3.1. Let \( v \) be any solution of the wave equation in double-null coordinates: \( v_{xt} = 0 \). Show that the two equations:

\[
u_x + v_x = \sqrt{2} \exp \left( \frac{u - v}{2} \right), \quad u_t - v_t = \sqrt{2} \exp \left( \frac{u + v}{2} \right),\]

are compatible if \( u \) satisfies Liouville’s equation \( u_{xt} = e^u \). These equations constitute a Bäcklund transformation. By considering the most general form of \( v = v(x,t) \), show that:

\[
u(x,t) = 2 \log \left( - \frac{\sqrt{2}}{\int_x \exp[-f(\xi)]d\xi + \int_t \exp[g(\tau)]d\tau} \right) + g(t) - f(x).
\]

3.2. Using the notation of qu. 3 on sheet II, show that if there exists a sequence \( \{H_n\}_{n=0}^{\infty} \) of functionals satisfying

\[
J_1 \frac{\delta H_{n+1}}{\delta u} = J_0 \frac{\delta H_n}{\delta u},
\]

then these are all 1st integrals (conserved quantities) for KdV.

3.3. i) Find the vector fields \( V_1, V_2, V_3 \) which generate the following smooth one-parameter groups of transformations of \( \mathbb{R} \):

\[
x \mapsto \psi^i_1 x = x + s, \quad x \mapsto \psi^2_1 x = e^s x, \quad x \mapsto \psi^3_1 x = \frac{x}{1 - sx}.
\]

ii) Deduce that these vector fields generate a group of transformations of the form

\[
x \mapsto \frac{ax + b}{cx + d}, \quad ad - bc = 1.
\]

iii) Compute the structure constants \( \{f_{ij}^k\}_{i,j,k=1}^3 \) defined by \( [V_i, V_j] = \sum_{k=1}^3 f_{ij}^k V_k \).

iv) (*) Show that these transformations can be understood as arising from a smooth left action of \( SL(2) = \{ A \in \text{mat}(2 \times 2) | \det A = 1 \} \) on \( \mathbb{R} \).

3.4. Compute the 1-parameter group of transformations generated by

\[
V = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.
\]

Find new coordinates \( (X, Y) \) with \( X = X(x,y) \) and \( Y = Y(x,y) \) such that \( V(X) = 1 \) and \( V(Y) = 0 \). Use your results to integrate the ODE

\[
x^2 \frac{dy}{dx} = F(xy),
\]

where \( F \) is an arbitrary function of one variable.

Please send any corrections to dmas2@cam.ac.uk

Questions marked (*) are optional and should not be attempted at the expense of unstarred questions.
3.5. Write each of the following 1-parameter groups of transformations as a composition of commutative 1-parameter transformations:

\[ \psi_1^s(x, t, u) = (x+s, t+2s, u+3s), \quad \psi_2^s(x, t, u) = (e^s x, e^s t, u-s), \quad \psi_3^s(x, t, u) = (e^s x, t+us, u). \]

Hence write down the vector fields which generate these transformations. Check your answers are correct by showing the relevant ODEs are satisfied. Show that

\[ \psi^s(x, t, u) = (x \cosh s + t \sinh s, x \sinh s + t \cosh s, u) \]

defines a 1-parameter group of transformations. Does the previous method fail in this case? Find the generator of \( \psi^s \) and comment on the aforementioned failure.

3.6. Let \( \tilde{x} = \psi^sx \) be a new set of coordinates where \( \psi^s \) is a 1-parameter group of transformations with generator \( V \). Use Taylor’s theorem to show (formally) that for nice functions \( f \)

\[ f(\tilde{x}) = f(x) + sV f(x) + \frac{s^2}{2!} V(V f)(x) + \frac{s^3}{3!} V(V(V f))(x) + \cdots = \sum_{n=0}^{\infty} \frac{s^n}{n!} (V^n f)(x). \]

Deduce that, at least formally, \( \psi^s \equiv \exp(sV) \). Show that \( \exp(s \partial_x)x = x + s \) and \( \exp(s x \partial_x)x = e^s x \).

3.7. Let \( \psi^s \) be a 1-parameter group of transformations generated by \( V \). A function \( F = F(x) \) is said to be an invariant of \( \psi^s \) if \( F(\psi^s x) = F(x) \) for all \( x \). Show that \( F \) is an invariant if and only if \( VF(x) = 0 \).

3.8. Compute the 1-parameter groups of transformations associated with the vector fields

\[ V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t}, \quad V_3 = \frac{\partial}{\partial u} + \alpha t \frac{\partial}{\partial x}, \quad V_4 = \beta x \frac{\partial}{\partial x} + \gamma t \frac{\partial}{\partial t} + \delta u \frac{\partial}{\partial u}. \]

Find the constants \( (\alpha, \beta, \gamma, \delta) \) for which these vector fields generate symmetries of the KdV equation. Determine the structure constants in the corresponding 4-dimensional Lie algebra of vector fields.

3.9. Let \( u = u(x) \). Calculate the first prolongation of the following 1-parameter groups of transformations

\[ \psi_1^s(x, u) = (x+s, u), \quad \psi_2^s(x, u) = (e^s x, u+s), \quad \psi_3^s(x, u) = (x \cos s - u \sin s, x \sin s + u \cos s). \]

Let \( V_1, V_2, V_3 \) be the corresponding generators. Using your answers to the previous part, show that

\[ \text{pr}^{(1)} V_1 = V_1, \quad \text{pr}^{(1)} V_2 = V_2 - u_x \frac{\partial}{\partial u_x}, \quad \text{pr}^{(1)} V_3 = V_3 + (1 + u_x^2) \frac{\partial}{\partial u_x}. \]

Without looking at your notes, derive the first prolongation formula and verify these are correct.
3.10. Let \( u = u(x,t) \). The vector field \( V = \xi \partial_x + \phi \partial_t + \eta \partial_u \) generates a 1-parameter group of transformations

\[
(x,t,u) \mapsto (\tilde{x}, \tilde{t}, \tilde{u}) = (x + s \xi(x,t,u), t + s \phi(x,t,u), u + s \eta(x,t,u)) + o(s).
\]

By considering the contact condition \( d\tilde{u} = \tilde{u}_t d\tilde{t} + \tilde{u}_x d\tilde{x} \) show that \( \text{pr}^{(1)} V = V + \eta^x \partial_{ux} + \eta^t \partial_{ut} \), where

\[
\eta^t = D_t \eta - u_t D_t \phi - u_x D_t \xi, \quad \eta^x = D_x \eta - u_x D_x \xi - u_t D_x \phi,
\]

where \( D_x \) and \( D_t \) are total derivatives.

3.11. The modified KdV equation is \( v_t + v_{xxx} - 6v^2v_x = 0 \). Find a Lie-point symmetry of the form

\[
\psi^s(x,t,v) = (e^{\alpha s} x, e^{\beta s} t, e^{\gamma s} v)
\]

for appropriate numbers \((\alpha, \beta, \gamma)\). Consider the group invariant solution \( v(x,t) = (3t)^{-1/3} w(z) \), where \( z = x(3t)^{-1/3} \), and construct a 3rd order differential equation for \( w \). Integrate this equation once to show that \( w \) satisfies Painlevé II.

3.12. Let \( u = u(x) \) and \( V = \xi \partial_x + \eta \partial_u \). Calculate \( \text{pr}^{(2)} V \). Show that the equation \( u_{xx} = 0 \) admits an 8 dimensional group of Lie-point symmetries. Can you give geometrical meaning to each of the generators?