

Principles of Quantum Mechanics - Problems 1

Please *email me* with any comments about these problems, particularly if you spot an error. Problems with an asterisk (*) are optional and may be more difficult.

1. Let $|\mathbf{x}\rangle$ and $|\mathbf{p}\rangle$ be eigenstates of the three-dimensional position and momentum operators, respectively.

i) Given that $\langle \mathbf{x} | \mathbf{p} \rangle = e^{i\mathbf{x} \cdot \mathbf{p} / \hbar} / (2\pi\hbar)^{3/2}$, show that

$$\langle \mathbf{x} | L_z | \psi \rangle = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \psi(\mathbf{x}) \quad \text{and} \quad \langle \mathbf{p} | L_z | \psi \rangle = -i\hbar \left(p_x \frac{\partial}{\partial p_y} - p_y \frac{\partial}{\partial p_x} \right) \tilde{\psi}(\mathbf{p}),$$

where $L_z = XP_y - YP_x$, $|\psi\rangle$ is a generic state with position- and momentum-space wavefunctions $\psi(\mathbf{x}) = \langle \mathbf{x} | \psi \rangle$ and $\tilde{\psi}(\mathbf{p}) = \langle \mathbf{p} | \psi \rangle$.

ii) Find the position space wavefunction of $e^{-i\mathbf{a} \cdot \mathbf{P} / \hbar} |\psi\rangle$ and the momentum space wavefunction of $e^{i\mathbf{k} \cdot \mathbf{X}} |\psi\rangle$, where \mathbf{a} and \mathbf{k} are constants.

2. Let A and B be any operators which each commute with $[A, B]$, and let $\lambda \in \mathbb{C}$.

i) Prove that $[A, B^n] = nB^{n-1}[A, B]$ for all $n \in \mathbb{N}_0$, and that $[A, e^B] = e^B[A, B]$.

ii) Define the operator-valued function $F(\lambda) = e^{\lambda A} e^{\lambda B} e^{-\lambda(A+B)}$. Show that $F'(\lambda) = \lambda[A, B]F(\lambda)$. Hence deduce that

$$e^A e^B = e^{A+B+\frac{1}{2}[A, B]} = e^B e^A e^{[A, B]}.$$

Now let A and B be any operators (not necessarily commuting with $[A, B]$).

iii) Prove that $d(e^{\lambda A} B e^{-\lambda A})/d\lambda = e^{\lambda A} [A, B] e^{-\lambda A}$. Hence deduce that

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2} [A, [A, B]] + \dots$$

3. Let $X(t) = e^{iHt/\hbar} X e^{-iHt/\hbar}$ and $P(t) = e^{iHt/\hbar} P e^{-iHt/\hbar}$ where X and P are the usual position and momentum operators, and H is the Hamiltonian of the $d = 1$ harmonic oscillator. Show that

$$X(t) = X \cos(\omega t) + \frac{1}{m\omega} P \sin(\omega t)$$

$$P(t) = P \cos(\omega t) - m\omega X \sin(\omega t).$$

and interpret this result. Evaluate $[X(t), P(t)]$.

4. A Fermi oscillator has Hilbert space $\mathcal{H} = \mathbb{C}^2$ and Hamiltonian $H = B^\dagger B$, where $B^2 = 0$ and

$$B^\dagger B + B B^\dagger = 1 \quad (\text{the anticommutator}).$$

Find the eigenvalues of H . If $|0\rangle$ is a state obeying $H|0\rangle = 0$ and $\langle 0|0\rangle = 1$, find $B|0\rangle$ and $B^\dagger|0\rangle$. Obtain a matrix representation of the operators B , B^\dagger and H .

5. Consider a $d = 1$ quantum harmonic oscillator with classical frequency ω . Define the *coherent state* $|\alpha\rangle$ by

$$|\alpha\rangle = e^{\alpha A^\dagger - \bar{\alpha} A} |0\rangle,$$

where A^\dagger and A are the usual raising & lowering operators, $|0\rangle$ is the ground state of the oscillator and $\alpha \in \mathbb{C}$ is a constant.

