Principles of Quantum Mechanics - Problems 3

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk ($*$) may be more difficult.

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1. Consider the addition of angular momentum 3 to angular momentum 1. Express the states

$$|4,4\rangle,\ |4,3\rangle,\ |3,3\rangle,\ |3,2\rangle,\ |2,2\rangle \quad \text{and} \quad |2,1\rangle$$

of the combined system in terms of the states of the subsystems.

2. States $|s,\sigma\rangle$ are formed by combining the states $|1,\sigma_1\rangle$, $|1,\sigma_2\rangle$ of two subsystems, each of spin 1.

   i) Show that the states of the combined system with $s = 2$ or 0 are symmetric under the interchange $\sigma_1 \leftrightarrow \sigma_2$, whereas those with $s = 1$ are antisymmetric.

   ii) Two identical spin 1 particles, whose centre of mass is at rest, have combined spin $\mathbf{S}$, relative orbital angular momentum $\mathbf{L}$ and total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$. Let (j, ℓ, s) denote the quantum numbers corresponding to the operators $(\mathbf{J}^2, \mathbf{L}^2, \mathbf{S}^2)$ of the combined system. Show that $\ell + s$ must be even. If $j = 1$, what are the possible values of ℓ and s?

3. Three spin 1 particles are governed by the Hamiltonian $H = \left(2\lambda/\hbar^2\right)(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_2 \cdot \mathbf{S}_3 + \mathbf{S}_3 \cdot \mathbf{S}_1)$, where $\mathbf{S}_i$ is the spin operator of the i^{th} particle and λ is a positive constant. Find the eigenvalues and corresponding degeneracies of the Hamiltonian in the cases that

   i) each particle is distinguishable,

   ii) all three particles are identical and all have the same spatial wavefunction.

4. Consider the isotropic harmonic oscillator in three dimensions, with potential $V(\mathbf{X}) = V_0 + \frac{1}{2}m\omega^2\mathbf{X}^2$.

   i) Write down the allowed energy eigenvalues. What is the degeneracy of the n^{th} level?

   ii*) In a simple model of the atomic nucleus, each nucleon (proton or neutron, each with spin-$\frac{1}{2}$) moves in the harmonic potential $V(\mathbf{X})$ above, interpreted as being created by the other nucleons. Suggest why the nuclear isotopes ^{4}He, ^{16}O and ^{40}Ca are especially stable. [*Helium, oxygen and calcium nuclei contain 2, 8 and 20 protons, respectively. All protons are identical.*]

In the next two questions, you should assume that both total angular momentum and parity are conserved in each decay process.

4. Show that a particle of spin 1 cannot decay into two identical particles of spin 0. The ρ-meson has spin-1 and can decay into two spinless π-mesons (pions) with different electric charges. If the intrinsic parity of any pion is negative, what is the intrinsic parity of the ρ-meson?

5. A particle X is observed to undergo the decays $X \to \rho^+ + \pi^+$ and $X \to K + K$, where the kaon K has spin-0. What is the lowest value of the spin of X that is consistent with this, and what is the corresponding intrinsic parity of X?

6. A $d = 1$ harmonic oscillator of mass m and frequency ω is perturbed by $\Delta = \lambda \mathrm{X}^4$. Show that the first order change in energy of the n^{th} excited state is

$$\delta E_n = 3\lambda\,(2n^2 + 2n + 1)\left(\frac{\hbar}{2m\omega}\right)^2 .$$

What is the radius of convergence (in λ) of the perturbation series?

7. A particle of mass m is confined by the infinite square well potential

$$V(x) = \begin{cases} 0 & \text{when } |x| < a \\ \infty & \text{when } |x| > a\,. \end{cases}$$

The potential is perturbed by $\delta V(x) = \lambda x/a$.

i) Show that the energy levels of this particle are unchanged to first order in λ.

ii) Show that the ground state wavefunction *is* changed, becoming

$$\psi_\lambda(x) = \frac{1}{\sqrt{a}}\cos(\pi x/2a) + \frac{16\lambda}{\pi^2 E_1\sqrt{a}}\sum_{n=2,4,\ldots}(-1)^{n/2}\frac{n}{(n^2-1)^3}\sin(n\pi x/2a) + O(\lambda^2)$$

where E_1 is the ground state energy. Explain why this wavefunction does not have well–defined parity.

iii) Write down a formula giving the second order correction to the energy of the ground state energy. Will this second order correction be positive or negative? [*Detailed calculation is not required.*]

iv*) Suppose now that the initial well is only finitely deep. Will perturbation theory converge? Comment on the relevance of this to the Stark effect in hydrogen.

8. An atomic nucleus has finite size and inside the electrostatic potential $\Phi(r)$ deviates from Ze/r. Take the proton's radius to be $a_{\mathrm{p}} \approx 10^{-15}\,$m and its charge density to be uniformly distributed over its surface. Treating the difference between $\Phi(r)$ and Ze/r as a perturbation of the Hamiltonian for the gross structure of Hydrogen, calculate the first order change in the energy of the ground state of hydrogen. [*Use the fact that $a_{\mathrm{p}} \ll a_0$ (the Bohr radius) to approximate the integral you encounter.*]

Why is the change in energy of any state with $\ell \geq 1$ extremely small?

9. An isotropic harmonic oscillator in $d = 2$ has mass m and frequency ω.

i) Write down a basis of its energy eigenstates.

ii) What is the energy and degeneracy of the n^{th} excited state?

iii) The oscillator is perturbed by a small potential $\delta V = \lambda xy$. Show that this perturbation lifts the degeneracy of the first excited state, producing states with energies $2\hbar\omega \pm \lambda\hbar/2m\omega$. What are the corresponding eigenstates?

iv) Which (if any) states remain degenerate in the presence of this perturbation?

v*) What is the radius of convergence (in λ) of this perturbation series?