Principles of Quantum Mechanics - Problems 1

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk (*) are optional and may be more difficult.

1. Let \(|x\rangle\) and \(|p\rangle\) be eigenstates of the three-dimensional position and momentum operators, respectively.

   i) Given that \(\langle x|p \rangle = e^{i\frac{p}{\hbar}(2\pi\hbar)^{3/2}}\), show that

   \[
   \langle x|L_z|\psi\rangle = -i\hbar \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \psi(x) \quad \text{and} \quad \langle p|L_z|\psi\rangle = -i\hbar \left( p_x \frac{\partial}{\partial p_y} - p_y \frac{\partial}{\partial p_x} \right) \tilde{\psi}(p),
   \]

   where \(L_z = XP_y - YP_x\), \(|\psi\rangle\) is a generic state with position- and momentum-space wavefunctions \(\psi(x) = \langle x|\psi\rangle\) and \(\tilde{\psi}(p) = \langle p|\psi\rangle\).

   ii) Find the position space wavefunction of \(e^{-ia\cdot P/\hbar}|\psi\rangle\) and the momentum space wavefunction of \(e^{ik\cdot X}|\psi\rangle\), where \(a\) and \(k\) are constants.

2. Let \(A\) and \(B\) be any operators which each commute with \([A, B]\), and let \(\lambda \in \mathbb{C}\).

   i) Prove that \([A, B^n] = nB^{n-1}[A, B]\) for all \(n \in \mathbb{N}_0\), and that \([A, e^B] = e^B[A, B]\).

   ii) Define the operator-valued function \(F(\lambda) = e^{\lambda A} e^{\lambda B} e^{-\lambda(A+B)}\). Show that \(F'(\lambda) = \lambda [A, B] F(\lambda)\).

   Hence deduce that

   \[
   e^A e^B = e^{A+B+\frac{1}{2}[A, B]} = e^B e^A e^{[A, B]}.
   \]

   Now let \(A\) and \(B\) be any operators (not necessarily commuting with \([A, B]\)).

   iii) Prove that \(d(e^{\lambda A} e^{-\lambda B})/d\lambda = e^{\lambda A} [A, B] e^{-\lambda A}\). Hence deduce that

   \[
   e^A B e^{-A} = B + [A, B] + \frac{1}{2} [A, [A, B]] + \cdots.
   \]

3. Let \(X(t) = e^{iHt/\hbar}X e^{-iHt/\hbar}\) and \(P(t) = e^{iHt/\hbar} P e^{-iHt/\hbar}\) where \(X\) and \(P\) are the usual position and momentum operators, and \(H\) is the Hamiltonian of the \(d = 1\) harmonic oscillator. Show that

   \[
   X(t) = X \cos(\omega t) + \frac{1}{m\omega} P \sin(\omega t)
   \]

   \[
   P(t) = P \cos(\omega t) - m\omega X \sin(\omega t).
   \]

   and interpret this result. Evaluate \([X(t), P(t)]\).

4. A Fermi oscillator has Hilbert space \(\mathcal{H} = \mathbb{C}^2\) and Hamiltonian \(H = B^\dagger B\), where \(B^2 = 0\) and \(B^\dagger B + BB^\dagger = 1\) (the anticommutator).

   Find the eigenvalues of \(H\). If \(|0\rangle\) is a state obeying \(H|0\rangle = 0\) and \(\langle 0|0\rangle = 1\), find \(B|0\rangle\) and \(B^\dagger|0\rangle\). Obtain a matrix representation of the operators \(B, B^\dagger\) and \(H\).
5. Consider a $d = 1$ quantum harmonic oscillator with classical frequency $\omega$. Define the coherent state $|\alpha\rangle$ by

$$|\alpha\rangle = e^{\alpha A^\dagger - \bar{\alpha} A}|0\rangle,$$

where $A^\dagger$ and $A$ are the usual raising & lowering operators, $|0\rangle$ is the ground state of the oscillator and $\alpha \in \mathbb{C}$ is a constant.

i) Show that $|\alpha\rangle$ is an eigenstate of $A$ and find its eigenvalue. Compute the inner product between two different coherent states $|\alpha\rangle$ and $|\beta\rangle$. [You may find it helpful to use the results of question 2.] Does the set $\{|\alpha\rangle\}_{\alpha \in \mathbb{C}}$ of all coherent states form a basis of the Hilbert space?

ii) A quantum oscillator is prepared to be in state $|\alpha\rangle$ at $t = 0$. Show that subsequently it evolves to become the new coherent state $e^{-i\omega t/2}|e^{-i\omega t}\alpha\rangle$.

iii) Suppose $\alpha \in \mathbb{R}$. By expressing $A$ and $A^\dagger$ in terms of $X$ and $P$, sketch the position space wavefunction of $|\alpha\rangle$ in this case.

v*) Now let $\alpha \in \mathbb{C}$ and compute $\langle\alpha|P|\alpha\rangle$. Hence give a physical interpretation of the coherent state for general complex $\alpha$. Without further calculation, describe the shape and motion of both the position space and momentum space wavefunctions of a general coherent state as time passes.

6*. In certain units where $\hbar = 2m = 1$, the relative motion of the atoms in a diatomic molecule can be modelled by the Hamiltonian

$$H_\nu = P^2 + \left(\nu + \frac{1}{2} - e^{-X}\right)^2,$$

where $\nu$ is a real parameter.

i) Sketch the potential. Suggest a reason why it gives a better description of the molecule’s vibrations than the harmonic oscillator does.

ii) Find a non-Hermitian operator $A_\nu$ that depends on $\nu$, such that

$$H_\nu = A^\dagger_\nu A_\nu + \nu + \frac{1}{4}.$$

What is the ground state energy of $H_\nu$? Calculate the position space wavefunction of the ground state. For what range of $\nu$ is this state normalizable?

iii) Let $|0,\nu\rangle$ be the ground state appropriate for parameter $\nu$. Show that

$$A_\nu A^\dagger_\nu = A^\dagger_{\nu-1} A_{\nu-1} + 2\nu - 1$$

and hence that $A^\dagger_\nu|0,\nu - 1\rangle$ is also an (unnormalized) eigenstate of $H_\nu$.

iv) Iterating this procedure, deduce an expression for the unnormalized $n^{th}$ eigenstate in terms of a sequence of raising operators acting on a ground state, for appropriate choices of the parameter. Hence find the bound state spectrum of $H_\nu$. Show that the number of bound states is $[\nu + 1]$. Do these states form a basis of the Hilbert space?

7. Let $P/h$ and $J/h$ be the generators of translations and rotations, respectively. By considering the effect of a rotation and translation on an arbitrary vector $v \in \mathbb{R}^3$, show that $[J_i, P_j] = i\hbar \epsilon_{ijk} P_k$. 
8*. A quantum particle is described by the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, d^3x)$ and has Hamiltonian $H = \frac{P^2}{2m}$. Galilean boosts with fixed velocity $v$ act on $\mathcal{H}$ through a time-independent unitary operator $U(v)$ such that

$$U^\dagger(v) X(t) U(v) = X(t) + vt,$$

where $X(t)$ is the position operator in the Heisenberg picture.

i) Show that $U(v_1) U(v_2) = U(v_1 + v_2)$ and that $U^\dagger(v) P U(v) = P + mv$. Hence express $U(v)$ in terms of $X$, $P$ and $v$.

ii) Let $T(a) = e^{-ia \cdot P / \hbar}$ be the translation operator. Evaluate the composition of operators

$$T^\dagger(a) U^\dagger(v) T(a) U(v)$$

Is this compatible with what you’d expect classically for the corresponding sequence of boosts and translations?