1. (a) Use the angular momentum commutation relations to verify that
\[
[ J_3, J_1^2 ] = -[ J_3, J_2^2 ] = i\hbar(J_1J_2 + J_2J_1) \quad \text{and} \quad [ J_3, J_1J_2 + J_2J_1 ] = 2i\hbar(J_2^2 - J_1^2) .
\]
(b) Show that if \( |\psi\rangle = |j m\rangle \), a simultaneous eigenstate of \( J^2 \) and \( J_3 \), then \( \langle A \rangle_\psi = \langle \psi | A | \psi \rangle = 0 \) for any operator of the form \( A = [B, J_3] \). Deduce that
\[
\langle J_1 \rangle_\psi = \langle J_2 \rangle_\psi = 0 \quad \text{and} \quad \langle J_1^2 \rangle_\psi = \langle J_2^2 \rangle_\psi = \frac{1}{2} \hbar^2 (j(j+1) - m^2) .
\]
(c) Re-derive the expectation values in part (b) by writing \( J_1 \) and \( J_2 \) in terms of \( J_\pm \).

2. A quantum system is in an angular momentum state \( |\psi\rangle = |j m\rangle \), as in example 1, when a measurement is made of the angular momentum component \( K = n \cdot J \), where the unit vector \( n \) lies in the \( x\)-\( z \) plane and makes an angle \( \theta \) with the \( z \)-axis.

For \( j = \frac{1}{2} \), find the probabilities that the measurement gives the results \( \pm \frac{1}{2} \hbar \) by first evaluating \( \langle K \rangle_\psi \). Show that in this case each probability is \( \cos^2 \frac{1}{2} \theta \) or \( \sin^2 \frac{1}{2} \theta \), depending on the value of \( m \).

For \( j = 1 \), consider the expectation values of \( K \) and \( K^2 \) and hence determine the probabilities for the measurement to yield each of the results \( \hbar, 0, -\hbar \).

3. A particle of spin \( \frac{1}{2} \) is at rest in a magnetic field \( B \) parallel to the \( z \)-axis. A small additional magnetic field \( B' \) is then switched on parallel to the \( x \)-axis, so that the Hamiltonian becomes
\[
H = -\frac{\hbar}{2} \gamma (B \sigma_3 + B' \sigma_1)
\]
(where \( \sigma_i \) are the Pauli matrices). Starting from the energy levels and eigenstates when \( B' = 0 \), use perturbation theory to calculate the corrections to the energies to order \( B'^2 \) and compare with the exact answer.

4. Show, without using explicit forms for the Pauli matrices, that the relations
\[
[\sigma_i, \sigma_j] = 2i \epsilon_{ijk} \sigma_k \quad \text{and} \quad (\mathbf{n} \cdot \sigma)^2 = 1 ,
\]
where \( \mathbf{n} \) is any unit vector, imply
\[
\mathbf{a} \cdot \sigma \mathbf{b} \cdot \sigma = \mathbf{a} \cdot \mathbf{b} + i \mathbf{a} \times \mathbf{b} \cdot \sigma ,
\]
for any vectors \( \mathbf{a} \) and \( \mathbf{b} \). Show also that
\[
\exp(i\phi \mathbf{n} \cdot \sigma) = \cos \phi + i \mathbf{n} \cdot \sigma \sin \phi .
\]
Hence deduce the results
\[
\mathbf{n} \cdot \sigma \mathbf{a} \cdot \sigma \mathbf{n} \cdot \sigma = 2 \mathbf{n} \cdot \mathbf{a} - \mathbf{n} \cdot \sigma ,
\]
\[
\exp(\frac{1}{2} i \theta \mathbf{n} \cdot \sigma) \exp(-\frac{1}{2} i \theta \mathbf{n} \cdot \sigma) = \sigma \cos \theta + \mathbf{n} \cdot \sigma (1 - \cos \theta) + \mathbf{n} \times \sigma \sin \theta .
\]

5. A particle of spin \( \frac{1}{2} \), with \( \mathbf{S} = \frac{1}{2} \hbar \sigma \), is at rest (its spatial degrees of freedom may be ignored) and interacts with a uniform homogeneous magnetic field \( B \mathbf{n} \) through its magnetic moment \( \mu = \gamma \mathbf{S} \), the Hamiltonian being
\[
H = -B \mathbf{n} \cdot \mu .
\]
Use the results given in example 4 to find an expression for the time evolution operator $\exp(-iH/t)$. Hence show that if the initial state of the particle is $|\chi\rangle$ then the probability of measuring it to be in an orthogonal state $|\chi'\rangle$ after a time $t$ is

$$p(t) = |\langle \chi'| \textbf{n} \cdot \sigma |\chi\rangle|^2 \sin^2 \frac{\omega t}{2}$$

where $\omega = \gamma B$.

