1. What is the unitary operator \( U(\alpha) \) corresponding to translation through \( \alpha \) for a one-dimensional quantum system with position \( \hat{x} \) and momentum \( \hat{p} \)? Calculate \([\hat{x}, U(\alpha)]\) and show that the result is consistent with the assumption that position eigenstates obey 
\(|x+\alpha) = U(\alpha)|x)\). Given this assumption, express the wavefunction for \( U(\alpha)|\psi) \) in terms of the wavefunction \( \psi(x) \) for \(|\psi)\).

If the system is a one-dimensional harmonic oscillator of mass \( m \) and frequency \( \omega \), show that
\[
U(\alpha) = e^{-\frac{1}{2}\gamma^2} e^{-\gamma\alpha} e^{-\gamma\alpha}
\]
where \( \gamma = \alpha (m\omega/2\hbar)^{\frac{1}{2}} \).

Deduce that if \( \psi_n(x) \) are wavefunctions for the usual normalised states with energies \( \hbar\omega(n+\frac{1}{2}) \), then
\[
\psi_0(x-\alpha) = e^{-\frac{1}{2}\gamma^2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \gamma^n \psi_n(x) .
\]

[Recall that \([A, e^B] = [A, B] e^B \) and \( e^A e^B = e^{A+B} e^{1/2[A,B]} \) provided \([A, B]\) commutes with \( A \) and \( B \).]

2. Write down the commutation relations for the components of a vector operator \( \mathbf{V} = (V_1, V_2, V_3) \) and the angular momentum operator \( \mathbf{J} = (J_1, J_2, J_3) \). Use these to show that
\[
\mathbf{V}(\theta) = \exp(i\theta\mathbf{n} \cdot \mathbf{J}/\hbar) \mathbf{V} \exp(-i\theta\mathbf{n} \cdot \mathbf{J}/\hbar)
\]
satisfies
\[
\mathbf{V}'(\theta) = \mathbf{n} \times \mathbf{V}(\theta)
\]
where \( \mathbf{n} \) is a unit vector and \( \theta \) a real parameter. Deduce that \( \mathbf{n} \cdot \mathbf{V}(\theta) = \mathbf{n} \cdot \mathbf{V} \) and hence that
\[
\mathbf{V}''(\theta) + \mathbf{V}(\theta) = (\mathbf{n} \cdot \mathbf{V}) \mathbf{n} .
\]
Solve this equation to find \( \mathbf{V}(\theta) \) in terms of \( \mathbf{V} \) and interpret your result.

3. Show that a particle of spin 1 cannot decay into two identical particles of spin 0. The \( \rho \)-meson has spin 1 and can decay into two spinless \( \pi \)-mesons, or pions, with different charges. If the intrinsic parity of any \( \pi \) is negative, what is the intrinsic parity of the \( \rho \)?

4. A particle \( X \) is observed to undergo the decays \( X \rightarrow \rho^+ + \pi^- \) and \( X \rightarrow K + \bar{K} \), where \( K \) is a particle of spin 0. What is the lowest value for the spin of \( X \) that is consistent with this, and what is the corresponding intrinsic parity of \( X \)? [Assume that total angular momentum and parity are conserved is all these processes.]

5. The Hamiltonian for a quantum mechanical system is \( H_0 + V(t) \), where \( H_0 \) has no explicit time dependence. At \( t = 0 \) the system is in an eigenstate \(|\alpha)\) of \( H_0 \). Show that the probability that a measurement at time \( t \) yields a result corresponding to an eigenstate \(|\alpha')\) of \( H_0 \) is 
\[
|\langle a'| T(t)|a)\rangle|^2 ,
\]
where \( T(t) \) satisfies the integral equation
\[
T(t) = 1 - \int_0^t \bar{V}(t') T(t') \, dt' , \quad \bar{V}(t) = e^{\hat{H}_0 t/\hbar} V(t) e^{-\hat{H}_0 t/\hbar} .
\]
Suppose a system has a basis of just two orthonormal states \(|1)\) and \(|2)\), with respect to which the total Hamiltonian has the time-independent matrix representation
\[
\begin{pmatrix}
E & V \\
V & E
\end{pmatrix}
\]
Show that, to lowest order in \( V \), the probability of a transition from state \(|1)\) to state \(|2)\) in the time interval \( t \) is \( V^2 t^2/\hbar^2 \). By comparing this with the exact result, state the conditions for such an approximation to be good.
6. A particle of mass \( m \) and charge \( e \) is contained within a cubical box of side \( a \). Initially the particle is in the stationary state of energy \( 3\pi^2\hbar^2/2ma^2 \). At time \( t = 0 \), a uniform electric field of strength \( E \) is switched on parallel to one of the edges of the cube. Obtain an expression to second order in \( E \) for the probability of measuring the particle to have energy \( 3\pi^2\hbar^2/ma^2 \) at time \( t \).

7. A harmonic oscillator, with angular frequency \( \omega \), is acted on by the time-dependent perturbation

\[
-e^{it^2/2\tau} \exp(-it^2/\tau^2)
\]

for all \( t \).

