Principles of Quantum Mechanics - Problems 2

Please email me with any comments about these problems, particularly if you spot an error.

1. Let \( n \) be the unit vector in the direction with polar coordinates \((\theta, \phi)\). Write down the matrix \( n \cdot \sigma \) and find its eigenvectors. Hence show that the state of a spin-\( \frac{1}{2} \) particle in which a measurement of the component of spin along \( n \) is certain to yield \( \hbar/2 \) is

\[
|\uparrow_n\rangle = \sin \frac{\theta}{2} e^{i\phi/2} |\downarrow\rangle + \cos \frac{\theta}{2} e^{-i\phi/2} |\uparrow\rangle.
\]

where \(|\uparrow\rangle, |\downarrow\rangle\) are the usual eigenstates of \( S_z \). Obtain the corresponding expression for \(|\downarrow_n\rangle\). Explain why each of the coefficients in (⋆) has modulus \(1/\sqrt{2} \) when \( \theta = \pi/2 \), and why \( \langle \uparrow | \uparrow_n \rangle = 0 \) at \( \theta = \pi \).

2. Write down the 3 \( \times \) 3 matrix that represents \( S_x \) for a spin-1 system in the basis in which \( S_z = \text{diag}(\hbar, 0, -\hbar) \).

An unpolarized beam of spin-1 particles enters a Stern–Gerlach filter that passes only particles with \( S_z = \hbar \). On exiting this filter, the beam enters a second filter that passes only particles with \( S_x = \hbar \) and then finally it encounters a filter that passes only particles with \( S_z = -\hbar \). What fraction of the initial particles make it right through?

3. Consider a \( d = 1 \) harmonic oscillator of mass \( m \) and frequency \( \omega \), with raising and lowering operators \( A^\dagger \) and \( A \). Show that the translation operator \( U(a) \) may be written as

\[
U(a) = e^{-\frac{1}{2} \gamma^2} e^{\gamma A^\dagger} e^{-\gamma A} \quad \text{where} \quad \gamma = \sqrt{a} \frac{\sqrt{m} \omega}{2\hbar}.
\]

Deduce that if \( \psi_n(x) \) are the normalised position space wavefunctions for states with energies \( \hbar \omega(n + \frac{1}{2}) \), then

\[
\psi_0(x - a) = e^{-\frac{1}{2} \gamma^2} \sum_{n=0}^{\infty} \frac{(-i\gamma)^n}{\sqrt{n!}} \psi_n(x).
\]

[Recall that \([A, e^B] = [A, B] e^B \) and \( e^A e^B = e^{A+B} e^{1/2[A,B]} \) provided \([A, B] \) commutes with \( A \) and \( B \).]

4. Show that \([L^2, X] = i\hbar (X \times L - L \times X) \) and that \( L \cdot X = 0 \), where \( L^2 \equiv L \cdot L \) is the total orbital angular momentum operator. Use these results to show that

\[
[L^2, [L^2, X]] = 2\hbar^2 (L^2 X + X L^2).
\]

By considering the matrix elements of this equation, show that

\[
((\beta - \beta')^2 - 2(\beta + \beta')) \langle \ell', m'|X|\ell, m \rangle = 0
\]
where \( \beta = \ell(\ell + 1) \), \( \beta' = \ell'(\ell' + 1) \) and \( |\ell, m\rangle \), \( |\ell', m'\rangle \) are orbital angular momentum eigenstates.

When hydrogen is immersed in a bath of radiation, transitions between the electron energy levels \( |n, \ell, m\rangle \) and \( |n', \ell', m'\rangle \) proceed at a rate proportional to \( |\langle n', \ell', m'|X|n, \ell, m\rangle|^2 \). Show that this transition rate vanishes unless \( |\ell - \ell'| = 1 \) or \( \ell' = \ell = 0 \). By considering the parity operator, show that the transition rate also vanishes if \( \ell = \ell' = 0 \). By constructing an appropriate commutator, show also that allowed transitions also have \( |m' - m| \leq 1 \). [This is a special case of the Wigner–Eckart theorem.]

5. Consider a free particle with mass \( m \). Show that energy eigenstates with definite total angular momentum obey \( H\ell|E, \ell\rangle = E|E, \ell\rangle \) where

\[
H\ell = \frac{1}{2m} \left( \mathbf{p}^2 + \frac{\ell(\ell + 1)\hbar^2}{|\mathbf{X}|^2} \right)
\]

and \( \mathbf{p}_r := (\mathbf{X} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{X})/2 \). Now let

\[
A\ell = \frac{1}{\sqrt{2m}} \left( i\mathbf{p}_r - \frac{(\ell + 1)\hbar}{|\mathbf{X}|} \right)
\]

Find \([A\ell, A^\dagger\ell]\) in terms of the \( H\ell \). What is the state \( A\ell|E, \ell\rangle \)? Show that for \( E > 0 \) there is no upper bound on \( \ell \) and interpret this result physically.

6. Consider the isotropic harmonic oscillator in three dimensions, with potential \( V(r) = V_0 + \frac{1}{2}m\omega^2r^2 \). Write down the allowed energy eigenvalues. What is the degeneracy of the \( n \)th level?

In a simple model of the atomic nucleus, each nucleon (proton or neutron, each with spin-\( \frac{1}{2} \)) moves in the harmonic potential \( V(r) \) above, interpreted as being created by the other nuclei. Explain why the nuclear isotopes \(^4\)He, \(^{16}\)O and \(^{40}\)Ca are especially stable. [Helium, oxygen and calcium have atomic numbers \( Z = 2, 8 \) and 20, respectively.]

7. A Fermi oscillator has Hamiltonian \( H = B^\dagger B \) where

\[
B^\dagger B + BB^\dagger = 1 \quad \text{(the anticommutator)}
\]

and \( B^2 = 0 \). Find the eigenvalues of \( H \). If \( |0\rangle \) is a state obeying \( H|0\rangle = 0 \) and \( \langle 0|0\rangle = 1 \), find \( B|0\rangle \) and \( B^\dagger|0\rangle \). Explain the connection between the spectrum of \( B^\dagger B \) and the Pauli exclusion principle.