Principles of Quantum Mechanics - Problems 2

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk (∗) are optional and may be more difficult.

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1. Let $\mathbf{n}$ be the unit vector in the direction with polar coordinates (θ, ϕ) and let $\boldsymbol{\sigma}$ be the Pauli matrices. Find the eigenvectors of $\mathbf{n}\cdot\boldsymbol{\sigma}$. Hence show that the state of a spin-$\frac{1}{2}$ particle in which a measurement of the component of spin along $\mathbf{n}$ is certain to yield $\hbar/2$ is

$$|\uparrow_{\mathbf{n}}\rangle = \sin\frac{\theta}{2}\,\mathrm{e}^{\mathrm{i}\phi/2}\,|\downarrow\rangle + \cos\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\phi/2}\,|\uparrow\rangle\,. \qquad (\star)$$

where $|\uparrow\rangle$, $|\downarrow\rangle$ are the usual eigenstates of S_z. Obtain the corresponding expression for $|\downarrow_{\mathbf{n}}\rangle$. Explain why $\langle\uparrow|\uparrow_{\mathbf{n}}\rangle = 0$ at $\theta = \pi$, and why each of the coefficients in $(\star)$ has modulus $1/\sqrt{2}$ when $\theta = \pi/2$.

2. For a spin-$\frac{1}{2}$ particle, the spin operator $\mathbf{S} = \hbar\boldsymbol{\sigma}/2$, where $\boldsymbol{\sigma}$ is the vector $(\sigma_1, \sigma_2, \sigma_3)$ each entry of which is a Pauli matrix.

   i) Using the commutation and anti-commutation relations, but without multiplying matrices, explain why the Pauli matrices obey
   $$\sigma_i\sigma_j = \delta_{ij}\,1_{2\times 2} + \mathrm{i}\epsilon_{ijk}\sigma_k$$
   for $i, j, k \in \{1, 2, 3\}$. Hence show that $\mathbf{a}\cdot\boldsymbol{\sigma}\;\mathbf{b}\cdot\boldsymbol{\sigma} = \mathbf{a}\cdot\mathbf{b} + \mathrm{i}(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma}$ for any two vectors $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^3$.

   ii) Hence show that $\mathrm{e}^{-\mathrm{i}\boldsymbol{\alpha}\cdot\mathbf{S}/\hbar} = \cos\left(\frac{\alpha}{2}\right)\mathrm{id} - \mathrm{i}\sin\left(\frac{\alpha}{2}\right)\hat{\boldsymbol{\alpha}}\cdot\boldsymbol{\sigma}$ for a spin-$\frac{1}{2}$ particle.

   iii) A particle of spin-$\frac{1}{2}$ interacts with a uniform, homogeneous magnetic field $\mathbf{B}$ via the Hamiltonian $H = -\gamma\,\mathbf{B}\cdot\mathbf{S}$, where γ is a constant. If the particle's spin is initially prepared to be in some state $|\chi\rangle$, show that the probability its spin is found to be in an orthogonal state $|\chi'\rangle$ a time t later is $|\langle\chi'|\hat{\mathbf{B}}\cdot\boldsymbol{\sigma}|\chi\rangle|^2\,\sin^2\omega t$, where ω is a frequency you should specify.

   iv) Obtain the Heisenberg equation of motion for $\mathbf{S}(t)$ with this Hamiltonian.

   v) Hence show that the spin operator in the Heisenberg picture is
   $$\mathbf{S}(t) = \cos 2\omega t\,\mathbf{S} + (1 - \cos 2\omega t)\,\hat{\mathbf{B}}\,(\hat{\mathbf{B}}\cdot\mathbf{S}) - \sin 2\omega t\;\hat{\mathbf{B}}\times\mathbf{S}\,.$$

3. A spin-$\frac{1}{2}$ particle interacts with a time-varying magnetic field such that

$$H = -\gamma\mathbf{B}(t)\cdot\mathbf{S} \qquad \text{where} \qquad \mathbf{B}(t) = B_0\hat{\mathbf{z}} + b\,(\hat{\mathbf{x}}\cos\omega_1 t + \hat{\mathbf{y}}\sin\omega_1 t)$$

Let $|\psi(t)\rangle$ be the state of the particle at time t and let $U(\omega_1 t\,\hat{\mathbf{z}})$ be the rotation operator around the $\hat{\mathbf{z}}$-axis through the time dependent angle $\omega_1 t$.

