

# Applications of Quantum Mechanics: Example Sheet 4

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1. The semi-classical equations of motion for an electron of charge  $-e$  and energy  $E(\mathbf{k})$  moving in a magnetic field  $\mathbf{B}$  are

$$\hbar \frac{d\mathbf{k}}{dt} = -e\mathbf{v} \times \mathbf{B} \quad \text{and} \quad \mathbf{v} = \frac{1}{\hbar} \frac{\partial E}{\partial \mathbf{k}}$$

Show that, in momentum space, the electrons orbit the Fermi surface  $E(k_F)$  in a plane perpendicular to  $\mathbf{B}$ . Show that the orbit of the electron in position space, projected onto the plane perpendicular to  $\mathbf{B}$ , traces out the perimeter of a cross-section of the Fermi surface. [*Hint: Consider the evolution of the position  $\mathbf{r}_\perp = \mathbf{r} - (\hat{\mathbf{B}} \cdot \mathbf{r})\hat{\mathbf{B}}$ , perpendicular to the magnetic field.*]

A free electron has  $E(k) = \hbar^2 k^2 / 2m$ . Use the results above to show that, for any value of  $\mathbf{k} \cdot \mathbf{B}$ , the electron orbits the Fermi surface with cyclotron frequency  $\omega_B = eB/m$ . Show that the time taken to orbit the Fermi surface can be written as

$$T = \frac{2\pi}{\omega_B} = \frac{\hbar^2}{eB} \left. \frac{\partial A(E)}{\partial E} \right|_{\mathbf{k} \cdot \mathbf{B}}$$

where  $A(E)$  is the cross-sectional area of the Fermi surface with Fermi energy  $E$ .

[An Aside: This formula is important because it holds for Fermi surfaces of any shape.]

2. A one-dimensional crystal comprises a chain of atoms of mass  $m$  equally spaced by a distance  $a$  when in equilibrium. The forces between the atoms are such that the effective spring constants are alternately  $\lambda$  and  $\alpha\lambda$ . Show that the dispersion relation for phonons has the form

$$\omega_\pm(k)^2 = \frac{\lambda}{m} \left[ (1 + \alpha) \pm \sqrt{1 + 2\alpha \cos 2ka + \alpha^2} \right]$$

where the wavenumber  $k$  satisfies  $-\pi/2a \leq k \leq \pi/2a$ . What is the speed of sound in this crystal?

3. The Schrödinger equation for a particle of mass  $m$  and charge  $q$  in an electromagnetic field is

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \left( \nabla - \frac{iq}{\hbar} \mathbf{A} \right)^2 \psi + q\phi\psi$$

Under a gauge transformation,

$$\phi \rightarrow \phi - \frac{\partial \alpha}{\partial t} \quad , \quad \mathbf{A} \rightarrow \mathbf{A} + \nabla \alpha .$$

Show that, with a suitable transformation of  $\psi$ , the Schrödinger equation transforms into itself. Show that the probability density  $|\psi|^2$  is gauge invariant. Show that the *mechanical momentum*  $\boldsymbol{\pi} = -i\hbar \nabla - q\mathbf{A}$  is gauge invariant. What is the physical interpretation of the mechanical momentum?

4. A particle of charge  $q$  moving in a magnetic field  $\mathbf{B} = \nabla \times \mathbf{A} = (0, 0, B)$  is described by the Hamiltonian

$$H = \frac{1}{2m} (\mathbf{p} - q\mathbf{A})^2$$

where  $\mathbf{p}$  is the canonical momentum. Show that the mechanical momentum  $\boldsymbol{\pi} = \mathbf{p} - q\mathbf{A}$  obeys

$$[\pi_x, \pi_y] = iq\hbar B$$

Define

$$a = \frac{1}{\sqrt{2q\hbar B}} (\pi_x + i\pi_y) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2q\hbar B}} (\pi_x - i\pi_y)$$

What commutation relations do  $a$  and  $a^\dagger$  obey? Write the Hamiltonian in terms of  $a$  and  $a^\dagger$  and hence solve for the spectrum.

