Applications of Quantum Mechanics: Example Sheet 4

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1. The semi-classical equations of motion for an electron of charge $-e$ and energy $E(k)$ moving in a magnetic field $\mathbf{B}$ are

$$\hbar \frac{dk}{dt} = -e \mathbf{v} \times \mathbf{B} \quad \text{and} \quad \mathbf{v} = \frac{1}{\hbar} \frac{\partial E}{\partial k}$$

Show that, in momentum space, the electrons orbit the Fermi surface $E(k_F)$ in a plane perpendicular to $\mathbf{B}$. Show that the orbit of the electron in position space, projected onto the plane perpendicular to $\mathbf{B}$, traces out the perimeter of a cross-section of the Fermi surface. [Hint: Consider the evolution of the position $\mathbf{r}_\perp = \mathbf{r} - (\mathbf{B} \cdot \mathbf{r}) \mathbf{B}$, perpendicular to the magnetic field.]

A free electron has $E(k) = \frac{\hbar^2 k^2}{2m}$. Use the results above to show that, for any value of $k \cdot \mathbf{B}$, the electron orbits the Fermi surface with cyclotron frequency $\omega_B = \frac{eB}{m}$. Show that the time taken to orbit the Fermi surface can be written as

$$T = \frac{2\pi}{\omega_B} = \frac{\hbar^2}{eB} \frac{\partial A(E)}{\partial E} \bigg|_{kB}$$

where $A(E)$ is the cross-sectional area of the Fermi surface with Fermi energy $E$.

[An Aside: This formula is important because it holds for Fermi surfaces of any shape.]

2. A one-dimensional crystal comprises a chain of atoms of mass $m$ equally spaced by a distance $a$ when in equilibrium. The forces between the atoms are such that the effective spring constants are alternately $\lambda$ and $\alpha \lambda$. Show that the dispersion relation for phonons has the form

$$\omega_{\pm}(k)^2 = \frac{\lambda}{m} \left[(1 + \alpha) \pm \sqrt{1 + 2\alpha \cos 2ka + \alpha^2}\right]$$

where the wavenumber $k$ satisfies $-\pi/2a \leq k \leq \pi/2a$. What is the speed of sound in this crystal?
3. The Schrödinger equation for a particle of mass $m$ and charge $q$ in an electromagnetic field is

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \left( \nabla - \frac{iq}{\hbar} A \right)^2 \psi + q\phi \psi$$

Under a gauge transformation,

$$\phi \rightarrow \phi - \frac{\partial \alpha}{\partial t}, \quad A \rightarrow A + \nabla \alpha.$$ 

Show that, with a suitable transformation of $\psi$, the Schrödinger equation transforms into itself. Show that the probability density $|\psi|^2$ is gauge invariant. Show that the mechanical momentum $\pi = -i\hbar \nabla - qA$ is gauge invariant. What is the physical interpretation of the mechanical momentum?

4. A particle of charge $q$ moving in a magnetic field $B = \nabla \times A = (0, 0, B)$ is described by the Hamiltonian

$$H = \frac{1}{2m} (p - qA)^2$$

where $p$ is the canonical momentum. Show that the mechanical momentum $\pi = p - qA$ obeys

$$[\pi_x, \pi_y] = i q \hbar B$$

Define

$$a = \frac{1}{\sqrt{2q\hbar B}} (\pi_x + i\pi_y) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2q\hbar B}} (\pi_x - i\pi_y)$$

What commutation relations do $a$ and $a^\dagger$ obey? Write the Hamiltonian in terms of $a$ and $a^\dagger$ and hence solve for the spectrum.

5a. Symmetric gauge is defined by $A = \frac{B}{2} (-y, x, 0)$. Confirm that this gives the magnetic field $B = (0, 0, B)$. Show that the Hamiltonian can be written as

$$H = -\frac{\hbar^2}{2m} \nabla^2 - \frac{qB}{2m} L_z + \frac{q^2B^2}{8m} (x^2 + y^2)$$

where $L_z$ is the component of the angular momentum parallel to $B$. 

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b. Show that the operator \( a \), defined in Question 4, takes the form

\[
a = -i\sqrt{2} \left( l_B \frac{\partial}{\partial w} + \frac{w}{4l_B} \right)
\]

where \( l_B = \sqrt{\hbar/qB} \) is the magnetic length and \( w = x + iy \) is a complex coordinate on the plane, with \( \partial_w = \frac{1}{2}(\partial_x + i\partial_y) \) so that \( \partial_w \omega = 0 \) and \( \partial_{\bar{w}} \bar{\omega} = 1 \). Hence show that the state

\[
\psi(w) = f(w)e^{-|w|^2/4l_B^2}
\]
sits in the lowest Landau level for any holomorphic function \( f(w) \).

6. In the presence of a magnetic field \( \mathbf{B} = (0, 0, B) \), a particle of charge \( q \) moves in the \((x, y)\)-plane on the trajectory,

\[
x(t) = X + R \sin(\omega_B t) \quad \text{and} \quad y(t) = Y + R \cos(\omega_B t)
\]

with \( \omega_B = qB/m \). Working in symmetric gauge \( \mathbf{A} = \frac{B}{2} (-y, x, 0) \), show that the centre of mass coordinates can be re-expressed as

\[
X = \frac{x}{2} + \frac{p_y}{m\omega_B} \quad \text{and} \quad Y = \frac{y}{2} - \frac{p_x}{m\omega_B}
\]

Viewed as quantum operators in the Heisenberg representation, show that both \( X \) and \( Y \) do not change in time. Show that

\[
[X, Y] = -il_B^2
\]

where \( l_B^2 = \hbar/qB \) is the magnetic length. Use the Heisenberg uncertainty relation for \( X \) and \( Y \) to estimate the number of states \( \mathcal{N} \) that can sit in a region of area \( A \).

7. A particle of charge \( e \) and spin \( \frac{1}{2} \) with g-factor \( g = 2 \) moves in the \((x, y)\)-plane in the presence of a magnetic field of the form \( \mathbf{B} = (0, 0, B) \). Show that the Hamiltonian can be written as

\[
H = \frac{1}{2m} Q^2 \quad \text{with} \quad Q = (\pi_x \sigma_x + \pi_y \sigma_y)
\]

where \( \sigma \) are the Pauli matrices and \( \pi \) is the mechanical momentum defined in earlier questions.

Confirm that \( Q \) is Hermitian. Show that zero energy states are annihilated by \( Q \). Show that \( |\psi\rangle \) and \( Q|\psi\rangle \) are degenerate and hence deduce that the lowest Landau level
contains half the states of the higher Landau levels. What is the physical interpretation of this? (Hint: consider the effect of Zeeman splitting on Landau levels.)

Working in Landau gauge, \( A = (0, Bx, 0) \) with \( B > 0 \), show that zero energy states have spin up and take the form

\[ \psi = \begin{pmatrix} f(w) e^{-x^2/2l_B^2} \\ 0 \end{pmatrix} \]

with \( w = x + i y \) and \( l_B^2 = \hbar q B \). Show by explicit calculation that there are no zero energy spin down states.

8*. Near the Dirac point, an electron in graphene is described by the Hamiltonian

\[ H = v_F Q \]

with \( v_F \) the Fermi velocity and \( Q \) the operator defined in Question 5. Working in Landau gauge \( A = (0, Bx, 0) \), show that the Landau level spectrum is given by

\[ E = \pm v_F \sqrt{2\hbar q B} \sqrt{n} \quad n = 0, 1, 2, \ldots \]