1. Starting from the appropriate equation of motion for either operators or states, verify that
\[ i\hbar \frac{d}{dt} \langle \phi | A | \psi \rangle = \langle \phi | [A, H] | \psi \rangle \]
holds in both the Heisenberg picture and the Schrödinger picture.

2. (a) Find and solve the Heisenberg equations of motion for the position \( \hat{x}(t) \) and momentum \( \hat{p}(t) \) of the harmonic oscillator. Write the solution in two forms, one involving \( \hat{x}(0) \) and \( \hat{p}(0) \) and the other involving the corresponding annihilation and creation operators \( a \) and \( a^\dagger \) (also defined at \( t = 0 \)).
Show that for any complex number \( \alpha \) there is a normalized state \( | \psi_\alpha \rangle \) such that
\[ \langle \psi_\alpha | \hat{x}(t) | \psi_\alpha \rangle = \alpha e^{-i\omega t} + \alpha^* e^{i\omega t}, \]
the general solution for the classical oscillator. [Use the results of example 6 on sheet 1.]

(b) The Hamiltonian for a point particle of mass \( m \) and charge \( e \) in a constant electric field \( E \) is
\[ H(\hat{x}, \hat{p}) = \frac{1}{2m} \hat{p}^2 - e \mathbf{E} \cdot \mathbf{x}. \]
Solve the Heisenberg equations of motion for \( \hat{x}(t) \) and \( \hat{p}(t) \), expressing your answer in terms of \( \hat{x}(0) \) and \( \hat{p}(0) \). Verify that \( H(\hat{x}(t), \hat{p}(t)) = H(\hat{x}(0), \hat{p}(0)) \).

3. Define annihilation, creation operators \( a_i, a_i^\dagger \) for the three-dimensional harmonic oscillator with Hamiltonian
\[ H = H_1 + H_2 + H_3 \quad \text{where} \quad H_i = \frac{\hat{p}_i^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}_i^2 \quad (i = 1, 2, 3). \]
Use these to find the energy levels and the space of states on which \( H \) acts. Assuming no internal structure, show that a complete set of commuting operators can be taken to be \( \{a_1^\dagger a_1, a_2^\dagger a_2, a_3^\dagger a_3\} \). Show that the degeneracy of the \( n \)-th excited state is \( \frac{1}{2}(n + 1)(n + 2) \).

4. Two quantum systems have spaces of states \( V \) and \( V' \). Each space is two-dimensional with an orthonormal basis for \( V \) given by \( \{|1\}, |2\rangle \) and for \( V' \) by \( \{|1\}', |2\rangle' \}. An operator \( A \) is defined on \( V \) by \( A|1\rangle = |2\rangle, A|2\rangle = |1\rangle \) and a similar operator \( A' \) is defined on \( V' \) by \( A'|1\rangle' = |2\rangle', A'|2\rangle' = |1\rangle' \). The identity operators on \( V \) and \( V' \) are \( I \) and \( I' \) respectively.

(a) Consider the combined system whose space of states is represented by the tensor product \( V \otimes V' \). With respect to a convenient basis, determine the matrix forms of the operators
\[ S = I \otimes I' + \lambda A \otimes A', \quad T = A \otimes I' + \mu I \otimes A', \]
where \( \lambda, \mu \) are real numbers. Find the eigenvalues of \( S \).
(b) Show that \( T^2 \propto S \) if \( \lambda \) and \( \mu \) are suitably related. Hence, or otherwise, find the eigenvalues of \( T \).

(c) Now suppose that \( V \) and \( V' \) are spaces of states for two indistinguishable fermions. What are the implications for the eigenvalues of \( S \)?

5. (a) The trace of an operator \( A \) acting on a vector space \( V \) is defined as \( \text{Tr}_V(A) = \sum_n \langle n|A|n \rangle \) where \( \{|n\} \) is any orthonormal basis for \( V \). If \( a, a^\dagger \) are annihilation, creation operators and \( V \) is the usual space of states for an oscillator, show that \( \text{Tr}(xx) = (1 - x)^{-1} \) for \( |x| < 1 \).

(b) Suppose \( V = V_1 \otimes V_2 \) and \( A = A_1 \otimes A_2 \) where \( A_1, A_2 \) act on \( V_1, V_2 \) respectively. Show that \( \text{Tr}_V(A) = \text{Tr}_{V_1}(A_1)\text{Tr}_{V_2}(A_2) \).