5a. Symmetric gauge is defined by  $\mathbf{A} = \frac{B}{2}(-y, x, 0)$ . Confirm that this gives the magnetic field  $\mathbf{B} = (0, 0, B)$ . Show that the Hamiltonian can be written as

$$H = -\frac{\hbar^2}{2m} \nabla^2 + \frac{qB}{2m} L_z + \frac{q^2 B^2}{8m} (x^2 + y^2)$$

where  $L_z$  is the component of the angular momentum parallel to  $B$ .

b. Show that the operator  $a$ , defined in Question 4, takes the form

$$a = -i\sqrt{2} \left( l_B \frac{\partial}{\partial \bar{w}} + \frac{w}{4l_B} \right)$$

where  $l_B = \sqrt{\hbar/qB}$  is the magnetic length and  $w = x + iy$  is a complex coordinate on the plane, with  $\partial_{\bar{w}} = \frac{1}{2}(\partial_x + i\partial_y)$  so that  $\partial_{\bar{w}}w = 0$  and  $\partial_{\bar{w}}\bar{w} = 1$ . Hence show that the state

$$\psi(w) = f(w)e^{-|w|^2/4l_B^2}$$

sits in the lowest Landau level for any holomorphic function  $f(w)$ .

6. In the presence of a magnetic field  $\mathbf{B} = (0, 0, B)$ , a particle of charge  $q$  moves in the  $(x, y)$ -plane on the trajectory,

$$x(t) = X + R \sin(\omega_B t) \quad \text{and} \quad y(t) = Y + R \cos(\omega_B t)$$

with  $\omega_B = qB/m$ . Working in symmetric gauge  $\mathbf{A} = \frac{B}{2}(-y, x, 0)$ , show that the centre of mass coordinates can be re-expressed as

$$X = \frac{x}{2} + \frac{p_y}{m\omega_B} \quad \text{and} \quad Y = \frac{y}{2} - \frac{p_x}{m\omega_B}$$

Viewed as quantum operators in the Heisenberg representation, show that both  $X$  and  $Y$  do not change in time. Show that

$$[X, Y] = -il_B^2$$

where  $l_B^2 = \hbar/qB$  is the magnetic length. Use the Heisenberg uncertainty relation for  $X$  and  $Y$  to estimate the number of states  $\mathcal{N}$  that can sit in a region of area  $A$ .

7. A particle of charge  $e$  and spin  $\frac{1}{2}$  with g-factor  $g = 2$  moves in the  $(x, y)$ -plane in the presence of a magnetic field of the form  $\mathbf{B} = (0, 0, B)$ . Show that the Hamiltonian can be written as

$$H = \frac{1}{2m} Q^2 \quad \text{with} \quad Q = (\pi_x \sigma_x + \pi_y \sigma_y)$$

where  $\sigma$  are the Pauli matrices and  $\boldsymbol{\pi}$  is the mechanical momentum defined in earlier questions.

Confirm that  $Q$  is Hermitian. Show that zero energy states are annihilated by  $Q$ . Show that  $|\psi\rangle$  and  $Q|\psi\rangle$  are degenerate and hence deduce that the lowest Landau level

contains half the states of the higher Landau levels. What is the physical interpretation of this? (Hint: consider the effect of Zeeman splitting on Landau levels.)

Working in Landau gauge,  $\mathbf{A} = (0, Bx, 0)$  with  $B > 0$ , show that zero energy states have spin up and take the form

$$\psi = \begin{pmatrix} f(w) e^{-x^2/2l_B^2} \\ 0 \end{pmatrix}$$

with  $w = x + iy$  and  $l_B^2 = \hbar/qB$ . Show by explicit calculation that there are no zero energy spin down states.

**8\***. Near the Dirac point, an electron in graphene is described by the Hamiltonian

$$H = v_F Q$$

with  $v_F$  the Fermi velocity and  $Q$  the operator defined in Question 5. Working in Landau gauge  $\mathbf{A} = (0, Bx, 0)$ , show that the Landau level spectrum is given by

$$E = \pm v_F \sqrt{2\hbar q B} \sqrt{n} \quad n = 0, 1, 2, \dots$$