Principles of Quantum Mechanics - Problems 2

Please email me with any comments about these problems, particularly if you spot an error.

1. Let $P/\hbar$ and $J/\hbar$ be the generators of translations and rotations, respectively. By considering the effect of a rotation and translation on an arbitrary vector $v \in \mathbb{R}^3$, show that $[J_i, P_j] = i\hbar \epsilon_{ijk} P_k$.

2. Consider a (spinless) free particle in $\mathbb{R}^3$. Galilean boosts with fixed velocity $v$ act on the particle’s Hilbert space through a time-independent unitary operator $U(v)$ defined by

$$U^{-1}(v) X(t) U(v) = X(t) + vt,$$

where $X(t)$ is the position operator in the Heisenberg picture.

   i) Show that $U(v_1) U(v_2) = U(v_1 + v_2)$ and that $U^{-1}(v) P U(v) = P + mv$. Hence express $U(v)$ in terms of $X$, $P$ and $v$.

   ii) Let $T(a) = e^{-ia \cdot P/\hbar}$ be the translation operator. Evaluate the composition of operators

$$T^{-1}(a) U^{-1}(v) T(a) U(v)$$

Is this compatible with what you’d expect classically for the corresponding sequence of boosts and translations?

3. For a spin-$1/2$ particle, the spin operator $S = \hbar\sigma/2$, where $\sigma$ is the vector $(\sigma_1, \sigma_2, \sigma_3)$ each entry of which is a Pauli matrix.

   i) Without picking a representation, explain why the Pauli matrices obey

$$\sigma_i \sigma_j = \frac{1}{2} \{\sigma_i, \sigma_j\} + \frac{1}{2} [\sigma_i, \sigma_j] = \delta_{ij} \text{id} + i\epsilon_{ijk} \sigma_k$$

for $i, j, k \in \{1, 2, 3\}$, where id is the identity matrix. Hence show that $\mathbf{a} \cdot \mathbf{\sigma} \mathbf{b} \cdot \mathbf{\sigma} = \mathbf{a} \cdot \mathbf{b} + i(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{\sigma}$ for any two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.

   ii) Show that the rotation operator can be written as

$$U(\alpha) = \cos \left( \frac{\alpha}{2} \right) \text{id} - i \sin \left( \frac{\alpha}{2} \right) \hat{\alpha} \cdot \mathbf{\sigma}.$$

when acting on a spin-$1/2$ particle. [Neglect the spatial wavefunction.]

   iii) A particle of spin-$1/2$ interacts with a uniform, homogeneous magnetic field $\mathbf{B}$ via the Hamiltonian $\hat{H} = -2\mu \mathbf{B} \cdot \mathbf{S}$, where $\mu$ is a constant. If the particle’s spin is initially prepared to be in some state $|\chi\rangle$, show that the probability its spin is found to be in an orthogonal state $|\chi'\rangle$ a time $t$ later is $|\langle \chi' | \mathbf{B} \cdot \mathbf{\sigma} |\chi\rangle|^2 \sin^2 \omega t$, where $\omega$ is a frequency you should specify.
iv) Show that with this Hamiltonian,
\[
S(t) = \cos 2\omega t \mathbf{S} + (1 - \cos 2\omega t) \mathbf{\hat{B}} (\mathbf{\hat{B}} \cdot \mathbf{S}) - \sin 2\omega t \mathbf{\hat{B}} \times \mathbf{S}
\]
is the spin operator in the Heisenberg picture.

4. Let \( \mathbf{n} \) be the unit vector in the direction with polar coordinates \((\theta, \phi)\). Write down the matrix \( \mathbf{n} \cdot \mathbf{\sigma} \) and find its eigenvectors. Hence show that the state of a spin-$\frac{1}{2}$ particle in which a measurement of the component of spin along \( \mathbf{n} \) is certain to yield \( \hbar/2 \) is
\[
|\uparrow_{\mathbf{n}}\rangle = \sin \frac{\theta}{2} e^{i\phi/2} |\downarrow\rangle + \cos \frac{\theta}{2} e^{-i\phi/2} |\uparrow\rangle.
\]
where \( |\uparrow\rangle \) and \( |\downarrow\rangle \) are the usual eigenstates of \( S_z \). Obtain the corresponding expression for \( |\downarrow_{\mathbf{n}}\rangle \). Explain why \( \langle \uparrow | \uparrow_{\mathbf{n}} \rangle = 0 \) at \( \theta = \pi \), and why each of the coefficients in (\*) has modulus \( 1/\sqrt{2} \) when \( \theta = \pi/2 \).

