Principles of Quantum Mechanics - Problems 2

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk (*) are optional and may be more difficult.

1. Let \( \mathbf{n} \) be the unit vector in the direction with polar coordinates \((\theta, \phi)\) and let \( \bm{\sigma} \) be the Pauli matrices. Find the eigenvectors of \( \mathbf{n} \cdot \bm{\sigma} \). Hence show that the state of a spin-\( \frac{1}{2} \) particle in which a measurement of the component of spin along \( \mathbf{n} \) is certain to yield \( \hbar / 2 \) is

\[
|\uparrow_{\mathbf{n}}\rangle = \sin \frac{\theta}{2} e^{i\phi/2} |\downarrow\rangle + \cos \frac{\theta}{2} e^{-i\phi/2} |\uparrow\rangle ,
\]

where \( |\uparrow\rangle , |\downarrow\rangle \) are the usual eigenstates of \( S_z \). Obtain the corresponding expression for \( |\downarrow_{\mathbf{n}}\rangle \). Explain why \( \langle \uparrow | \uparrow_{\mathbf{n}} \rangle = 0 \) at \( \theta = \pi \), and why each of the coefficients in (\( \star \)) has modulus \( 1/\sqrt{2} \) when \( \theta = \pi/2 \).

2. For a spin-\( \frac{1}{2} \) particle, the spin operator \( \bm{S} = \hbar \bm{\sigma}/2 \), where \( \bm{\sigma} \) is the vector \((\sigma_1, \sigma_2, \sigma_3)\) each entry of which is a Pauli matrix.

i) Using the commutation and anti-commutation relations, but without multiplying matrices, explain why the Pauli matrices obey

\[
\sigma_i \sigma_j = \delta_{ij} 1_{2 \times 2} + i \epsilon_{ijk} \sigma_k
\]

for \( i, j, k \in \{1, 2, 3\} \). Hence show that \( \mathbf{a} \cdot \bm{\sigma} \mathbf{b} \cdot \bm{\sigma} = \mathbf{a} \cdot \mathbf{b} + i(\mathbf{a} \times \mathbf{b}) \cdot \bm{\sigma} \) for any two vectors \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \).

ii) Hence show that \( e^{-i\mathbf{a} \cdot \bm{\sigma} / \hbar} = \cos \left( \frac{\alpha}{2} \right) \mathbf{id} - i \sin \left( \frac{\alpha}{2} \right) \hat{\mathbf{a}} \cdot \bm{\sigma} \) for a spin-\( \frac{1}{2} \) particle.

iii) A particle of spin-\( \frac{1}{2} \) interacts with a uniform, homogeneous magnetic field \( \mathbf{B} \) via the Hamiltonian

\[
H = -\gamma \mathbf{B} \cdot \mathbf{S},
\]

where \( \gamma \) is a constant. If the particle’s spin is initially prepared to be in some state \( |\chi\rangle \), show that the probability its spin is found to be in an orthogonal state \( |\chi'\rangle \) a time \( t \) later is

\[
|\langle \chi' | \hat{\mathbf{B}} \cdot \bm{\sigma} |\chi\rangle|^2 \sin^2 \omega t ,
\]

where \( \omega \) is a frequency you should specify.

iv) Obtain the Heisenberg equation of motion for \( \mathbf{S}(t) \) with this Hamiltonian.

v) Hence show that the spin operator in the Heisenberg picture is

\[
\mathbf{S}(t) = \cos 2\omega t \mathbf{S} + (1 - \cos 2\omega t) \hat{\mathbf{B}} (\mathbf{B} \cdot \mathbf{S}) - \sin 2\omega t \hat{\mathbf{B}} \times \mathbf{S} .
\]

3. A spin-\( \frac{1}{2} \) particle interacts with a time-varying magnetic field such that

\[
H = -\gamma \mathbf{B}(t) \cdot \mathbf{S} \quad \text{where} \quad \mathbf{B}(t) = B_0 \hat{\mathbf{z}} + b (\hat{x} \cos \omega_1 t + \hat{y} \sin \omega_1 t)
\]

Let \( |\psi(t)\rangle \) be the state of the particle at time \( t \) and let \( U(\omega_1 t \hat{\mathbf{z}}) \) be the rotation operator around the \( \hat{\mathbf{z}} \)-axis through the time dependent angle \( \omega_1 t \).