- i) Show that $|\alpha\rangle$ is an eigenstate of A and find its eigenvalue. Compute the inner product between two different coherent states $|\alpha\rangle$ and $|\beta\rangle$. [You may find it helpful to use the results of question 2.] Does the set $\{|\alpha\rangle\}_{\alpha \in \mathbb{C}}$ of all coherent states form a basis of the Hilbert space?
- ii) A quantum oscillator is prepared to be in state $|\alpha\rangle$ at $t = 0$. Show that subsequently it evolves to become the new coherent state $e^{-i\omega t/2} |e^{-i\omega t}\alpha\rangle$.
- iii) Suppose $\alpha \in \mathbb{R}$. By expressing A and A^\dagger in terms of X and P , sketch the position space wavefunction of $|\alpha\rangle$ in this case.
- iv*) Now let $\alpha \in \mathbb{C}$ and compute $\langle\alpha|P|\alpha\rangle$. Hence give a physical interpretation of the coherent state for general complex α . Without further calculation, describe the shape and motion of both the position space and momentum space wavefunctions of a general coherent state as time passes.

- 6*. In certain units where $\hbar = 2m = 1$, the relative motion of the atoms in a diatomic molecule can be modelled by the Hamiltonian

$$H_\nu = P^2 + \left(\nu + \frac{1}{2} - e^{-X} \right)^2,$$

where ν is a real parameter.

- i) Sketch the potential. Suggest a reason why it gives a better description of the molecule's vibrations than the harmonic oscillator does.
- ii) Find a non-Hermitian operator A_ν that depends on ν , such that

$$H_\nu = A_\nu^\dagger A_\nu + \nu + \frac{1}{4}$$

What is the ground state energy of H_ν ? Calculate the position space wavefunction of the ground state. For what range of ν is this state normalizable?

- iii) Let $|0, \nu\rangle$ be the ground state appropriate for parameter ν . Show that

$$A_\nu A_\nu^\dagger = A_{\nu-1}^\dagger A_{\nu-1} + 2\nu - 1$$

and hence that $A_\nu^\dagger |0, \nu - 1\rangle$ is also an (unnormalized) eigenstate of H_ν .

- iv) Iterating this procedure, deduce an expression for the unnormalized n^{th} eigenstate in terms of a sequence of raising operators acting on a ground state, for appropriate choices of the parameter. Hence find the bound state spectrum of H_ν . Show that the number of bound states is $\lfloor \nu + 1 \rfloor$. Do these states form a basis of the Hilbert space?

7. Let \mathbf{P}/\hbar and \mathbf{J}/\hbar be the generators of translations and rotations, respectively. By considering the effect of a rotation and translation on an arbitrary vector $\mathbf{v} \in \mathbb{R}^3$, show that $[J_i, P_j] = i\hbar \epsilon_{ijk} P_k$.

- 8*. A quantum particle is described by the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, d^3x)$ and has Hamiltonian $H = \mathbf{P}^2/2m$. Galilean boosts with fixed velocity \mathbf{v} act on \mathcal{H} through a time-independent unitary operator $U(\mathbf{v})$ such that

$$U^\dagger(\mathbf{v}) \mathbf{X}(t) U(\mathbf{v}) = \mathbf{X}(t) + \mathbf{v}t,$$

where $\mathbf{X}(t)$ is the position operator in the Heisenberg picture.

- i) Show that $U(\mathbf{v}_1)U(\mathbf{v}_2) = U(\mathbf{v}_1 + \mathbf{v}_2)$ and that $U^\dagger(\mathbf{v})\mathbf{P}U(\mathbf{v}) = \mathbf{P} + m\mathbf{v}$. Hence express $U(\mathbf{v})$ in terms of \mathbf{X} , \mathbf{P} and \mathbf{v} .
- ii) Let $T(\mathbf{a}) = e^{-i\mathbf{a}\cdot\mathbf{P}/\hbar}$ be the translation operator. Evaluate the composition of operators

$$T^\dagger(\mathbf{a})U^\dagger(\mathbf{v})T(\mathbf{a})U(\mathbf{v})$$

Is this compatible with what you'd expect classically for the corresponding sequence of boosts and translations?