Show that in first order perturbation theory the only allowed transition from the ground state is to the first excited state and find the probability that this takes place to first order in \( e^2 \) if the perturbation acts from very early times to very late times.

8. (a) Recall that the trace of an operator \( A \) on a space of states \( V \) is defined by \( \text{Tr}(A) = \sum_i \langle n|A|n \rangle \) for any orthonormal basis \( \{ |n \rangle \} \) in \( V \). Show that \( \text{Tr}(AB) = \text{Tr}(BA) \) and hence \( \text{Tr}([A,B]) = 0 \) (assuming all relevant sums converge). Use this to demonstrate that for an operator \( \rho(t) \)

\[
i\hbar \frac{d}{dt} \rho = [H, \rho] \quad \Rightarrow \quad \frac{d}{dt} \text{Tr}(\rho^n) = 0.
\]

Verify that the first of these equations is satisfied if \( \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \) provided the states \( |\psi_i\rangle \) obey the time-dependent Schrödinger equation and the \( p_i \) are constants. Given that \( |\psi_i\rangle \) are also orthonormal, calculate \( \text{Tr}(\rho^n) \) explicitly in terms of \( p_i \).

(b) Consider a normalised state in \( V \otimes W \) of the form

\[
|\Psi⟩ = \lambda_1 |\psi_1⟩ |\phi_1⟩ + \mu_1 |\psi_1⟩ |\phi_2⟩ + \lambda_2 |\psi_2⟩ |\phi_1⟩ + \mu_2 |\psi_2⟩ |\phi_2⟩
\]

where \( |\psi_i⟩ \) are orthonormal states in \( V \), as in part (a) above, and \( |\phi_i⟩ \) are orthonormal states in \( W \). If \( A \) acts only on \( V \) and not on \( W \), show that \( \langle \Psi|A|\Psi \rangle = \text{Tr}(\rho A) \) where the (reduced) density operator \( \rho = \sum_i p_i |\psi_i⟩ \langle \psi_i| + p_2 |\psi_2⟩ \langle \psi_2| \) for \( p_1 \) and \( p_2 \) to be determined. Verify that \( \text{Tr}_V\rho = 1 \). If, in addition, \( \text{Tr}_V\rho^2 = 1 \), prove that either \( p_1 \) or \( p_2 \) must be zero. Is \( |\Psi⟩ \) an entangled state in this case?

9. A spin-half system has space of states \( V \) and spin operators given by the Pauli matrices \( \sigma_x, \sigma_y, \sigma_z \). What are the states obtained by applying \( \sigma_x, \sigma_y \) to the eigenstates \( |↑⟩ \) and \( |↓⟩ \) of \( \sigma_z \)?

A combination of three such systems has states belonging to \( V(1) \otimes V(2) \otimes V(3) \) and spin operators acting on each subsystem denoted by \( \sigma_x^{(i)}, \sigma_y^{(i)} \) with \( i = 1, 2, 3 \). Show that the state

\[
|\Psi⟩ = \frac{1}{\sqrt{2}} \left[ |↑⟩_1 |↑⟩_2 |↑⟩_3 − |↓⟩_1 |↓⟩_2 |↓⟩_3 \right]
\]

satisfies

\[
\sigma_x^{(1)} \sigma_x^{(2)} \sigma_y^{(3)} |\Psi⟩ = \sigma_y^{(1)} \sigma_x^{(2)} \sigma_y^{(3)} |\Psi⟩ = \sigma_y^{(1)} \sigma_y^{(2)} \sigma_x^{(3)} |\Psi⟩ = |\Psi⟩ \quad \text{and} \quad \sigma_x^{(1)} \sigma_x^{(2)} \sigma_x^{(3)} |\Psi⟩ = -|\Psi⟩.
\]

Consider now whether these outcomes for measurements of particular combinations of the operators \( \sigma_x^{(i)}, \sigma_y^{(i)} \) in the state \( |\Psi⟩ \) could be reproduced by replacing the spin operators with classical variables \( s_x^{(i)}, s_y^{(i)} \) which take values \( \pm 1 \) according to some probabilities. To agree with the results above for \( \sigma_x^{(1)} \sigma_y^{(2)} \sigma_y^{(3)}, \sigma_y^{(1)} \sigma_x^{(2)} \sigma_y^{(3)}, \sigma_y^{(1)} \sigma_y^{(2)} \sigma_x^{(3)} \) on \( |\Psi⟩ \),

\[
\left( s_x^{(1)} s_x^{(2)} s_y^{(3)} \right) = \left( s_y^{(1)} s_y^{(2)} s_y^{(3)} \right) = \left( s_y^{(1)} s_y^{(2)} s_x^{(3)} \right) = 1.
\]

Show that classically, this implies

\[
s_x^{(1)} s_x^{(2)} s_x^{(3)} = 1,
\]

which does not agree with the quantum measurement of \( \sigma_x^{(1)} \sigma_x^{(2)} \sigma_x^{(3)} \) in the state \( |\Psi⟩ \).