corresponding to the same discrete eigenvalue then $\psi \propto \psi'$. Deduce that the discrete eigenvalues are non-degenerate, i.e. each discrete eigenvalue corresponds to exactly one bound state.

(ii) Show that the reflection and transmission coefficients obey $|R(k)|^2 + |T(k)|^2 = 1$ for all $k$. \textit{[Hint:} if $L\Psi = k^2 \Phi$ then $L^\dagger \Phi = k^2 \Psi$, also, the bar denoting complex conjugation.\textit{]}

2.8. For $u$ a real function, define the linear operators

$$L = -\frac{d^2}{dx^2} + u(x,t), \quad A = 4\frac{d^3}{dx^3} - 3u\frac{d}{dx} - 3\frac{d}{dx}u.$$  

Show that the KdV equation is equivalent to Lax’s equation $L_t = [L, A]$.

Show that $L$ is self-adjoint and $A$ is skew-adjoint: $\langle \varphi, L\psi \rangle = \langle L\varphi, \psi \rangle, \langle \varphi, A\psi \rangle = -\langle A\varphi, \psi \rangle$ for any smooth, rapidly decaying functions $\psi$ and $\varphi$. If $\psi$ is a real function with $|\psi(t)| = 1$ and $\dot{\psi}(t) = \psi_t(t) + A\psi(t)$, show that $\psi$ and $\dot{\psi}$ are orthogonal, i.e. $\langle \psi, \dot{\psi} \rangle = 0$. Conclude that if $u$ satisfies the KdV equation and $\psi$ is a bound state for $L$ then $\psi_t + A\psi = 0$. \textit{[Hint:} use question 2.7(i),\textit{]} and obtain the time dependence of the discrete part of the scattering data associated to the potential $u$.

2.9. Recall from lectures that if $A = \partial_x + m \tanh mx$ and $A^\dagger = -\partial_x + m \tanh mx$, then

$$-\frac{d^2}{dx^2} + m^2 = AA^\dagger \quad \text{and} \quad -\frac{d^2}{dx^2} + m^2 - 2m^2 \text{sech}^2 mx = A^\dagger A,$$

from which we found the scattering data for the potential $-2m^2 \text{sech}^2 mx$ was $R(k) = 0$ and $\chi_1^2 = -m^2$ and $c_1 = \sqrt{2m}$. Now by considering $B = \partial_x + 2m \tanh mx$ and $B^\dagger = -\partial_x + 2m \tanh mx$, and computing $BB^\dagger$ and $B^\dagger B$, find the scattering data for the potential $-6m^2 \text{sech}^2 mx$. \textit{[Hint:} consider $B^\dagger A^\dagger e^{ikx} = (-k^2 - 3imk \tanh mx +
\( 2m^2 - 3m^2 \text{sech}^2 mx \) \( e^{ikx} \) to find the reflection coefficient, and argue similarly to the case \( -2m^2 \text{sech}^2 mx \) treated in lectures for the bound state scattering data. 

(*) Verify that solving Gelfand-Levitan-Marcenko equation for \( K = K(x, y) \) and defining \( u(x) = -2 \frac{d}{dx} K(x, x) \) leads back to \( u = -6m^2 \text{sech}^2 mx \).

2.10. The \( N = 2 \) soliton solution to the KdV is given by \( (\chi_1 > \chi_2) \)

\[
 u(x, t) = -8 \left[ \frac{(\chi_1^2 e^{\eta_1} + \chi_2^2 e^{\eta_2}) + 2(\chi_1 - \chi_2)^2 e^{\eta_1+\eta_2} + \alpha_{12}(\chi_1^2 e^{2\eta_1+2\eta_2} + \chi_2^2 e^{2\eta_1+\eta_2})}{(1 + e^{\eta_1} + e^{\eta_2} + \alpha_{12} e^{\eta_1+\eta_2})^2} \right]
\]

where \( \eta_i(x, t) = 2\chi_i x - 8\chi_i^3 t + \beta_i \) for \( i = 1, 2 \) and \( \alpha_{12} = (\chi_1 - \chi_2)^2(\chi_1 + \chi_2)^{-2} \). By setting \( \eta_1 = \text{const} \) and taking the limit \( t \to \infty \) show that in a frame of reference travelling at speed \( 4\chi_1^2 \) the 2-soliton reduces to a one soliton solution

\[
 u(x, t) = -2\chi_1^2 \text{sech}^2 [\chi_1(x - 4\chi_1^2 t) + \phi_\infty]
\]

where you should determine the constant \( \phi_\infty \). By instead taking the limit \( t \to -\infty \), calculate the phase shift \( \Delta \phi = \phi_\infty - \phi_{-\infty} \) induced by the soliton interaction.