Principles of Quantum Mechanics - Problems 3

Please email me with any comments about these problems, particularly if you spot an error.

1. States \(|s, \sigma\rangle\) are formed by combining the states \(|1, \sigma_1\rangle, |1, \sigma_2\rangle\) of two subsystems, each having spin \(s = 1\). Show that the states of the combined system with \(s = 2, 0\) are symmetric under the interchange \(\sigma_1 \leftrightarrow \sigma_2\), whereas those with \(s = 1\) are antisymmetric.

Two identical spin 1 particles, whose centre of mass is at rest, have combined spin \(S\), relative orbital angular momentum \(L\) and total angular momentum \(J = L + S\). Let \((j, \ell, s)\) denote the quantum numbers corresponding to the operators \((J^2, L^2, S^2)\) of the combined system. Show that \(\ell + s\) must be even. If \(j = 1\), what are the possible values of \(\ell\) and \(s\)?

2. Consider the addition of angular momentum 3 to angular momentum 1. Express the states \(|j, m\rangle\in\{|4, 4\rangle, |4, 3\rangle, |3, 3\rangle, |3, 2\rangle, |2, 2\rangle, |2, 1\rangle\}\) in terms of the states of the subsystems.

In the next two questions, you should assume that total angular momentum and parity are each conserved in the decay process.

3. Show that a particle of spin 1 cannot decay into two identical particles of spin 0. The \(\rho\)-meson has spin-1 and can decay into two spinless \(\pi\)-mesons (pions) with different electric charges. If the intrinsic parity of any pion is negative, what is the intrinsic parity of the \(\rho\)-meson?

4. A particle \(X\) is observed to undergo the decays \(X \to \rho^+ + \pi^+\) and \(X \to K + K\), where the kaon \(K\) has spin-0. What is the lowest value of the spin of \(X\) that is consistent with this, and what is the corresponding intrinsic parity of \(X\)?

5. The interaction between neighbouring spin-\(\frac{1}{2}\) atoms in a certain crystal is described by the Hamiltonian

\[
H = K \left( \frac{\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}}{|\mathbf{a}|} - 3 \frac{(\mathbf{a} \cdot \mathbf{S}^{(1)})(\mathbf{a} \cdot \mathbf{S}^{(2)})}{|\mathbf{a}|^3} \right),
\]

where \(\mathbf{a}\) is the separation between the atoms, \(K\) is a constant and \(\mathbf{S}^{(1)}\) is the first atom’s spin operator. Explain the physical idea underlying this form of \(H\).

Let \(\mathbf{S} = \mathbf{S}^{(1)} + \mathbf{S}^{(2)}\) be the total spin operator. Show that we can find a complete set of mutual eigenstates of \(\mathbf{S} \cdot \mathbf{S}, \mathbf{a} \cdot \mathbf{S}\) and \(H\). By showing that

\[
S_x^{(1)} S_x^{(2)} + S_y^{(1)} S_y^{(2)} = \frac{1}{2} \left( S_+^{(1)} S_-^{(2)} + S_-^{(1)} S_+^{(2)} \right),
\]

find the energy eigenvalues. [Make an appropriate choice for the direction of \(\hat{\mathbf{a}}\).]

At time \(t = 0\), particle 1 has its spin parallel to \(\mathbf{a}\), while the other particle’s spin is antiparallel to \(\mathbf{a}\). Find the time required for both spins to reverse their directions.
6. A $d = 1$ harmonic oscillator of mass $m$ and frequency $\omega$ is perturbed by $\Delta = \lambda X^4$. By expressing the position operator $X$ in terms of the raising and lowering operators $A^\dagger$ & $A$, show that the first order change in energy of the $n^{th}$ excited state is

$$\delta E_n = 3\lambda (2n^2 + 2n + 1) \left( \frac{\hbar}{2m\omega} \right)^2.$$

What is the radius of convergence in $\lambda$ of perturbation theory here?

7. A particle of mass $m$ is confined by the infinite square well potential

$$V(x) = \begin{cases} 0 & \text{when } |x| < a \\ \infty & \text{when } |x| > a. \end{cases}$$

Show that, to first order in $\lambda$, the energy levels of this particle are unchanged when the potential is perturbed by $\delta V(x) = \lambda x/a$. Show that the ground–state wavefunction is changed, becoming

$$\psi_\lambda(x) = \frac{1}{\sqrt{a}} \cos(\pi x/2a) + \frac{16\lambda}{\pi^2 E_1 \sqrt{a}} \sum_{n=2,4,...} \left( -1 \right)^{n/2} \frac{n}{(n^2 - 1)^3} \sin(n\pi x/2a) + O(\lambda^2)$$

where $E_1$ is the ground state energy. Explain why this wavefunction does not have well–defined parity. Will the second order correction to the ground–state energy be positive or negative? [Detailed calculation is not required.]

Suppose now that the initial well is only finitely deep. Will perturbation theory converge? Comment on the relevance of this to the Stark effect in hydrogen.

8. An atomic nucleus has finite size and inside the electrostatic potential $\Phi(r)$ deviates from $Ze/r$. Take the proton’s radius to be $a_p \approx 10^{-15}$ m and its charge density to be uniform. Treating the difference between $\Phi(r)$ and $Ze/r$ as a perturbation of the gross structure Hamiltonian of the hydrogen, calculate the first order change in the energy of the ground state of hydrogen. [Use the fact that $a_p \ll a_0$ (the Bohr radius) to approximate the integral you encounter.]

Why is the change in energy of any state with $\ell \geq 1$ extremely small?

9. An isotropic harmonic oscillator in $d = 2$ has mass $m$ and frequency $\omega$. Write down a basis of its energy eigenstates. What is the energy and degeneracy of the $n^{th}$ excited state?

This oscillator is perturbed by small potential $\delta V = \lambda xy$. Show that this perturbation lifts the degeneracy of the first excited state, producing states with energies $2\hbar \omega \pm \lambda \hbar/2m\omega$. What are the corresponding eigenstates? Which states remain degenerate in the presence of this perturbation? What is the radius of convergence (in $\lambda$) of this perturbation series?