Principles of Quantum Mechanics - Problems 4

Please email me with any comments about these problems, particularly if you spot an error.

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1. Suppose a system has a basis of just two orthonormal states $|1\rangle$ and $|2\rangle$, with respect to which the total Hamiltonian has the matrix representation

$$\begin{pmatrix} E_1 & V_0\,\mathrm{e}^{\mathrm{i}\omega t} \\ V_0\,\mathrm{e}^{-\mathrm{i}\omega t} & E_2 \end{pmatrix},$$

where V_0 is independent of time. At $t = 0$, the system is in state $|1\rangle$. Show that the probability of a transition from state $|1\rangle$ to state $|2\rangle$ in time interval t is

$$P(t) = \frac{4V_0^2}{(E_1 - E_2 + \hbar\omega)^2}\sin^2((E_1 - E_2 + \hbar\omega)t/2\hbar)^2 + O(V^4), \qquad (\star)$$

to lowest non–trivial order in V_0. Solve this two–state problem exactly to find the true value of $P(t)$ and hence state conditions necessary for the perturbative approach to be valid here.

2. A particle of mass m and charge e is contained within a cubical box of side a. Initially the particle is in the stationary state of energy $3\pi^2\hbar^2/2ma^2$. At time $t = 0$ a uniform electric field of strength E is switched on parallel to one of the edges of the cube. Obtain an expression to second order in e for the probability of measuring the particle to have energy $3\pi^2\hbar^2/ma^2$ at time t.

3. A harmonic oscillator of angular frequency ω is acted on by the time–dependent perturbation

$$\frac{q\mathcal{E}X}{\sqrt{\pi}\,\tau}\exp\left(-\frac{t^2}{\tau^2}\right) \qquad \text{for all } t,$$

where X is the position operator and q, $\mathcal{E}$ and τ are constants. Show that in first–order perturbation theory, the only allowed transition from the ground state is to the first excited state. If the perturbation acts from very early times to very late times, find the probability that this transition takes place, correct to order $\mathcal{E}^2$.

By expanding $U_{\mathrm{I}}(t)$ to *second* non–trivial order, calculate the corresponding probability for a transition from the ground state to the second excited state.

4. A particle travelling in one dimension with momentum $p = \hbar k > 0$ encounters the steep–sided potential well $V(x) = -V_0 < 0$ for $|x| < a$. Use Fermi's golden rule to show that the probability the particle will be reflected by the well is

$$P_{\text{reflect}} \approx \frac{V_0^2}{4E^2}\sin^2(2ka),$$

where $E = p^2/2m$. Show that in the limit $E \gg V_0$ this result is consistent with the exact result for the reflection probability. [*Hint: adopt periodic boundary conditions to normalise the wavefunctions of the initial and final states.*]

5. Consider the driven quantum harmonic oscillator with Hamiltonian

$$H = \hbar\omega\left(A^\dagger A + \frac{1}{2}\right) + \hbar\left(f^*(t)A + f(t)A^\dagger\right).$$

Taking H_0 to be the standard oscillator Hamiltonian, show that the perturbation in the interaction picture is

$$V_{\mathrm{I}}(t) = \hbar\left(\tilde{f}^*(t)A + \tilde{f}(t)A^\dagger\right) \qquad \text{where} \qquad \tilde{f}(t) = \mathrm{e}^{\mathrm{i}\omega t} f(t)\,.$$

Show that $U(g) := \mathrm{e}^{gA^\dagger - g^* A} = \mathrm{e}^{-|g|^2/2}\,\mathrm{e}^{gA^\dagger}\,\mathrm{e}^{-g^*A}$ where $g = g(t)$. By taking the time–derivative of this expression, deduce that the time evolution operator in the interaction picture can be written as

$$U_{\mathrm{I}}(t) = U(g)\,\mathrm{e}^{-\int_0^t \mathrm{Im}(\dot{g}^* g)\,dt'} \qquad \text{for the choice} \qquad g(t) = -\mathrm{i}\int_0^t \tilde{f}(t')\,dt'\,.$$

At $t = 0$ the oscillator is initially in its ground state $|0\rangle$. In the case that $f(t) = \mathrm{e}^{-\mathrm{i}\omega t} f_0$ where f_0 is constant, show that at time t the oscillator is in the coherent state

$$|\psi_{\mathrm{S}}(t)\rangle = \mathrm{e}^{-|f_0|^2 t^2/2}\,\mathrm{e}^{-\mathrm{i}\omega t/2}\,\mathrm{e}^{-\mathrm{i}tf(t)A^\dagger}|0\rangle$$

in the Schrödinger picture. Comment on the relevance of this model to the operation of a laser.

6. A certain quantum system has N distinct energy levels, with corresponding states $|n\rangle$ for $n = 1, \ldots, N$. Let

$$|\psi(\boldsymbol{\alpha})\rangle = \sum_n \sqrt{p_n}\,\mathrm{e}^{\mathrm{i}\alpha_n}|n\rangle\,,$$

where the α_n are real phases and p_n is the probability of being in the state $|n\rangle$. If we know the values of the p_n but not the phases, show that the density operator is

$$\rho = \int_0^{2\pi} \frac{d^N\boldsymbol{\alpha}}{(2\pi)^N}\,|\psi(\boldsymbol{\alpha})\rangle\langle\psi(\boldsymbol{\alpha})| = \sum_n p_n |n\rangle\langle n|\,.$$

Suppose we expand $|\psi(\boldsymbol{\alpha})\rangle$ in some other basis as

$$|\psi(\boldsymbol{\alpha})\rangle = \sum_r \sqrt{P_r}\,\mathrm{e}^{\mathrm{i}\beta_r}\,|q_r\rangle$$

where P_r is the probability of measuring q_r on measuring the observable Q, and again the phases $\beta_r(\boldsymbol{\alpha})$ are unknown. Is ρ diagonal in the Q representation?

7. Let $\rho = \sum_{i,j} \rho_{ij}|i\rangle\langle j|$ be the density operator of some quantum system. Show that the system is in a pure state if and only if every row of the matrix ρ_{ij} is a multiple of the first row, and every column is a multiple of the first column.

8. Show also that the entropy $s(\rho) = -\mathrm{tr}_{\mathcal{H}}(\rho \ln \rho)$ obeys $s(U\rho U^\dagger) = s(\rho)$ for any unitary operator U, and hence that the entropy is both time independent and independent of the choice of basis on $\mathcal{H}$.

   A composite system is formed from two uncorrelated subsystems A and B. Both subsystems are in impure states, with the numbers $\{p_{Ai}\}$ and $\{p_{Br}\}$ being the probabilities of the members of the complete sets of states $\{|A;i\rangle\}$ and $\{|B;r\rangle\}$, respectively. Show that the entropy of the composite system is the sum of the entropies of the two subsystems. What is the relevance of this result for thermodynamics?