*(c) Let \( a_i, a_i^\dagger \) be a set of \( D \) annihilation, creation operators obeying \([a_i, a_j^\dagger] = \delta_{ij}, \ [a_i, a_j] = 0, \ [a_i^\dagger, a_j^\dagger] = 0\). Show that

\[
\text{Tr}(x^N) = (1 - x)^{-D} \quad \text{where} \quad N = \sum_{i=1}^{D} a_i^\dagger a_i .
\]

Find the degeneracy \( d_n \) of the eigenvalue \( n \) of the operator \( N \), by first explaining why

\[
\text{Tr}(x^N) = \sum_{n \geq 0} d_n x^n .
\]

6. A particle of mass \( m \) is confined to a three-dimensional box with sides \( a < b < c \) (the wavefunction for the particle vanishes on the boundary of the box). Show that, with a suitable choice of coordinates, the allowed eigenfunctions and energy levels are

\[
\psi_{npq}(x, y, z) = \left( \frac{8}{abc} \right)^{\frac{1}{2}} \sin \frac{n\pi x}{a} \sin \frac{p\pi y}{b} \sin \frac{q\pi z}{c}, \quad E_{npq} = \frac{\hbar^2 \pi^2}{2m} \left( \frac{n^2}{a^2} + \frac{p^2}{b^2} + \frac{q^2}{c^2} \right);
\]

where \( n, p, q = 1, 2, \ldots \).

What are the two lowest energy levels, their degeneracies and associated wavefunctions, for two identical non-interacting spin-0 particles confined to the box? What are the corresponding results for two non-interacting, identical, spin-\( \frac{1}{2} \) particles?

7. A harmonic oscillator of mass \( m \), frequency \( \omega \) and charge \( e \) is perturbed by a constant electric field of strength \( E \), resulting in a new term \( \mathcal{H}' = -eE \hat{x} \) in the Hamiltonian. Calculate the change in the energy levels to order \( E^2 \) and compare with the exact result.

8. The quantum-mechanical observable \( Q \) has just three eigenstates, \(|1\rangle, \ |2\rangle, \ |3\rangle \) which are orthogonal and correspond to eigenvalues 2, 1, 1 respectively. The operator \( Q' \) is defined by \( Q' = Q + \epsilon T \), where

\[
\langle j | T | i \rangle = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & \sqrt{6} \\ 0 & \sqrt{6} & 1 \end{pmatrix} .
\]

Deriving the necessary formulae from first principles, apply perturbation theory to find two of the eigenvalues of \( Q' \) correct to order \( \epsilon \) and the third correct to order \( \epsilon^2 \).
9. Two harmonic oscillators of frequency $\omega$ and mass $m$ interact through a potential $m\omega^2 \lambda x_1 x_2$ where $x_1$ and $x_2$ are the displacements of the oscillators. Use perturbation theory to calculate the ground state energy of the system correct to second order in powers of $\lambda$ by adding the interaction potential on to the Hamiltonian. Find the energies of the first two excited states, working to lowest non-trivial order in perturbation theory.

Verify your results by using the change of variables $\sqrt{2} q_1 = x_1 + x_2$, $\sqrt{2} q_2 = x_1 - x_2$ to obtain an exact expression for the energy eigenvalues.

10. The first excited energy level of the hydrogen atom is four-fold degenerate (ignoring electron spin). The four wave functions may be taken to be

$$\psi_0(x) = \frac{1}{(8\pi a^3)^{1/2}} \left(1 - \frac{r}{2a}\right) e^{-r/2a}, \quad \psi_i(x) = \frac{1}{(32\pi a^5)^{1/2}} x_i e^{-r/2a}, \quad a = \frac{4\pi \varepsilon_0 \hbar^2}{e^2 m},$$

where $x_i$ are the Cartesian components of the vector $x$. The atom is perturbed by a weak electric field $\mathcal{E}$ parallel to the 3-axis. Show that the only non-zero matrix elements of the perturbation $H'$ are $\langle \psi_0 | H' | \psi_3 \rangle$ and $\langle \psi_3 | H' | \psi_0 \rangle$. Calculate the new energy levels to first order in $\mathcal{E}$. 