Principles of Quantum Mechanics - Problems 1

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk (∗) may be more difficult.

1. Let |x⟩ and |p⟩ be eigenstates of the three-dimensional position and momentum operators, respectively. Given that ⟨x|p⟩ = e^(ix·p/ℏ)/(2πℏ)^3/2, show that

⟨x|L_z|ψ⟩ = −iℏ(∂/∂y − y∂/∂x)⟨x|ψ⟩

and

⟨p|L_z|ψ⟩ = −iℏ(px∂/∂p_y − py∂/∂p_x)⟨p|ψ⟩,

where L_z = XP_y − YP_x and |ψ⟩ is a generic state.

2. Let A and B be any operators which each commute with [A, B], and let λ ∈ C.


ii) Define the operator-valued function F(λ) = e^{λA}e^{λB}e^{-λ(A+B)}. Show that F′(λ) = λ[A, B]F(λ). Hence deduce that

e^A e^B = e^{A+B+1/2[A,B]} = e^B e^A e^{[A,B]}.

3. Consider a d = 1 quantum harmonic oscillator with classical frequency ω. For any α ∈ C, define the coherent state |α⟩ by

|α⟩ = e^{αA^† − ∗A}|0⟩,

where A^† and A are the usual raising & lowering operators and |0⟩ is the ground state of the oscillator.

i) Show that |α⟩ is an eigenstate of A and find its eigenvalue. Compute the inner product between two different coherent states |α⟩ and |β⟩. [You may find it helpful to use the results of the previous question.] Does the set {⟨α⟩}_{α∈C} of all coherent states form a basis of the Hilbert space?

ii) A quantum oscillator is prepared to be in state |α⟩ at t = 0. Show that subsequently it evolves to become the new coherent state e^{-iωt/2} e^{-iωt|α⟩}.

iii) By expressing A and A^† in terms of X and P, sketch the position space wavefunction of |α⟩ in the case α ∈ R.

iv*) Give a physical interpretation of the coherent state for general complex α. [You may wish to first compute the expectation value ⟨α|P|α⟩.] Without further calculation, describe the shape and motion of both the position space and momentum space wavefunctions of a general coherent state as time passes.
4. A Fermi oscillator has Hamiltonian $H = B^\dagger B$ where $B^2 = 0$

$$B^\dagger B + BB^\dagger = 1 \quad \text{(the anti-commutator).}$$

Find the eigenvalues of $H$. If $|0\rangle$ is a state obeying $H|0\rangle = 0$ and $\langle 0|0 \rangle = 1$, find $B|0\rangle$ and $B^\dagger |0\rangle$. Explain the connection between the spectrum of $B^\dagger B$ and the Pauli exclusion principle.

5. In certain units where $\hbar = 2m = 1$, the relative motion of the atoms in a diatomic molecule can be modelled by the Hamiltonian

$$H_\nu = P^2 + \left(\nu + \frac{1}{2} - e^{-x}\right)^2,$$

where $\nu$ is a real parameter.

i) Sketch the potential. Suggest a reason why it better describes the molecule’s vibrations than the harmonic oscillator does.

ii) Find a non-Hermitian operator $A_\nu$ that depends on $\nu$, such that

$$H_\nu = A_\nu^\dagger A_\nu + \nu + \frac{1}{4}$$

What is the ground state energy of $H_\nu$? Calculate the position space wavefunction of the ground state. For what range of $\nu$ is this state normalizable?

iii) Let $|0, \nu\rangle$ be the ground state appropriate for parameter $\nu$. Show that

$$A_\nu A_\nu^\dagger = A_{\nu-1}^\dagger A_{\nu-1} + 2\nu - 1$$

and hence that $A_\nu^\dagger |0, \nu - 1\rangle$ is also an (unnormalized) eigenstate of $H_\nu$.

iv*) Iterating this procedure, deduce an expression for the unnormalized $n$th eigenstate in terms of a sequence of raising operators acting on a ground state, for appropriate choices of the parameter. Hence find the bound state spectrum of $H_\nu$. Show that the number of bound states is $[\nu + 1]$. Do these states form a basis of the Hilbert space?

6. An electron moves along an infinite chain of potential wells. For sufficiently low energies we can assume that the set $\{|n\rangle\}$ is complete, where $|n\rangle$ is the state of definitely being in the $n$th well. Assume that the only non-vanishing matrix elements of the Hamiltonian are $\langle n|H|n\rangle = \mathcal{E}$ and $\langle n \pm 1|H|n\rangle = A$. Give an interpretation of $\mathcal{E}$ and $A$. What does the assumption that $\langle n + r|H|n\rangle$ are negligible for $|r| > 1$ mean?

Expanding an energy eigenstate $|E\rangle$ in this basis as $|E\rangle = \sum_n c_n |n\rangle$, show that

$$c_m (E - \mathcal{E}) - A(c_{m+1} + c_{m-1}) = 0.$$ 

Obtain solutions of these equations in which $c_m \propto e^{ikm}$ and thus find the corresponding energies $E_k$. Why is there an upper limit to the values of $k$ that need be considered?

Initially, the electron is in the state $|\psi\rangle = (|E_k\rangle + |E_{k+\Delta}\rangle)/\sqrt{2}$, where $0 < \Delta \ll k \ll 1$. Describe the electron’s subsequent motion.