Principles of Quantum Mechanics - Problems 1

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk (∗) are optional and may be more difficult.

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1. Let $|\mathbf{x}\rangle$ and $|\mathbf{p}\rangle$ be eigenstates of the three-dimensional position and momentum operators, respectively.

   i) Given that $\langle\mathbf{x}|\mathbf{p}\rangle = \mathrm{e}^{\mathrm{i}\mathbf{x}\cdot\mathbf{p}/\hbar}/(2\pi\hbar)^{3/2}$, show that
   $$\langle\mathbf{x}|L_z|\psi\rangle = -\mathrm{i}\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\psi(\mathbf{x}) \qquad \text{and} \qquad \langle\mathbf{p}|L_z|\psi\rangle = -\mathrm{i}\hbar\left(p_x\frac{\partial}{\partial p_y} - p_y\frac{\partial}{\partial p_x}\right)\tilde{\psi}(\mathbf{p})\,,$$
   where $L_z = XP_y - YP_x$, $|\psi\rangle$ is a generic state with position- and momentum-space wavefunctions $\psi(\mathbf{x}) = \langle\mathbf{x}|\psi\rangle$ and $\tilde{\psi}(\mathbf{p}) = \langle\mathbf{p}|\psi\rangle$.

   ii) Find the position space wavefunction of $\mathrm{e}^{-\mathrm{i}\mathbf{a}\cdot\mathbf{P}/\hbar}|\psi\rangle$ and the momentum space wavefunction of $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{X}}|\psi\rangle$, where $\mathbf{a}$ and $\mathbf{k}$ are constants.

2. Let A and B be any operators which each commute with $[A,B]$, and let $\lambda \in \mathbb{C}$.

   i) Prove that $[A, B^n] = nB^{n-1}[A,B]$ for all $n \in \mathbb{N}_0$, and that $[A, e^B] = e^B[A,B]$.

   ii) Define the operator-valued function $F(\lambda) = \mathrm{e}^{\lambda A}\,\mathrm{e}^{\lambda B}\,\mathrm{e}^{-\lambda(A+B)}$. Show that $F'(\lambda) = \lambda\,[A,B]\,F(\lambda)$. Hence deduce that
   $$\mathrm{e}^A\,\mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]} = \mathrm{e}^B\,\mathrm{e}^A\,\mathrm{e}^{[A,B]}\,.$$

   Now let A and B be any operators (not necessarily commuting with $[A,B]$).

   iii) Prove that $d\left(\mathrm{e}^{\lambda A}B\mathrm{e}^{-\lambda A}\right)/d\lambda = \mathrm{e}^{\lambda A}\,[A,B]\,\mathrm{e}^{-\lambda A}$. Hence deduce that
   $$\mathrm{e}^A B\,\mathrm{e}^{-A} = B + [A,B] + \frac{1}{2}\,[A,[A,B]] + \cdots\,.$$

3. Let $X(t) = \mathrm{e}^{\mathrm{i}Ht/\hbar}X\mathrm{e}^{-\mathrm{i}Ht/\hbar}$ and $P(t) = \mathrm{e}^{\mathrm{i}Ht/\hbar}\,P\,\mathrm{e}^{-\mathrm{i}Ht/\hbar}$ where X and P are the usual position and momentum operators, and H is the Hamiltonian of the $d=1$ harmonic oscillator. Show that
   $$X(t) = X\cos(\omega t) + \frac{1}{m\omega}P\sin(\omega t)$$
   $$P(t) = P\cos(\omega t) - m\omega X\sin(\omega t)\,.$$
   and interpret this result. Evaluate $[X(t), P(t)]$.

4. A Fermi oscillator has Hilbert space $\mathcal{H} = \mathbb{C}^2$ and Hamiltonian $H = B^\dagger B$, where $B^2 = 0$ and
   $$B^\dagger B + BB^\dagger = 1 \qquad (\text{the } \textit{anti}\text{commutator})\,.$$
   Find the eigenvalues of H. If $|0\rangle$ is a state obeying $H|0\rangle = 0$ and $\langle 0|0\rangle = 1$, find $B|0\rangle$ and $B^\dagger|0\rangle$. Obtain a matrix representation of the operators B, $B^\dagger$ and H.