Show also that in the Heisenberg picture the spin operators at time $t$ are

$$\textbf{S}(t) = \textbf{S} \cos \omega t + \textbf{n} \cdot \textbf{S} (1 - \cos \omega t) - \textbf{n} \times \textbf{S} \sin \omega t,$$

where $\textbf{S}(0) = \textbf{S}$, the Schrödinger picture operators (the pictures are defined to coincide at $t = 0$).

Simplify the expressions for $p(t)$ and $\textbf{S}(t)$ in the special case where the magnetic field points along the $y$-axis and $|\chi\rangle$ and $|\chi'\rangle$ are eigenstates of spin up and spin down along the $z$-axis. Are these simplified expressions compatible with the results of examples 2 and 4, above?

6. Consider the addition of angular momentum 3 to angular momentum 1. Find the composition of combined states $|J M\rangle = \{|4 4\}, |4 3\rangle, |3 3\rangle, |3 2\rangle, |2 2\rangle, |2 1\rangle\}$ in terms of the initial angular momentum states. Try to explicitly calculate $|1 1\rangle$, finding the reason for the fact that it is zero. [You may set $\hbar = 1$ and assume the result $J_{-}|jm\rangle = \{(j+m)(j-m+1)\}^{\frac{1}{2}} |jm-1\rangle$.]

7. Show that, in the expression for states $|SM\rangle$ of spin $S$ formed from the product of two spin-1 states, $|1m_1\rangle|1m_2\rangle$, the states with $S = 0$ and $S = 2$ are symmetric under the interchange of $m_1$ and $m_2$, whereas those with $S = 1$ are antisymmetric.

Two identical spin-1 particles, whose centre of mass is at rest, have combined spin $\textbf{S}$, relative orbital angular momentum $\textbf{L}$ and total angular momentum $\textbf{J} = \textbf{L} + \textbf{S}$, with corresponding quantum numbers $S$, $L$ and $J$. Show that $L + S$ must be even. If $J = 1$, what are the possible values of $L$ and $S$?

8. (a) Let $\textbf{J} = (J_1, J_2, J_3)$ and $|jm\rangle$ denote the standard angular momentum operators and states, so that, using units in which $\hbar = 1,$

$$\textbf{J}^2 |jm\rangle = j(j+1) |jm\rangle, \quad J_3 |jm\rangle = m |jm\rangle.$$

Show that $U(\theta) = \exp(-i\theta J_2)$ is unitary and define

$$J_i(\theta) = U(\theta) J_i U(\theta)^{-1} \quad \text{for} \quad i = 1, 3.\$$

Using the commutation relations for angular momentum show that

$$\frac{d^2 J_i(\theta)}{d\theta^2} + J_i(\theta) = 0 \quad \text{for} \quad i = 1, 3.\$$

Hence show that

$$J_1(\theta) = J_1 \cos \theta - J_3 \sin \theta, \quad J_3(\theta) = J_1 \sin \theta + J_3 \cos \theta.$$

Deduce that $U\left(\frac{\pi}{2}\right) |jm\rangle$ are eigenstates of $J_1$.

(b) For $j = 1/2$, use the Pauli representation of operators and states to show that

$$U(\theta)|\uparrow\rangle = \cos \frac{1}{2} \theta |\uparrow\rangle + \sin \frac{1}{2} \theta |\downarrow\rangle, \quad U(\theta)|\downarrow\rangle = -\sin \frac{1}{2} \theta |\uparrow\rangle + \cos \frac{1}{2} \theta |\downarrow\rangle$$

where $|\uparrow\rangle = |\frac{1}{2} \frac{1}{2}\rangle$ and $|\downarrow\rangle = |\frac{1}{2} -\frac{1}{2}\rangle$. Verify in this representation that $U\left(\frac{\pi}{2}\right) |\uparrow\rangle$ and $U\left(\frac{\pi}{2}\right) |\downarrow\rangle$ are eigenstates of $J_1$.

(c) Show that for two spin-$\frac{1}{2}$ particles the composite state

$$\frac{1}{\sqrt{2}} (|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle)$$

is unchanged by a transformation $|\uparrow\rangle \mapsto U(\theta)|\uparrow\rangle$ and $|\downarrow\rangle \mapsto U(\theta)|\downarrow\rangle$ applied to all one-particle states. How does this relate to the angular momentum properties of the two-particle state?