D18a  Principles of Quantum Mechanics: Sheet 1  Michaelmas 2016

1. A Hamiltonian has the form

\[ H = \sum_n (E_0|n\rangle\langle n| + E_1|n+1\rangle\langle n| + E_1|n\rangle\langle n+1|) \]

where \( n \) runs over all integers (positive and negative) and \( |n\rangle = \delta_{nm} \). Show that the states \(|\theta\rangle = \sum_n e^{in\theta}|n\rangle\) are eigenvectors of \( H \) and determine the corresponding energy eigenvalues.

2. If \( H = p^2/2m + V(\hat{x}) \), evaluate \([\hat{x}, H], \hat{x} \) by using the basic commutation relation \([\hat{x}, p] = i\hbar \). Show that \(|s\rangle = (E_r - E_s)\langle s|\) are normalized eigenstates of \( H \) with energies \( E_n \). Use the completeness relation \( \sum_r |r\rangle\langle r| = 1 \) to obtain the sum rule

\[ \sum_r (E_r - E_s)|r\rangle\langle r|s\rangle = \frac{\hbar^2}{2m}. \]

3. The position space Schrödinger equation for the one-dimensional harmonic oscillator is

\[ H \psi_n(x) = E_n \psi_n(x) \quad \text{with} \quad H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2. \]

Find the corresponding equation in momentum space and deduce that the associated eigenfunctions \( \psi_n(p) \) can be obtained from the functions \( \psi_n(x) \) by replacing the constants \( m \) and \( \omega \) appropriately.

4. Obtain the Schrödinger equation for the hydrogen atom in the momentum space representation. Show that \( \psi(p) = N/(p^2 + \alpha^2)^2 \) is a solution for a suitable choice of the constant \( \alpha \) and find its energy eigenvalue. The following integrals may be used:

\[ \frac{4\pi}{\mathbf{k}^2} = \int d^3x \ e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{1}{|\mathbf{x}|}, \quad \int d^3p' \frac{1}{(\mathbf{p} - \mathbf{p}')^2} = \frac{1}{\alpha^3}. \]

5. If \( A \) and \( B \) are any operators which each commute with their commutator \([A, B]\), prove that \([A, B^n] = nB^{n-1}[A, B] \) and \([A, e^B] = e^B[A, B] \). Let \( F(\lambda) = e^{\lambda A}e^{\lambda B}e^{-\lambda(A+B)} \), an operator-valued function of the parameter \( \lambda \). Show that \( F'(\lambda) = \lambda[A, B]F(\lambda) \) and hence deduce that

\[ e^Ae^B = e^{A+B+\frac{1}{2}[A,B]} = e^Be^Ae^{[A,B]} \cdot \]

6. (a) Write down the position \( \hat{x} \) and momentum \( \hat{p} \) of a one-dimensional harmonic oscillator in terms of the annihilation and creation operators \( a \) and \( a^\dagger \). Use \([a, a^\dagger] = 1 \) to evaluate \([a, a^\dagger]^n\). If \(|0\rangle \) is a normalized state such that \( a|0\rangle = 0 \), verify that \( a^\dagger|n\rangle \) is an eigenstate of \( N = a^\dagger a \) and find its norm. Calculate the expectation values of \( \hat{x} \), \( \hat{p} \), \( \hat{x}^2 \) and \( \hat{p}^2 \) in this state and show that \( \Delta x\Delta p = (n+\frac{1}{2})\hbar \).

(b) Prove that \(|z\rangle = e^{za^\dagger}|0\rangle \) is an eigenstate of \( a \) with eigenvalue \( z \) for any complex number \( z \) and calculate \( \langle z_1|z_2 \rangle \). Show that the expectation values of position and momentum in the state \(|z\rangle \) are

