Applications of Quantum Mechanics: Example Sheet 4

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1. The Schrödinger equation for a particle of mass m and charge q in an electromagnetic field is

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\nabla - \frac{iq}{\hbar}\mathbf{A}\right)^2\psi + q\phi\psi$$

Under a gauge transformation,

$$\phi \to \phi - \frac{\partial\alpha}{\partial t} \quad , \quad \mathbf{A} \to \mathbf{A} + \nabla\alpha \ .$$

Show that, with a suitable transformation of ψ, the Schrödinger equation transforms into itself. Show that the probability density $|\psi|^2$ is gauge invariant. Show that the *mechanical momentum* $\boldsymbol{\pi} = -i\hbar\nabla - q\mathbf{A}$ is gauge invariant. What is the physical interpretation of the mechanical momentum?

2. A particle of charge q moving in a magnetic field $\mathbf{B} = \nabla\times\mathbf{A} = (0,0,B)$ is described by the Hamiltonian

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2$$

where $\mathbf{p}$ is the canonical momentum. Show that the mechanical momentum $\boldsymbol{\pi} = \mathbf{p} - q\mathbf{A}$ obeys

$$[\pi_x, \pi_y] = iq\hbar B$$

Define

$$a = \frac{1}{\sqrt{2q\hbar B}}(\pi_x + i\pi_y) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2q\hbar B}}(\pi_x - i\pi_y)$$

What commutation relations do a and $a^\dagger$ obey? Write the Hamiltonian in terms of a and $a^\dagger$ and hence solve for the spectrum.

3a. Symmetric gauge is defined by $\mathbf{A} = \frac{B}{2}(-y, x, 0)$. Confirm that this gives the magnetic field $\mathbf{B} = (0,0,B)$. Show that the Hamiltonian can be written as

$$H = -\frac{\hbar^2}{2m}\nabla^2 - \frac{qB}{2m}L_z + \frac{q^2B^2}{8m}\left(x^2 + y^2\right)$$

where L_z is the component of the angular momentum parallel to B.

b. Show that the operator a, defined in Question 2, takes the form

$$a = -i\sqrt{2}\left(l_B \frac{\partial}{\partial \bar{w}} + \frac{w}{4l_B}\right)$$

where $l_B = \sqrt{\hbar/qB}$ is the magnetic length and $w = x + iy$ is a complex coordinate on the plane, with $\partial_{\bar{\omega}} = \frac{1}{2}(\partial_x + i\partial_y)$ so that $\partial_{\bar{\omega}}\omega = 0$ and $\partial_{\bar{\omega}}\bar{\omega} = 1$. Hence show that the state

$$\psi(w) = f(w)e^{-|w|^2/4l_B^2}$$

sits in the lowest Landau level for any holomorphic function $f(w)$.

4. In the presence of a magnetic field $\mathbf{B} = (0, 0, B)$, a particle of charge q moves in the (x, y)-plane on the trajectory,

$$x(t) = X + R\sin(\omega_B t) \quad \text{and} \quad y(t) = Y + R\cos(\omega_B t)$$

with $\omega_B = qB/m$. Working in symmetric gauge $\mathbf{A} = \frac{B}{2}(-y, x, 0)$, show that the centre of mass coordinates can be re-expressed as

$$X = \frac{x}{2} + \frac{p_y}{m\omega_B} \quad \text{and} \quad Y = \frac{y}{2} - \frac{p_x}{m\omega_B}$$

Viewed as quantum operators in the Heisenberg representation, show that both X and Y do not change in time. Show that

$$[X, Y] = -il_B^2$$

where $l_B^2 = \hbar/qB$ is the magnetic length. Use the Heisenberg uncertainty relation for X and Y to estimate the number of states $\mathcal{N}$ that can sit in a region of area A.

5. A particle of charge e and spin $\frac{1}{2}$ with g-factor $g = 2$ moves in the (x, y)-plane in the presence of a magnetic field of the form $\mathbf{B} = (0, 0, B)$. Show that the Hamiltonian can be written as

$$H = \frac{1}{2m}Q^2 \quad \text{with} \quad Q = (\pi_x\sigma_x + \pi_y\sigma_y)$$

where $\boldsymbol{\sigma}$ are the Pauli matrices and $\boldsymbol{\pi}$ is the mechanical momentum defined in earlier questions.

Confirm that Q is Hermitian. Show that zero energy states are annihilated by Q. Show that $|\psi\rangle$ and $Q|\psi\rangle$ are degenerate and hence deduce that the lowest Landau level contains half the states of the higher Landau levels. What is the physical interpretation of this? (Hint: consider the effect of Zeeman splitting on Landau levels.)

Working in Landau gauge, $\mathbf{A} = (0, Bx, 0)$ with $B > 0$, show that zero energy states have spin up and take the form

$$\psi = \begin{pmatrix} f(w)\, e^{-x^2/2l_B^2} \\ 0 \end{pmatrix}$$

with $w = x + iy$ and $l_B^2 = \hbar/qB$. Show by explicit calculation that there are no zero energy spin down states.

6. The semi-classical equations of motion for an electron of charge $-e$ and energy $E(\mathbf{k})$ moving in a magnetic field $\mathbf{B}$ are

$$\hbar\frac{d\mathbf{k}}{dt} = -e\mathbf{v} \times \mathbf{B} \quad \text{and} \quad \mathbf{v} = \frac{1}{\hbar}\frac{\partial E}{\partial \mathbf{k}}$$

Show that, in momentum space, the electrons orbit the Fermi surface $E(k_F)$ in a plane perpendicular to $\mathbf{B}$. Show that the orbit of the electron in position space, projected onto the plane perpendicular to $\mathbf{B}$, traces out the perimeter of a cross-section of the Fermi surface. [*Hint: Consider the evolution of the position* $\mathbf{r}_\perp = \mathbf{r} - (\hat{\mathbf{B}} \cdot \mathbf{r})\hat{\mathbf{B}}$, *perpendicular to the magnetic field.*]

A free electron has $E(k) = \hbar^2 k^2/2m$. Use the results above to show that, for any value of $\mathbf{k} \cdot \mathbf{B}$, the electron orbits the Fermi surface with cyclotron frequency $\omega_B = eB/m$. Show that the time taken to orbit the Fermi surface can be written as

$$T = \frac{2\pi}{\omega_B} = \frac{\hbar^2}{eB}\left.\frac{\partial A(E)}{\partial E}\right|_{\mathbf{k}\cdot\mathbf{B}}$$

where $A(E)$ is the cross-sectional area of the Fermi surface with Fermi energy E.

[An Aside: This formula is important because it holds for Fermi surfaces of any shape.]

7. A one-dimensional crystal comprises a chain of atoms of mass m equally spaced by a distance a when in equilibrium. The forces between the atoms are such that the effective spring constants are alternately λ and $\alpha\lambda$. Show that the dispersion relation for phonons has the form

$$\omega_\pm(k)^2 = \frac{\lambda}{m}\left[(1+\alpha) \pm \sqrt{1 + 2\alpha\cos 2ka + \alpha^2}\right]$$

where the wavenumber k satisfies $-\pi/2a \le k \le \pi/2a$. What is the speed of sound in this crystal?