   i) Define $|\chi(t)\rangle$ by $|\psi(t)\rangle = U(\omega_1 t\,\hat{\mathbf{z}})|\chi(t)\rangle$. Show that $|\chi(t)\rangle$ obeys the TDSE with Hamiltonian $H_{\text{eff}} = -\gamma\,\mathbf{B}_{\text{eff}}\cdot\mathbf{S}$ where $\mathbf{B}_{\text{eff}}$ is a time-independent magnetic field you should specify.

   ii) Hence show that $\langle\mathbf{S}\rangle_{\psi(t)} = \mathbf{R}(\omega_1 t\,\hat{\mathbf{z}})\,\mathbf{R}(-\gamma t\,\mathbf{B}_{\text{eff}})\,\langle\mathbf{S}\rangle_{\psi(0)}$, where $\mathbf{R}(\boldsymbol{\alpha})$ is a rotation matrix in $\mathbb{R}^3$

   iii) Draw a sketch to illustrate how $\langle\mathbf{S}\rangle_{\psi(t)}$ varies in time.

4. Write down the 3×3 matrix that represents S_x for a spin-1 system in the basis in which $S_z = \mathrm{diag}(\hbar, 0, -\hbar)$.

   An unpolarized beam of spin-1 particles enters a Stern–Gerlach filter that passes only particles with $S_z = \hbar$. On exiting this filter, the beam enters a second filter that passes only particles with $S_x = \hbar$ and then finally it encounters a filter that passes only particles with $S_z = -\hbar$. What fraction of the initial particles make it right through?

5. Let $R = |\mathbf{X}|$ and $P_r = (\hat{\mathbf{X}} \cdot \mathbf{P} + \mathbf{P} \cdot \hat{\mathbf{X}})/2$ be operators corresponding to the radial coordinate and momenta. Evaluate $[R, P_r]$.

   The radial Hamiltonian of a Hydrogen atom in a state of definite total angular momentum labelled by ℓ is

$$H_\ell = \frac{P_r^2}{2\mu} + V_{\text{eff}}(R) = \frac{\hbar^2}{2\mu}\left(\frac{P_r^2}{\hbar^2} + \frac{\ell(\ell+1)}{R^2} - \frac{2}{a_0 R}\right),$$

   where μ is the reduced mass and $a_0 = 4\pi\epsilon_0\hbar^2/\mu e^2$ is the Bohr radius.

   i) Sketch the effective potential for various $\ell \in \mathbb{N}_0$. By considering their minima, show that for any fixed $E < 0$ there is a maximum value of ℓ beyond which bound states of energy E cannot exist.

   ii) Let A_ℓ be the dimensionless operator

$$A_\ell = \frac{a_0}{\sqrt{2}}\left(\frac{1}{(\ell+1)a_0} - \frac{\ell+1}{R} + \frac{\mathrm{i}}{\hbar}P_r\right)$$

   Express H_ℓ in terms of A_ℓ and its Hermitian adjoint, and show that

$$[A_\ell, A_\ell^\dagger] = \frac{a_0^2\mu}{\hbar^2}(H_{\ell+1} - H_\ell)\,.$$

   Hence show that $A_\ell H_\ell = H_{\ell+1}A_\ell$.

   iii) Let $|E,\ell\rangle$ obey $H_\ell|E,\ell\rangle = E|E,\ell\rangle$. Show that $A_\ell|E,\ell\rangle$ is an eigenstate of $H_{\ell+1}$ with the same eigenvalue E. What is the interpretation of the radial wavefunction $\langle r|A_\ell|E,\ell\rangle$?

   iv) Explain why there must exist an $\ell_{\max} \in \mathbb{N}_0$ s.t. $A_{\ell_{\max}}|E,\ell_{\max}\rangle = 0$. Deduce that $|E,\ell_{\max}\rangle$ has energy

$$E = -\frac{\hbar^2}{2\mu a_0^2}\,\frac{1}{(\ell_{\max}+1)^2}$$

   and hence that the energy levels of Hydrogen are

$$E = -\frac{\hbar^2}{2\mu a_0^2}\frac{1}{n^2}$$

   with orbital angular momentum $\ell \in \{0, 1, \ldots, n-1\}$.

6*. Consider a $d = 2$ isotropic harmonic oscillator of frequency ω.

   i) Construct combinations of the raising and lowering operators of the $2d$ oscillator that obey the same $\mathfrak{so}(3)$ algebra as angular momentum in $3d$.

   ii) Show how all the states of the $2d$ oscillator fit into representations of $\mathfrak{so}(3)$.