Applications of Quantum Mechanics: Example Sheet 4

Nick Dorey, March 2022

1. The Schrödinger equation for a particle of mass $m$ and charge $q$ in an electromagnetic field is

\[ \frac{i\hbar}{\partial t} \psi = -\frac{\hbar^2}{2m} \left( \nabla - \frac{iq}{\hbar} A \right)^2 \psi + q\phi \psi \]

Under a gauge transformation,

\[ \phi \rightarrow \phi - \frac{\partial \alpha}{\partial t}, \quad A \rightarrow A + \nabla \alpha. \]

Show that, with a suitable transformation of $\psi$, the Schrödinger equation transforms into itself. Show that the probability density $|\psi|^2$ is gauge invariant. Show that the mechanical momentum $\pi = -i\hbar \nabla - qA$ is gauge invariant. What is the physical interpretation of the mechanical momentum?

2. A particle of charge $q$ moving in a magnetic field $B = \nabla \times A = (0, 0, B)$ is described by the Hamiltonian

\[ H = \frac{1}{2m}(p - qA)^2 \]

where $p$ is the canonical momentum. Show that the mechanical momentum $\pi = p - qA$ obeys

\[ [\pi_x, \pi_y] = i\hbar B \]

Define

\[ a = \frac{1}{\sqrt{2q\hbar B}}(\pi_x + i\pi_y) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2q\hbar B}}(\pi_x - i\pi_y) \]

What commutation relations do $a$ and $a^\dagger$ obey? Write the Hamiltonian in terms of $a$ and $a^\dagger$ and hence solve for the spectrum.

3a. Symmetric gauge is defined by $A = \frac{B}{2}(-y, x, 0)$. Confirm that this gives the magnetic field $B = (0, 0, B)$. Show that the Hamiltonian can be written as

\[ H = -\frac{\hbar^2}{2m} \nabla^2 - \frac{qB}{2m} L_z + \frac{q^2B^2}{8m} (x^2 + y^2) \]

where $L_z$ is the component of the angular momentum parallel to $B$. 
b. Show that the operator $a$, defined in Question 4, takes the form

$$a = -i\sqrt{2} \left( l_B \frac{\partial}{\partial \bar{w}} + \frac{w}{4l_B} \right)$$

where $l_B = \sqrt{\hbar/qB}$ is the magnetic length and $w = x + iy$ is a complex coordinate on the plane, with $\partial_{\omega} = \frac{1}{2}(\partial_x + i\partial_y)$ so that $\partial_{\omega}\omega = 0$ and $\partial_{\omega}\bar{\omega} = 1$. Hence show that the state

$$\psi(w) = f(w)e^{-|w|^2/4l_B^2}$$

sits in the lowest Landau level for any holomorphic function $f(w)$.

4. In the presence of a magnetic field $B = (0, 0, B)$, a particle of charge $q$ moves in the $(x, y)$-plane on the trajectory,

$$x(t) = X + R\sin(\omega_B t) \quad \text{and} \quad y(t) = Y + R\cos(\omega_B t)$$

with $\omega_B = qB/m$. Working in symmetric gauge $A = \frac{B}{2}(-y, x, 0)$, show that the centre of mass coordinates can be re-expressed as

$$X = \frac{x}{2} + \frac{py}{m\omega_B} \quad \text{and} \quad Y = \frac{y}{2} - \frac{px}{m\omega_B}$$

Viewed as quantum operators in the Heisenberg representation, show that both $X$ and $Y$ do not change in time. Show that

$$[X, Y] = -il_B^2$$

where $l_B^2 = \hbar/qB$ is the magnetic length. Use the Heisenberg uncertainty relation for $X$ and $Y$ to estimate the number of states $\mathcal{N}$ that can sit in a region of area $A$.

5. A particle of charge $e$ and spin $\frac{1}{2}$ with g-factor $g = 2$ moves in the $(x, y)$-plane in the presence of a magnetic field of the form $B = (0, 0, B)$. Show that the Hamiltonian can be written as

$$H = \frac{1}{2m} Q^2 \quad \text{with} \quad Q = (\pi_x \sigma_x + \pi_y \sigma_y)$$

where $\sigma$ are the Pauli matrices and $\pi$ is the mechanical momentum defined in earlier questions.

Confirm that $Q$ is Hermitian. Show that zero energy states are annihilated by $Q$. Show that $|\psi\rangle$ and $Q|\psi\rangle$ are degenerate and hence deduce that the lowest Landau level
contains half the states of the higher Landau levels. What is the physical interpretation of this? (Hint: consider the effect of Zeeman splitting on Landau levels.)

Working in Landau gauge, $A = (0, B x, 0)$ with $B > 0$, show that zero energy states have spin up and take the form

$$
\psi = \begin{pmatrix} f(w) e^{-x^2/2l_B^2} \\ 0 \end{pmatrix}
$$

with $w = x + i y$ and $l_B^2 = h/q B$. Show by explicit calculation that there are no zero energy spin down states.

6. The semi-classical equations of motion for an electron of charge $-e$ and energy $E(k)$ moving in a magnetic field $B$ are

$$
\frac{\hbar k}{dt} = -e v \times B \quad \text{and} \quad v = \frac{1}{\hbar} \frac{\partial E}{\partial k}
$$

Show that, in momentum space, the electrons orbit the Fermi surface $E(k_F)$ in a plane perpendicular to $B$. Show that the orbit of the electron in position space, projected onto the plane perpendicular to $B$, traces out the perimeter of a cross-section of the Fermi surface. [Hint: Consider the evolution of the position $r_\perp = r - (\hat{B} \cdot r)\hat{B}$, perpendicular to the magnetic field.]

A free electron has $E(k) = \hbar^2 k^2/2m$. Use the results above to show that, for any value of $k \cdot B$, the electron orbits the Fermi surface with cyclotron frequency $\omega_B = eB/m$. Show that the time taken to orbit the Fermi surface can be written as

$$
T = \frac{2\pi}{\omega_B} = \frac{\hbar^2}{eB} \left. \frac{\partial A(E)}{\partial E} \right|_{k_B}
$$

where $A(E)$ is the cross-sectional area of the Fermi surface with Fermi energy $E$.

[An Aside: This formula is important because it holds for Fermi surfaces of any shape.]

7. A one-dimensional crystal comprises a chain of atoms of mass $m$ equally spaced by a distance $a$ when in equilibrium. The forces between the atoms are such that the effective spring constants are alternately $\lambda$ and $\alpha \lambda$. Show that the dispersion relation for phonons has the form

$$
\omega_{\pm}(k)^2 = \frac{\lambda}{m} \left[ (1 + \alpha) \pm \sqrt{1 + 2\alpha \cos 2ka + \alpha^2} \right]
$$

where the wavenumber $k$ satisfies $-\pi/2a \leq k \leq \pi/2a$. What is the speed of sound in this crystal?