Part II Integrable Systems, Sheet Three

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1. **Gauge invariance of zero curvature equations.** Let $g = g(\tau, \rho)$ be an arbitrary invertible matrix. Show that the transformation

$$\widetilde{U} = gUg^{-1} + \frac{\partial g}{\partial \rho} g^{-1}, \quad \widetilde{V} = gVg^{-1} + \frac{\partial g}{\partial \tau} g^{-1}$$

maps solutions to the zero curvature equation into new solutions: if the matrices (U, V) satisfy

$$\frac{\partial}{\partial \tau} U(\lambda) - \frac{\partial}{\partial \rho} V(\lambda) + [U(\lambda), V(\lambda)] = 0$$

then so do $\widetilde{U}(\lambda), \widetilde{V}(\lambda)$. What is the relationship between the solutions of the associated linear problems?

2. **Finite gap integration.** Consider solutions to the KdV hierarchy which are stationary with respect to

$$c_0 \frac{\partial}{\partial t_0} + c_1 \frac{\partial}{\partial t_1}$$

where the kth KdV flow is generated by the Hamiltonian $(-1)^k I_k[u]$ and $I_k[u]$ are the first integrals constructed in lectures.

Show that the resulting solution to KdV is

$$F(u) = c_1 x - c_0 t,$$

where $F(u)$ is given by an integral which should be determined and $t_0 = x, t_1 = t$.

Find the zero curvature representation for the ODE characterising the stationary solutions.

3. **Nonlinear Schrödinger equation.** Consider the zero curvature representation with

$$\begin{aligned} U &= i\lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i \begin{pmatrix} 0 & \overline{\phi} \\ \phi & 0 \end{pmatrix}, \\ V &= 2i\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2i\lambda \begin{pmatrix} 0 & \overline{\phi} \\ \phi & 0 \end{pmatrix} + \begin{pmatrix} 0 & \overline{\phi}_\rho \\ -\phi_\rho & 0 \end{pmatrix} - i \begin{pmatrix} |\phi|^2 & 0 \\ 0 & -|\phi|^2 \end{pmatrix} \end{aligned}$$

and show that complex valued function $\phi = \phi(\tau, \rho)$ satisfies the nonlinear Schrödinger equation

$$i\phi_\tau + \phi_{\rho\rho} + 2|\phi|^2\phi = 0.$$

[This is another famous soliton equation which can be solved by inverse scattering transform.]

4. **From group action to vector fields.** Consider three one–parameter groups of transformations of $\mathbb{R}$

$$x \to x + \varepsilon_1, \quad x \to e^{\varepsilon_2} x, \quad x \to \frac{x}{1 - \varepsilon_3 x},$$

and find the vector fields V_1, V_2, V_3 generating these groups. Deduce that these vector fields generate a three-parameter group of transformations

$$x \to \frac{ax + b}{cx + d}, \qquad ad - bc = 1.$$

Show that

$$[V_\alpha, V_\beta] = \sum_{\gamma=1}^{3} f_{\alpha\beta}^{\gamma} V_\gamma, \qquad \alpha, \beta = 1, 2, 3$$

for some constants $f_{\alpha\beta}^{\gamma}$ which should be determined.

5. **ODE with symmetry.** Consider a vector field

$$V = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}$$

and find the corresponding one parameter group of transformations of $\mathbb{R}^2$. Sketch the integral curves of this vector field.

Find the invariant coordinates, i.e. functions $s(x,u), g(x,u)$ such that

$$V(s) = 1, \qquad V(g) = 0$$

[These are not unique. Make sure that that s, g are functionally independent in a domain of $\mathbb{R}^2$ which you should specify.]

Use your results to integrate the ODE

$$x^2\frac{du}{dx} = F(xu)$$

where F is arbitrary function of one variable.

6. **Lie point symmetries of KdV.** Consider the vector fields

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t}, \quad V_3 = \frac{\partial}{\partial u} + \alpha t\frac{\partial}{\partial x}, \quad V_4 = \beta x\frac{\partial}{\partial x} + \gamma t\frac{\partial}{\partial t} + \delta u\frac{\partial}{\partial u}$$

where $(\alpha, \beta, \gamma, \delta)$ are constants and find the corresponding one parameter groups of transformations of $\mathbb{R}^3$ with coordinates (x, t, u).

Find $(\alpha, \beta, \gamma, \delta)$ such that these are symmetry groups of KdV and deduce the existence of a four–parameter symmetry group.

Determine the structure constants of the corresponding Lie algebra of vector fields.

7. **Painlevé II from modified KdV.** Consider the modified KdV equation

$$v_t - 6v^2v_x + v_{xxx} = 0.$$

Find a Lie point symmetry of this equation of the form

$$(\tilde{v}, \tilde{x}, \tilde{t}) = (c^\alpha v, c^\beta x, c^\gamma t), \qquad c \neq 0$$

for some (α, β, γ) which should be found, and find the corresponding vector field generating this group.

Consider the group invariant solution of the form

$$v(x,t) = (3t)^{-1/3}w(z), \quad \text{where} \quad z = x(3t)^{-1/3}$$

and obtain a third order ODE for $w(z)$. Integrate this ODE once to show that $w(z)$ satisfies the second Painlevé equation.

8. **Symmetry reduction of Sine–Gordon.** Show that the transformation

$$(\tilde{\rho}, \tilde{\tau}) = (c\rho, \frac{1}{c}\tau), \qquad c \neq 0$$

is a one–parameter symmetry of the Sine–Gordon equation and find its generating vector field.

Consider the group invariant solutions of the form $\phi(\rho, \tau) = F(z)$ where $z = \rho\tau$. Substitute $w(z) = \exp(iF(z))$ and demonstrate that the ODE arising from a symmetry reduction is one of the Painlevé equations.

9. **First prolongation.** Let $u = u(x)$. Calculate the first prolongation of the following 1-parameter groups of transformations

$$\psi_1^s(x,u) = (x+s,u), \quad \psi_2^s(x,u) = (e^s x, u+s), \quad \psi_3^s(x,u) = (x\cos s - u\sin s, x\sin s + u\cos s).$$

Let V_1, V_2, V_3 be the corresponding generators. Using your answers to the previous part, show that

$$\mathrm{pr}^{(1)}\, V_1 = V_1, \quad \mathrm{pr}^{(1)}\, V_2 = V_2 - u_x \frac{\partial}{\partial u_x}, \quad \mathrm{pr}^{(1)} V_3 = V_3 + (1+u_x^2)\frac{\partial}{\partial u_x}.$$

Without looking at your notes, derive the first prolongation formula and verify these are correct.

10*. **Symmetries of a trivial second order ODE.** Let $u = u(x)$ and $V = \xi\partial_x + \eta\partial_u$. Calculate $\mathrm{pr}^{(2)}V$. Show that the equation $u_{xx} = 0$ admits an 8 dimensional group of Lie-point symmetries. Can you give geometrical meaning to each of the generators?