\[ \langle \hat{x} \rangle = (2\hbar/m\omega)^{\frac{1}{2}} \text{Re} z, \quad \langle \hat{p} \rangle = (2\hbar m\omega)^{\frac{1}{2}} \text{Im} z. \]
7. Show that in position space the annihilation operator for the simple harmonic oscillator has the representation 
\[ a = \frac{1}{\sqrt{2}} \left( \frac{d}{dy} + y \right) \] 
where \( y = \left( \frac{m\omega}{\lambda} \right)^{\frac{1}{2}} x \). Use the relation \( a|0\rangle = 0 \) to calculate the wave function \( \psi_0(y) \) of the ground state. Use the position space representation of the creation operator to show that the wave function for the \( n \)-th excited state is given by
\[ \psi_n(y) = h_n(y) \psi_0(y) \] 
where
\[ h_n(y) = \left( \frac{-1}{(2\pi n!)^\frac{1}{2}} \right) e^{y^2 \frac{d^n}{dy^n} e^{-y^2}} \] is a polynomial of degree \( n \).

8. The operators \( a \) and \( a^\dagger \) satisfy the relations \( aa^\dagger + a^\dagger a = 1 \) and \( a^2 = a_{\dagger}^2 = 0 \). Show that the eigenvalues of \( N = a^\dagger a \) can only be 0 or 1. Assuming the existence of a normalized state \( |0\rangle \) with eigenvalue 0, construct a normalized state \( |1\rangle \) with eigenvalue 1, and show how \( |0\rangle \) can be expressed in terms of \( |1\rangle \). Obtain the matrix elements \( \langle n|a|n'\rangle \) and \( \langle n|a^\dagger|n'\rangle \) (for \( n, n' = 0, 1 \)) and check that these matrices obey the correct algebra (i.e. the relations for the operators given above).

9. Prove that, for any operators \( A \) and \( B \),
\[ \frac{d}{d\lambda} (e^{\lambda A}Be^{-\lambda A}) = e^{\lambda A}[A, B]e^{-\lambda A} \]
By repeated use of this relation, find a series expansion in \( \lambda \) for \( e^{\lambda A}Be^{-\lambda A} \) and deduce that
\[ e^{\lambda A}Be^{-\lambda A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \ldots \]

10. (a) Starting from the canonical commutation relation \([\hat{x}, \hat{p}] = i\hbar\) for one-dimensional position and momentum, show that \([\hat{p}, g(\hat{x})] = -i\hbar g'(\hat{x})\) where \( g \) is any function expressible as a power series. Use this in conjunction with the last example (9, above) to show that if \( f \) is also a function with a power series expansion then \( \hat{p} - f(\hat{x}) = U\hat{p}U^{-1} \) and \( U\hat{x}U^{-1} = \hat{x} \) for some operator \( U \).

*(b) Consider three-dimensional position and momentum operators \( \hat{x}_i \) and \( \hat{p}_j \) with \([\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}\) and all other commutators vanishing. Suppose the momenta \( \hat{p}_j \) are replaced by \( \hat{P}_j = \hat{p}_j - f_j(\hat{x}_i) \); what conditions on the functions \( f_j(\hat{x}_i) \) will ensure that all the commutation relations are unaltered? Show that if these conditions hold there is an operator \( U \) such that \( \hat{P}_j = U\hat{p}_jU^{-1} \) and \( U\hat{x}U^{-1} = \hat{x}_j \).

11. Determine the energy eigenvalues of the Hamiltonian \( H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2(\hat{x} - c)^2 \) by setting \( \hat{x} = c + (\hbar/2m\omega)^{\frac{1}{2}}(a + a^\dagger) \) and \( \hat{p} = (\hbar/m\omega)^{\frac{1}{2}}i(a^\dagger - a) \). Why are the energies independent of the constant \( c \)?

The Hamiltonian for a particle of charge \( e \) in a constant magnetic field \( B \) along the \( z \) direction is
\[ H = \frac{1}{2m}(\hat{p} - eA(\hat{x}))^2, \quad \text{with} \quad A_x = A_z = 0, \quad A_y = B\hat{x}. \]

Show that \( \hat{p}_y, \hat{p}_z, H \) are a set of commuting operators. Consider states on which \( \hat{p}_y, \hat{p}_z \) take the definite values \( p_y, p_z \) and show that the possible energy eigenvalues are
\[ E = \frac{p_y^2}{2m} + (n + \frac{1}{2})\hbar\omega, \quad \omega = \frac{|eB|}{m}, \]
independent of \( p_y \).