3.1. Let $g = g(x,t)$ be a non-singular matrix. Show that if the matrices (U, V) are solutions to the zero curvature equations $U_t - V_x + [U, V] = 0$ then so are

$$\tilde{U} = gUg^{-1} + \frac{\partial g}{\partial x}g^{-1}, \quad \tilde{V} = gVg^{-1} + \frac{\partial g}{\partial t}g^{-1}.$$

What is the relationship between the associated linear problems?

3.2. Let $v = v(x,t)$ be a complex valued function. Show that the matrices (U, V) defined by

$$U(\lambda) = i\lambda \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + i \begin{bmatrix} 0 & \bar{v} \\ v & 0 \end{bmatrix}, \quad V(\lambda) = 2i\lambda^2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 2i\lambda \begin{bmatrix} 0 & \bar{v} \\ v & 0 \end{bmatrix} + \begin{bmatrix} 0 & \bar{v}_x \\ -v_x & 0 \end{bmatrix} - i \begin{bmatrix} |v|^2 & 0 \\ 0 & -|v|^2 \end{bmatrix}$$

satisfy the zero curvature equations iff v satisfies the *nonlinear Schrödinger* (NLS) equation

$$iv_t + v_{xx} + 2|v|^2 v = 0.$$

This is another integrable equation which arises in optics and the mathematical theory of water waves. Show that the small amplitude solutions to the NLS equation are dispersive.

3.3. i) Find the vector fields V_1, V_2, V_3 which generate the following smooth one-parameter groups of transformations of $\mathbb{R}$:

$$x \mapsto \psi_1^s x = x + s, \quad x \mapsto \psi_2^s x = e^s x, \quad x \mapsto \psi_3^s x = \frac{x}{1 - sx}.$$

ii) Deduce that these vector fields generate a group of transformations of the form

$$x \mapsto \frac{ax + b}{cx + d}, \quad ad - bc = 1.$$

iii) Compute the structure constants $\{f_{ij}^k\}_{i,j,k=1}^3$ defined by $[V_i, V_j] = \sum_{k=1}^3 f_{ij}^k V_k$.

iv) (*) Show that these transformations can be understood as arising from a smooth left action of $SL(2) = \{A \in \mathrm{mat}(2 \times 2) | \det A = 1\}$ on $\mathbb{R}$.

3.4. Compute the 1-parameter group of transformations generated by

$$V = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

Find new coordinates (X, Y) with $X = X(x,y)$ and $Y = Y(x,y)$ such that $V(X) = 1$ and $V(Y) = 0$. Use your results to integrate the ODE

$$x^2 \frac{dy}{dx} = F(xy),$$

where F is an arbitrary function of one variable.

Please send any corrections to cmw50@cam.ac.uk
Questions marked $(*)$ are optional and should not be attempted at the expense of unstarred questions

3.5. Let ψ_1^s and ψ_2^s be *commuting* 1-parameter groups of transformations generated by the vector fields V_1 and V_2 respectively. Show that $\psi^s = \psi_1^s \circ \psi_2^s$ also defines a 1-parameter group of transformations and show that it is generated by $V = V_1 + V_2$.

Conversely, show that if a 1-parameter group of transformations ψ^s is generated by $V = V_1 + V_2$ where $[V_1, V_2] = 0$, then $\psi^s = \psi_1^s \circ \psi_2^s$ where the ψ_1^s and ψ_2^s are generated by V_1 and V_2 as before.

3.6. Write each of the following 1-parameter groups of transformations as a composition of commutative 1-parameter transformations:

$$\psi_1^s(x,t,u) = (x{+}s, t{+}2s, u{+}3s), \quad \psi_2^s(x,t,u) = (e^s x, e^s t, u{-}s), \quad \psi_3^s(x,t,u) = (e^s x, t{+}us, u).$$

Hence *write down* the vector fields which generate these transformations. Check your answers are correct by showing the relevant ODEs are satisfied. Show that

$$\psi^s(x,t,u) = (x\cosh s + t\sinh s, x\sinh s + t\cosh s, u)$$

defines a 1-parameter group of transformations. Does the previous method fail in this case? Find the generator of ψ^s and comment on the aforementioned failure.

3.7. Let $\tilde{\mathbf{x}} = \psi^s \mathbf{x}$ be a new set of coordinates where ψ^s is a 1-parameter group of transformations with generator V. Use Taylor's theorem to show (formally) that for nice functions f

$$f(\tilde{\mathbf{x}}) = f(\mathbf{x}) + sVf(\mathbf{x}) + \frac{s^2}{2!}V(Vf)(\mathbf{x}) + \frac{s^3}{3!}V(V(Vf))(\mathbf{x}) + \cdots = \sum_{n=0}^{\infty} \frac{s^n}{n!}(V^n f)(\mathbf{x}).$$

Deduce that, at least formally, $\psi^s \equiv \exp(sV)$. Show that $\exp(s\partial_x)x = x + s$ and $\exp(sx\partial_x)x = e^s x$.