5. Write down the \( 3 \times 3 \) matrix that represents \( S_x \) for a spin-1 system in the basis in which \( S_z = \text{diag}(h, 0, -h) \).

An unpolarized beam of spin-1 particles enters a Stern–Gerlach filter that passes only particles with \( S_z = h \). On exiting this filter, the beam enters a second filter that passes only particles with \( S_x = h \) and then finally it encounters a filter that passes only particles with \( S_z = -h \). What fraction of the initial particles make it right through?

6. i) Show that \( [L^2, X] = i\hbar (X \times L - L \times X) \) and that \( L \cdot X = 0 \), where \( L \) is the orbital angular momentum operator and \( L^2 \equiv L \cdot L \).

ii*) Use these results to show that
\[
[L^2, [L^2, X]] = 2\hbar^2 (L^2 X + X L^2).
\]
By considering the matrix elements of the above equation, show that
\[
((\beta - \beta')^2 - 2(\beta + \beta')) \langle \ell', m' | X | \ell, m \rangle = 0
\]
where \( \beta = \ell(\ell+1) \), \( \beta' = \ell'(\ell'+1) \) and \( |\ell, m\rangle, |\ell', m'\rangle \) are orbital angular momentum eigenstates.

iii) When a Hydrogen atom is immersed in a bath of radiation, transitions between the electron energy levels \( |n, \ell, m\rangle \) and \( |n', \ell', m'\rangle \) proceed at a rate proportional to
\[
|\langle n', \ell', m' | \varepsilon \cdot X | n, \ell, m \rangle|^2
\]
where \( \varepsilon \) is the polarization vector of the electromagnetic field. Show that this transition rate vanishes unless \( |\ell - \ell'| = 1 \) or \( \ell' = \ell = 0 \).

iv) By considering the parity operator, show that the transition rate also vanishes if \( \ell = \ell' = 0 \).

v*) By constructing appropriate commutators, show also that allowed transitions also have \( |m' - m| \leq 1 \).
7*. Let \( R = |X| \) and \( P_r = (\dot{X} \cdot P + P \cdot \dot{X})/2 \) be operators corresponding to the radial coordinate and momenta. Evaluate \([R, P_r]\).

The radial Hamiltonian of a Hydrogen atom in a state of definite total angular momentum labelled by \( \ell \) is

\[
H_\ell = \frac{P_r^2}{2\mu} + V_{\text{eff}}(R) = \frac{\hbar^2}{2\mu} \left( \frac{P_r^2}{\hbar^2} + \frac{\ell(\ell + 1)}{R^2} - \frac{2}{a_0 R} \right),
\]

where \( \mu \) is the reduced mass and \( a_0 = 4\pi\varepsilon_0\hbar^2/\mu e^2 \) is the Bohr radius.

i) Sketch the effective potential for various \( \ell \in \mathbb{N}_0 \). By considering their minima, show that for any fixed \( E < 0 \) there is a maximum value of \( \ell \) beyond which bound states of energy \( E \) cannot exist.

ii) Let \( A_\ell \) be the dimensionless operator

\[
A_\ell = \frac{a_0}{\sqrt{2}} \left( \frac{1}{(\ell + 1)a_0} - \frac{\ell + 1}{R} + \frac{i}{\hbar} P_r \right)
\]

Express \( H_\ell \) in terms of \( A_\ell \) and its Hermitian adjoint, and show that

\[
[A_\ell, A_\ell^\dagger] = \frac{a_0\mu}{\hbar^2} (H_{\ell+1} - H_\ell).
\]

Hence show that \( A_\ell H_\ell = H_{\ell+1}A_\ell \).

iii) Let \( |E,\ell\rangle \) obey \( H_\ell |E,\ell\rangle = E |E,\ell\rangle \). Show that \( A_\ell |E,\ell\rangle \) is an eigenstate of \( H_{\ell+1} \) with the same eigenvalue \( E \). What is the interpretation of the radial wavefunction \( \langle r | A_\ell | E,\ell \rangle \)?

iv) Explain why there must exist an \( \ell_{\text{max}} \in \mathbb{N}_0 \) s.t. \( A_{\ell_{\text{max}}} |E,\ell_{\text{max}}\rangle = 0 \). Deduce that \( |E,\ell_{\text{max}}\rangle \) has energy

\[
E = -\frac{\hbar^2}{2\mu a_0^2} \frac{1}{(\ell_{\text{max}} + 1)^2}
\]

and hence that the energy levels of Hydrogen are

\[
E = -\frac{\hbar^2}{2\mu a_0^2} \frac{1}{n^2}
\]

with orbital angular momentum \( \ell \in \{0, 1, \ldots, n-1\} \).