Applications of Quantum Mechanics: Example Sheet 3

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1. The Schrödinger equation for a particle of mass $m$ and charge $q$ in an electromagnetic field is

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \left( \nabla - \frac{iq}{\hbar} A \right)^2 \psi + q\phi \psi$$

Under a gauge transformation,

$$\phi \to \phi - \frac{\partial \alpha}{\partial t}, \quad A \to A + \nabla \alpha.$$

Show that, with a suitable transformation of $\psi$, the Schrödinger equation transforms into itself. Show that the probability density $|\psi|^2$ is gauge invariant. Show that the mechanical momentum $\pi = -i\hbar \nabla - qA$ is gauge invariant. What is the physical interpretation of the mechanical momentum?

2. A particle of charge $q$ moving in a magnetic field $B = \nabla \times A = (0, 0, B)$ is described by the Hamiltonian

$$H = \frac{1}{2m} (p - qA)^2$$

where $p$ is the canonical momentum. Show that the mechanical momentum $\pi = p - qA$ obeys

$$[\pi_x, \pi_y] = i\hbar B$$

Define

$$a = \frac{1}{\sqrt{2q\hbar B}}(\pi_x + i\pi_y) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2q\hbar B}}(\pi_x - i\pi_y)$$

What commutation relations do $a$ and $a^\dagger$ obey? Write the Hamiltonian in terms of $a$ and $a^\dagger$ and hence solve for the spectrum.
3i. Symmetric gauge is defined by $A = \frac{B}{2} (-y, x, 0)$. Confirm that this gives the magnetic field $B = (0, 0, B)$. Show that the Hamiltonian can be written as

$$H = -\frac{\hbar^2}{2m} \nabla^2 + \frac{qB}{2m} L_z + \frac{q^2B^2}{8m} (x^2 + y^2)$$

where $L_z$ is the component of the angular momentum parallel to $B$.

ii. Show that the operator $a$, defined in Question 2, takes the form

$$a = -i\sqrt{2} \left( l_B \frac{\partial}{\partial w} + \frac{w}{4l_B} \right)$$

where $l_B = \sqrt{\hbar/qB}$ is the magnetic length and $w = x + iy$ is a complex coordinate on the plane, with $\partial_\omega = \frac{1}{2}(\partial_x + i\partial_y)$ so that $\partial_\omega \omega = 0$ and $\partial_\omega \bar{\omega} = 1$. Hence show that the state

$$\psi(w) = f(w)e^{-|w|^2/4l_B^2}$$

sits in the lowest Landau level for any holomorphic function $f(w)$.

4. In the presence of a magnetic field $B = (0, 0, B)$, a particle of charge $q$ moves in the $(x, y)$-plane on the trajectory,

$$x(t) = X + R\sin(\omega_B t) \quad \text{and} \quad y(t) = Y + R\cos(\omega_B t)$$

with $\omega_B = qB/m$. Working in symmetric gauge $A = \frac{B}{2} (-y, x, 0)$, show that the centre of mass coordinates can be re-expressed as

$$X = \frac{x}{2} + \frac{p_y}{m\omega_B} \quad \text{and} \quad Y = \frac{y}{2} - \frac{p_x}{m\omega_B}$$

Viewed as quantum operators in the Heisenberg representation, show that both $X$ and $Y$ do not change in time. Show that

$$[X, Y] = -il_B^2$$

where $l_B^2 = \hbar / qB$ is the magnetic length. Use the Heisenberg uncertainty relation for $X$ and $Y$ to estimate the number of states $N$ that can sit in a region of area $A$. 

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5. A particle of charge $e$ and spin $\frac{1}{2}$ with g-factor $g = 2$ moves in the $(x, y)$-plane in the presence of a magnetic field of the form $\mathbf{B} = (0, 0, B)$. Show that the Hamiltonian can be written as

$$H = \frac{1}{2m} Q^2 \quad \text{with} \quad Q = (\pi_x \sigma_x + \pi_y \sigma_y)$$

where $\sigma$ are the Pauli matrices and $\pi$ is the mechanical momentum defined in earlier questions.

Confirm that $Q$ is Hermitian. Show that zero energy states are annihilated by $Q$. Show that $|\psi\rangle$ and $Q|\psi\rangle$ are degenerate and hence deduce that the lowest Landau level contains half the states of the higher Landau levels. What is the physical interpretation of this? (Hint: consider the effect of Zeeman splitting on Landau levels.)

Working in Landau gauge, $\mathbf{A} = (0, Bx, 0)$ with $B > 0$, show that zero energy states have spin up and take the form

$$\psi = \begin{pmatrix} f(w) e^{-x^2/2l_B^2} \\ 0 \end{pmatrix}$$

with $w = x + iy$ and $l_B^2 = \hbar/qB$. Show by explicit calculation that there are no zero energy spin down states.

6*. Near the Dirac point, an electron in graphene is described by the Hamiltonian

$$H = v_F Q$$

with $v_F$ the Fermi velocity and $Q$ the operator defined in Question 5. Working in Landau gauge $\mathbf{A} = (0, Bx, 0)$, show that the Landau level spectrum is given by

$$E = \pm v_F \sqrt{2\hbar qB} \sqrt{n} \quad n = 0, 1, 2, \ldots$$
7. A particle of mass $m$ is scattered by the symmetric square well potential given by

$$V(x) = \begin{cases} 
-V_0 & |x| < a/2 \\
0 & |x| > a/2 
\end{cases} \quad \text{with } V_0 > 0$$

The incoming and outgoing plane waves of even $(+)$ and odd $(-)$ parity are

$$I_+(k; x) = e^{-ik|x|}, \quad I_-(k; x) = \text{sign}(x) e^{-ik|x|}$$

$$O_+(k; x) = e^{+ik|x|}, \quad O_-(k; x) = -\text{sign}(x) e^{+ik|x|}$$

The corresponding scattering states for $|x| > a/2$ are

$$\psi_+(k; x) = I_+(k; x) + S_{++}(k) O_+(k; x)$$
$$\psi_-(k; x) = I_-(k; x) + S_{--}(k) O_-(k; x)$$

where $S_{++}$ and $S_{--}$ are the diagonal elements of the S-matrix in the parity basis. By imposing the boundary condition that $(d\psi/dx)/\psi$ is continuous at $x = a/2$ show that

$$S_{++} = -e^{-ika} q \tan(qa/2) - ik \over q \tan(qa/2) + ik$$
$$S_{--} = e^{-ika} q + ik \tan(qa/2) \over q - ik \tan(qa/2)$$

where $q^2 = k^2 + U_0$ and $U_0 = 2mV_0/\hbar^2$. Interpret the poles and zeros of $S_{++}$ and $S_{--}$ in terms of bound states.

8. Carry out a similar analysis to that in Question 7, this time for the potential

$$V(x) = V_0 \delta(x - 1) + V_0 \delta(x + 1)$$

with $V_0 > 0$. Interpret the poles and zeros of $S_{++}$ and $S_{--}$ in the complex $k$-plane as resonances, in the case where $V_0 \gg 1$. Show that, approximately, the pole position in $S_{++}$ with the smallest real part lies at

$$k = \frac{\pi}{2} - \frac{\pi}{2U_0} + \frac{\pi}{2U_0^2} - i \frac{\pi^2}{4U_0^2}$$

where $U_0 = 2mV_0/\hbar^2$. 