5. Consider a $d = 1$ quantum harmonic oscillator with classical frequency ω. Define the *coherent state* $|\alpha\rangle$ by
$$|\alpha\rangle = \mathrm{e}^{\alpha A^\dagger - \bar{\alpha} A}|0\rangle\,,$$
where $A^\dagger$ and A are the usual raising & lowering operators, $|0\rangle$ is the ground state of the oscillator and $\alpha \in \mathbb{C}$ is a constant.

   i) Show that $|\alpha\rangle$ is an eigenstate of A and find its eigenvalue. Compute the inner product between two different coherent states $|\alpha\rangle$ and $|\beta\rangle$. *[You may find it helpful to use the results of question 2.]* Does the set $\{|\alpha\rangle\}_{\alpha\in\mathbb{C}}$ of all coherent states form a basis of the Hilbert space?

   ii) A quantum oscillator is prepared to be in state $|\alpha\rangle$ at $t = 0$. Show that subsequently it evolves to become the new coherent state $\mathrm{e}^{-\mathrm{i}\omega t/2}\,|\mathrm{e}^{-\mathrm{i}\omega t}\alpha\rangle$.

   iii) Suppose $\alpha \in \mathbb{R}$. By expressing A and $A^\dagger$ in terms of X and P, sketch the position space wavefunction of $|\alpha\rangle$ in this case.

   v*) Now let $\alpha \in \mathbb{C}$ and compute $\langle\alpha|P|\alpha\rangle$. Hence give a physical interpretation of the coherent state for general complex α. Without further calculation, describe the shape and motion of both the position space and momentum space wavefunctions of a general coherent state as time passes.

6*. In certain units where $\hbar = 2m = 1$, the relative motion of the atoms in a diatomic molecule can be modelled by the Hamiltonian
$$H_\nu = P^2 + \left(\nu + \frac{1}{2} - \mathrm{e}^{-X}\right)^2\,,$$
where ν is a real parameter.

   i) Sketch the potential. Suggest a reason why it gives a better description of the molecule's vibrations than the harmonic oscillator does.

   ii) Find a non-Hermitian operator A_ν that depends on ν, such that
$$H_\nu = A_\nu^\dagger A_\nu + \nu + \frac{1}{4}$$
What is the ground state energy of H_ν? Calculate the position space wavefunction of the ground state. For what range of ν is this state normalizable?

   iii) Let $|0,\nu\rangle$ be the ground state appropriate for parameter ν. Show that
$$A_\nu A_\nu^\dagger = A_{\nu-1}^\dagger A_{\nu-1} + 2\nu - 1$$
and hence that $A_\nu^\dagger|0,\nu-1\rangle$ is also an (unnormalized) eigenstate of H_ν.

   iv) Iterating this procedure, deduce an expression for the unnormalized n^{th} eigenstate in terms of a sequence of raising operators acting on a ground state, for appropriate choices of the parameter. Hence find the bound state spectrum of H_ν. Show that the number of bound states is $\lfloor \nu + 1 \rfloor$. Do these states form a basis of the Hilbert space?

7. Let $\mathbf{P}/\hbar$ and $\mathbf{J}/\hbar$ be the generators of translations and rotations, respectively. By considering the effect of a rotation and translation on an arbitrary vector $\mathbf{v} \in \mathbb{R}^3$, show that $[J_i, P_j] = \mathrm{i}\hbar\,\epsilon_{ijk}P_k$.

8*. A quantum particle is described by the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, \mathrm{d}^3x)$ and has Hamiltonian $H = \mathbf{P}^2/2m$. Galilean boosts with fixed velocity $\mathbf{v}$ act on $\mathcal{H}$ through a time-independent unitary operator $U(\mathbf{v})$ such that

$$U^\dagger(\mathbf{v})\,\mathbf{X}(t)\,U(\mathbf{v}) = \mathbf{X}(t) + \mathbf{v}t\,,$$

where $\mathbf{X}(t)$ is the position operator in the Heisenberg picture.

i) Show that $U(\mathbf{v}_1)\,U(\mathbf{v}_2) = U(\mathbf{v}_1 + \mathbf{v}_2)$ and that $U^\dagger(\mathbf{v})\,\mathbf{P}\,U(\mathbf{v}) = \mathbf{P} + m\mathbf{v}$. Hence express $U(\mathbf{v})$ in terms of $\mathbf{X}$, $\mathbf{P}$ and $\mathbf{v}$.

ii) Let $T(\mathbf{a}) = \mathrm{e}^{-\mathrm{i}\mathbf{a}\cdot\mathbf{P}/\hbar}$ be the translation operator. Evaluate the composition of operators

$$T^\dagger(\mathbf{a})\,U^\dagger(\mathbf{v})\,T(\mathbf{a})\,U(\mathbf{v})$$

Is this compatible with what you'd expect classically for the corresponding sequence of boosts and translations?