i) Define \( |\chi(t)\rangle \) by \( |\psi(t)\rangle = U(\omega_1 t \hat{\mathbf{z}}) |\chi(t)\rangle \). Show that \( |\chi(t)\rangle \) obeys the TDSE with Hamiltonian

\[
H_{\text{eff}} = -\gamma \mathbf{B}_{\text{eff}} \cdot \mathbf{S}
\]

where \( \mathbf{B}_{\text{eff}} \) is a time-independent magnetic field you should specify.

ii) Hence show that \( \langle \mathbf{S} \rangle_{\psi(t)} = \mathbf{R}(\omega_1 t \hat{\mathbf{z}}) \mathbf{R}(\gamma t \mathbf{B}_{\text{eff}}) (\mathbf{S})_{\psi(0)} \), where \( \mathbf{R}(\alpha) \) is a rotation matrix in \( \mathbb{R}^3 \).

iii) Draw a sketch to illustrate how \( \langle \mathbf{S} \rangle_{\psi(t)} \) varies in time.
4. Write down the $3 \times 3$ matrix that represents $S_x$ for a spin-1 system in the basis in which $S_z = \text{diag}(h, 0, -h)$.

An unpolarized beam of spin-1 particles enters a Stern–Gerlach filter that passes only particles with $S_z = h$. On exiting this filter, the beam enters a second filter that passes only particles with $S_x = h$ and then finally it encounters a filter that passes only particles with $S_z = -h$. What fraction of the initial particles make it right through?

5. Let $R = \lvert X \rangle$ and $P_r = (\hat{X} \cdot \hat{P} + \hat{P} \cdot \hat{X})/2$ be operators corresponding to the radial coordinate and momenta. Evaluate $[R, P_r]$.

The radial Hamiltonian of a Hydrogen atom in a state of definite total angular momentum labelled by $\ell$ is

$$H_\ell = \frac{P_r^2}{2\mu} + V_{\text{eff}}(R) = \frac{\hbar^2}{2\mu} \left( \frac{P_r^2}{\hbar^2} + \frac{\ell(\ell + 1)}{R^2} - \frac{2}{a_0 R} \right),$$

where $\mu$ is the reduced mass and $a_0 = 4\pi\varepsilon_0 \hbar^2/\mu e^2$ is the Bohr radius.

i) Sketch the effective potential for various $\ell \in \mathbb{N}_0$. By considering their minima, show that for any fixed $E < 0$ there is a maximum value of $\ell$ beyond which bound states of energy $E$ cannot exist.

ii) Let $A_\ell$ be the dimensionless operator

$$A_\ell = \frac{a_0}{\sqrt{2}} \left( \frac{1}{(\ell + 1)a_0} - \frac{\ell + 1}{R} + \frac{i}{\hbar} P_r \right).$$

Express $H_\ell$ in terms of $A_\ell$ and its Hermitian adjoint, and show that

$$[A_\ell, A_\ell^\dagger] = \frac{a_0^2 \mu}{\hbar^2} (H_{\ell+1} - H_\ell).$$

Hence show that $A_\ell H_\ell = H_{\ell+1} A_\ell$.

iii) Let $\lvert E, \ell \rangle$ obey $H_\ell \lvert E, \ell \rangle = E \lvert E, \ell \rangle$. Show that $A_\ell \lvert E, \ell \rangle$ is an eigenstate of $H_{\ell+1}$ with the same eigenvalue $E$. What is the interpretation of the radial wavefunction $\langle r | A_\ell | E, \ell \rangle$?

iv) Explain why there must exist an $\ell_{\text{max}} \in \mathbb{N}_0$ s.t. $A_{\ell_{\text{max}}} \lvert E, \ell_{\text{max}} \rangle = 0$. Deduce that $\lvert E, \ell_{\text{max}} \rangle$ has energy

$$E = -\frac{\hbar^2}{2\mu a_0^2} \frac{1}{(\ell_{\text{max}} + 1)^2}$$

and hence that the energy levels of Hydrogen are

$$E = -\frac{\hbar^2}{2\mu a_0^2} \frac{1}{n^2}$$

with orbital angular momentum $\ell \in \{0, 1, \ldots, n - 1\}$.

6*. Consider a $d = 2$ isotropic harmonic oscillator of frequency $\omega$.

i) Construct combinations of the raising and lowering operators of the $2d$ oscillator that obey the same $\mathfrak{so}(3)$ algebra as angular momentum in $3d$.

ii) Show how all the states of the $2d$ oscillator fit into representations of $\mathfrak{so}(3)$. 