Mathematical Tripos Part II Christopher Thomas

STATISTICAL PHYSICS Examples 2 Lent 2024

1. Derive the Sackur-Tetrode formula for the entropy of an ideal monatomic gas with $Z = Z_1^N/N!$. Show that the entropy is not extensive if we fail to include the $N!$ factor.

2. A particle moving in one dimension has Hamiltonian

$$H = \frac{p^2}{2m} + \lambda q^4$$

where λ is a constant. Show that the heat capacity for a gas of N such particles is $C_V = \frac{3}{4}Nk_B$. Explain why the heat capacity is the same regardless of whether the particles are distinguishable or indistinguishable.

3. Consider an ultra-relativistic gas of N spinless particles obeying the energy-momentum relation $E = pc$, where c is the speed of light. [Here ultra-relativistic means that $pc \gg mc^2$ where m is the mass of the particle]. Show that the canonical partition function is given by

$$Z(V,T) = \frac{1}{N!}\left[\frac{V}{\pi^2}\left(\frac{k_BT}{\hbar c}\right)^3\right]^N$$

Hence show that an ultra-relativistic gas also obeys the familiar ideal gas law $pV = Nk_BT$.

4. * Consider a perfect classical gas of diatomic molecules for which each molecule has a magnetic moment m aligned along its axis. Let there be a magnetic field B, so that each molecule has a potential energy $-mB\cos\theta$ (θ being the angle between the axis of the molecule and the magnetic field). Show that the rotational part of the partition function is $Z_{\text{rot}} = (z_{\text{rot}})^N$ where

$$z_{\text{rot}} = \left[\frac{2I}{\hbar^2 mB\beta^2}\right]\sinh(m\beta B)\,.$$

Evaluate the total magnetisation, $M = -\partial F/\partial B$ and sketch its dependence upon $m\beta B$. Show that, for large $m\beta B$, the average value of the potential energy is $Nk_BT - NmB(1 + 2e^{-2m\beta B} + \ldots)$.

5. A classical gas of atoms in three dimensions is constrained by a wall to move in the $x \geq 0$ region of space. A potential

$$V(x) = \frac{1}{2}\alpha x^2$$

attracts the atoms to the wall. The atoms are free to move in an area A in the y and z directions. If the gas is at uniform temperature T, show that the number density of atoms varies as

$$\rho(x) = 2N_0\sqrt{\frac{\alpha\beta}{2\pi}}e^{-\alpha\beta x^2/2}$$

where N_0 is the total number of atoms per unit area. By considering the balance of forces on a slab of gas between x and $x + \Delta x$, show that locally the gas continues to obey the ideal gas law. Hence determine the pressure that the gas exerts on the wall.

6. Consider the neutral gas of electrons, protons and hydrogen discussed in Question 11 of Examples Sheet 1. You know from Quantum Mechanics that the hydrogen atom has binding energy $E = -\Delta$ (where $\Delta = 13.6$ eV). Let the number of hydrogen atoms be $N_H = (1-x)N$ and the number of electrons and protons be $N_e = N_p = xN$ with $x \in [0, 1]$. By treating the system as three ideal gases in the grand canonical ensemble, use the equilibrium condition $\mu_H = \mu_e + \mu_p$ to show that

$$\frac{x^2}{1-x} = \frac{V}{N}\left(\frac{m_e m_p}{2\pi\hbar^2 m_H}\right)^{3/2}(k_B T)^{3/2}\, e^{-\Delta/k_B T}\,.$$

7. Compute the equation of state, including the second virial coefficient, for a gas of non-interacting hard discs of radius $r_0/2$ in two dimensions. [For hard discs, the potential is infinite if the discs overlap and zero otherwise.]

8. Suppose that the potential between two atoms in a gas is $U(r) = \alpha/r^n$ where $n > 3$ and $\alpha > 0$. Find the second virial coefficient. Evaluate the special case $n = 6$. [For general n, the numerical factor in the integral reduces to a gamma function, but it simplifies for $n = 6$.]

9. Determine the density of states for non-relativistic particles in $d = 2$ and $d = 1$ dimensions. [You should find that the density is constant for particles on a plane and decreases with energy for particles on a line.]

10. In many experiments, particles are not trapped in a box, but instead in a quadratic potential. In d-dimensions, the potential energy felt by a single particle is

$$V(\vec{x}) = \frac{1}{2}\sum_{i=1}^{d} \omega_i^2\, x_i^2\,.$$

Compute the density of states $g(E)$ in $d = 3$ and $d = 2$ dimensions assuming that E is large enough that the spectrum may be treated as a continuum (i.e. $E \gg \hbar\omega_i$). [Hint: First determine $G(E)$, the number of states with energy less than E.]

11. Consider blackbody radiation at temperature T. Show that the average number of photons grows as T^3. What is the mean photon energy? What is the most likely energy of a photon?

12. A black body at temperature T absorbs all the radiation that falls on it and emits radiation at the rate $\mathcal{E} = \sigma T^4$ per unit area, where σ is Stefan's constant. A black, perfectly conducting sphere orbits a star of radius 7×10^5 km at a distance of 1.5×10^8 km. The star radiates like a black body at temperature 6000 K. Can you make a gin and tonic on this sphere?

13. * The purpose of this question is to explain why the microwave background radiation still has a black body spectrum, even though it has not been in thermal equilibrium with matter since very early in the universe's history.

    Consider a region of volume V in the cosmos containing black body radiation of temperature T. Suppose the cosmos expands (slowly) by a scale factor α, so that the wavevector $\vec{k}$ and angular frequency ω of each electromagnetic radiation mode are rescaled by $1/\alpha$. Explain why you should expect the mean number of photons in each mode not to change. Show that the Planck distribution is valid after the expansion provided the temperature is also rescaled by $1/\alpha$.

    Verify, from the formula for the entropy of black body radiation, that the entropy in the expanded volume is the same as the original entropy, thus confirming the adiabatic character of the expansion.

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