3.1. Let $v$ be any solution of the wave equation in double-null coordinates: $v_{xt} = 0$. Show that the two equations:

\[ u_x + v_x = \sqrt{2} \exp \left( \frac{u - v}{2} \right), \quad u_t - v_t = \sqrt{2} \exp \left( \frac{u + v}{2} \right), \]

are compatible iff $u$ satisfies Liouville’s equation $u_{xt} = e^u$. These equations constitute a Bäcklund transformation. By considering the most general form of $v = v(x,t)$, show that:

\[ u(x,t) = 2 \log \left( -\frac{\sqrt{2}}{\int_x \exp \left[-f(\xi)\right]d\xi + \int_t \exp \left[g(\tau)\right]d\tau} \right) + g(t) - f(x). \]

3.2. Using the notation of qu. 3 on sheet II, show that if there exists a sequence $\{H_n\}_{n=0}^{\infty}$ of functionals satisfying

\[ J_1 \frac{\delta H_{n+1}}{\delta u} = J_0 \frac{\delta H_n}{\delta u}, \]

then these are all 1st integrals (conserved quantities) for KdV.

3.3. i) Find the vector fields $V_1, V_2, V_3$ which generate the following smooth one-parameter groups of transformations of $\mathbb{R}$:

\[ x \mapsto \psi_s^1 x = x + s, \quad x \mapsto \psi_s^2 x = e^s x, \quad x \mapsto \psi_s^3 x = \frac{x}{1 - sx}. \]

ii) Deduce that these vector fields generate a group of transformations of the form

\[ x \mapsto \frac{ax + b}{cx + d}, \quad ad - bc = 1. \]

iii) Compute the structure constants $\{f_{ij}^k\}_{i,j,k=1}^3$ defined by $[V_i, V_j] = \sum_{k=1}^3 f_{ij}^k V_k$.

iv) (*) Show that these transformations can be understood as arising from a smooth left action of $SL(2) = \{ A \in \text{mat}(2 \times 2) | \det A = 1 \}$ on $\mathbb{R}$.

3.4. Compute the 1-parameter group of transformations generated by

\[ V = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}. \]

Find new coordinates $(X, Y)$ with $X = X(x, y)$ and $Y = Y(x, y)$ such that $V(X) = 1$ and $V(Y) = 0$. Use your results to integrate the ODE

\[ x^2 \frac{dy}{dx} = F(xy), \]

where $F$ is an arbitrary function of one variable.
3.5. Write each of the following 1-parameter groups of transformations as a composition of commutative 1-parameter transformations:

\[ \psi_1^s(x, t, u) = (x + s, t, u + 3s), \quad \psi_2^s(x, t, u) = (e^s x, e^s t, u - s), \quad \psi_3^s(x, t, u) = (e^s x, t + us, u). \]

Hence write down the vector fields which generate these transformations. Check your answers are correct by showing the relevant ODEs are satisfied. Show that \( \psi^s(x, t, u) = (x \cosh s + t \sinh s, x \sinh s + t \cosh s, u) \) defines a 1-parameter group of transformations. Does the previous method fail in this case? Find the generator of \( \psi^s \) and comment on the aforementioned failure.

3.6. Let \( \tilde{x} = \psi^s x \) be a new set of coordinates where \( \psi^s \) is a 1-parameter group of transformations with generator \( V \). Use Taylor’s theorem to show (formally) that for nice functions \( f \)

\[
f(\tilde{x}) = f(x) + sVf(x) + \frac{s^2}{2!}V(Vf)(x) + \frac{s^3}{3!}V(V(Vf))(x) + \cdots = \sum_{n=0}^{\infty} \frac{s^n}{n!}(V^n f)(x).
\]

Deduce that, at least formally, \( \psi^s \equiv \exp(sv) \). Show that \( \exp(s x \partial_x)x = e^s x \).

3.7. Let \( \psi^s \) be a 1-parameter group of transformations generated by \( V \). A function \( F = F(x) \) is said to be an invariant of \( \psi^s \) if \( F(\psi^s x) = F(x) \) for all \( x \). Show that \( F \) is an invariant if and only if \( VF(x) = 0 \).

3.8. Compute the 1-parameter groups of transformations associated with the vector fields

\[ V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t}, \quad V_3 = \frac{\partial}{\partial u} + \alpha t \frac{\partial}{\partial x}, \quad V_4 = \beta x \frac{\partial}{\partial x} + \gamma t \frac{\partial}{\partial t} + \delta u \frac{\partial}{\partial u}. \]

Find the constants \( (\alpha, \beta, \gamma, \delta) \) for which these vector fields generate symmetries of the KdV equation. Determine the structure constants in the corresponding 4-dimensional Lie algebra of vector fields.

3.9. Let \( u = u(x) \). Calculate the first prolongation of the following 1-parameter groups of transformations

\[ \psi_1^s(x, u) = (x + s, u), \quad \psi_2^s(x, u) = (e^s x, u + s), \quad \psi_3^s(x, u) = (x \cos s - u \sin s, x \sin s + u \cos s). \]

Let \( V_1, V_2, V_3 \) be the corresponding generators. Using your answers to the previous part, show that

\[ \text{pr}^{(1)} V_1 = V_1, \quad \text{pr}^{(1)} V_2 = V_2 - u_x \frac{\partial}{\partial u_x}, \quad \text{pr}^{(1)} V_3 = V_3 + (1 + u_x^2) \frac{\partial}{\partial u_x}. \]

Without looking at your notes, derive the first prolongation formula and verify these are correct.
3.10. Let $u = u(x,t)$. The vector field $V = \xi \partial_x + \phi \partial_t + \eta \partial_u$ generates a 1-parameter group of transformations

$$(x, t, u) \mapsto (\tilde{x}, \tilde{t}, \tilde{u}) = (x + s\xi(x,t,u), t + s\phi(x,t,u), u + s\eta(x,t,u)) + o(s).$$

By considering the contact condition $d\tilde{u} = \tilde{u}_t d\tilde{t} + \tilde{u}_x d\tilde{x}$ show that $pr^{(1)} V = V + \eta^t \partial_{u_x} + \eta^t \partial_{u_t}$, where

$$\eta^t = D_t \eta - u_tD_t \phi - u_x D_t \xi, \quad \eta^x = D_x \eta - u_x D_x \xi - u_tD_x \phi,$$

where $D_x$ and $D_t$ are total derivatives.

3.11. The modified KdV equation is $v_t + v_{xxx} - 6v^2v_x = 0$. Find a Lie-point symmetry of the form

$$\psi^s(x,t,v) = (e^{\alpha x} x, e^{\beta t} t, e^{\gamma s} v)$$

for appropriate numbers $(\alpha, \beta, \gamma)$. Consider the group invariant solution $v(x,t) = (3t)^{-1/3} w(z)$, where $z = x(3t)^{-1/3}$, and construct a 3rd order differential equation for $w$. Integrate this equation once to show that $w$ satisfies Painlevé II.

3.12. Let $u = u(x)$ and $V = \xi \partial_x + \eta \partial_u$. Calculate $pr^{(2)} V$. Show that the equation $u_{xx} = 0$ admits an 8 dimensional group of Lie-point symmetries. Can you give geometrical meaning to each of the generators?