3.8. Let ψ^s be a 1-parameter group of transformations generated by V. A function $F = F(\mathbf{x})$ is said to be an *invariant* of ψ^s if $F(\psi^s \mathbf{x}) = F(\mathbf{x})$ for all $\mathbf{x}$. Show that F is an invariant if and only if $VF(\mathbf{x}) = 0$.

3.9. Compute the 1-parameter groups of transformations associated with the vector fields

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial t}, \quad V_3 = \frac{\partial}{\partial u} + \alpha t\frac{\partial}{\partial x}, \quad V_4 = \beta x\frac{\partial}{\partial x} + \gamma t\frac{\partial}{\partial t} + \delta u\frac{\partial}{\partial u}.$$

Find the constants $(\alpha, \beta, \gamma, \delta)$ for which these vector fields generate symmetries of the KdV equation. Determine the structure constants in the corresponding 4-dimensional Lie algebra of vector fields.

3.10. Let $u = u(x)$. Calculate the first prolongation of the following 1-parameter groups of transformations

$$\psi_1^s(x,u) = (x{+}s, u), \quad \psi_2^s(x,u) = (e^s x, u{+}s), \quad \psi_3^s(x,u) = (x\cos s{-}u\sin s, x\sin s{+}u\cos s).$$

Let V_1, V_2, V_3 be the corresponding generators. Using your answers to the previous part, show that

$$\mathrm{pr}^{(1)} V_1 = V_1, \quad \mathrm{pr}^{(1)} V_2 = V_2 - u_x \frac{\partial}{\partial u_x}, \quad \mathrm{pr}^{(1)} V_3 = V_3 + (1+u_x^2)\frac{\partial}{\partial u_x}.$$

Without looking at your notes, derive the first prolongation formula and verify these are correct.

3.11. Let $u = u(x,t)$. The vector field $V = \xi\partial_x + \phi\partial_t + \eta\partial_u$ generates a 1-parameter group of transformations

$$(x,t,u) \mapsto (\tilde{x},\tilde{t},\tilde{u}) = (x + s\xi(x,t,u), t + s\phi(x,t,u), u + s\eta(x,t,u)) + o(s).$$

By considering the contact condition $d\tilde{u} = \tilde{u}_{\tilde{t}} d\tilde{t} + \tilde{u}_{\tilde{x}} d\tilde{x}$ show that $\mathrm{pr}^{(1)} V = V + \eta^x \partial_{u_x} + \eta^t \partial_{u_t}$ where

$$\eta^t = \mathrm{D}_t\eta - u_t \mathrm{D}_t\phi - u_x \mathrm{D}_t\xi, \quad \eta^x = \mathrm{D}_x\eta - u_x \mathrm{D}_x\xi - u_t \mathrm{D}_x\phi,$$

where D_x and D_t are total derivatives.

3.12. The modified KdV equation is $v_t + v_{xxx} - 6v^2 v_x = 0$. Find a Lie-point symmetry of the form

$$\psi^s(x,t,v) = (e^{\alpha s}x, e^{\beta s}t, e^{\gamma s}v)$$

for appropriate numbers (α, β, γ). Consider the group invariant solution $v(x,t) = (3t)^{-1/3}w(z)$, where $z = x(3t)^{-1/3}$, and construct a 3rd order differential equation for w. Integrate this equation once to show that w satisfies Painlevé II.

3.13 (*)**.** Let $\tilde{\mathbf{x}} = \psi^s\mathbf{x}$ be a 1-parameter group of coordinate transformations generated by V. Show that ψ^s is a Lie-Point symmetry of the equation $\Delta[\mathbf{x}] = 0$ iff $V(\Delta) = 0$ on solutions to $\Delta[\mathbf{x}] = 0$.

3.14 (*)**.** Let $u = u(x)$ and $V = \xi\partial_x + \eta\partial_u$. Calculate $\mathrm{pr}^{(2)}V$. Show that the equation $u_{xx} = 0$ admits an 8 dimensional group of Lie-point symmetries. Can you give geometrical meaning to each of the generators?

3.15 (*)**.** Establish the following properties of the matrix exponential for all $A \in \mathrm{mat}(n \times n)$ and $t, s \in \mathbb{R}$:

i) $(e^A)^T = e^{A^T}$.

ii) $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$.

iii) $e^{At}e^{As} = a^{A(t+s)} = e^{As}e^{At}$

iv) $(e^A)^{-1} = e^{-A}$.

v) $\lim_{n\to\infty}\left(1 + \frac{A}{n}\right)^n = e^A$

vi) $\lim_{n\to\infty}\left[\Gamma\left(\frac{1}{n}\right)\right]^n = e^{\dot{\Gamma}(0)}$ for any C^1 curve $\Gamma : (-\epsilon, \epsilon) \to \mathrm{mat}(n \times n)$.