1. A Wigner crystal is a triangular lattice of electrons in a two-dimensional plane. The longitudinal vibration modes of this crystal are bosons with dispersion relation $\omega = \alpha \sqrt{|k|}$ for small $|k|$. Show that, at low temperatures, these modes provide a contribution to the heat capacity that scales as $C \sim T^4$.

2. A system has two energy levels with energies 0 and $\epsilon$. These can be occupied by (spinless) fermions from a particle and heat bath with temperature $T$ and chemical potential $\mu$. The fermions are non-interacting. Show that there are four possible microstates, and show that the grand partition function is $Z(T, \mu) = 1 + \xi + \xi e^{-\epsilon/T} + \xi^2 e^{-\epsilon/T}$ where $\xi = e^{\mu/T}$. Verify that $Z$ factorises into a product of partition functions for the two energy levels separately. Evaluate the mean occupation number of the state of energy $\epsilon$, and show that this is compatible with the result of the calculation of the mean energy of the system using the Fermi-Dirac distribution. How could you take account of fermion interactions?

3. In an ideal Fermi gas the mean occupation number of the single particle state $|r\rangle$ is $n_r$. Show that the entropy
$$S = \frac{\partial}{\partial T} (T \log Z) \bigg|_{\mu}$$
can be written as
$$S = -\sum_r \left[ (1 - n_r) \log(1 - n_r) + n_r \log n_r \right].$$
Find the corresponding expression for an ideal Bose gas.

4. The Gamma function is defined as
$$\Gamma(\nu) = \int_0^\infty y^{\nu-1} e^{-y} \, dy \quad (\nu > 0).$$
Verify that $\Gamma(\nu + 1) = \nu \Gamma(\nu)$, and that $\Gamma(\nu + 1) = \nu!$ for integer $\nu$. Establish the values
$$\Gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2 \sqrt{\pi}}, \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{4 \sqrt{\pi}}.$$

5. As a simple model of a semiconductor, suppose that there are $N$ bound electron states, each having energy $-\Delta < 0$, which are filled at zero temperature. At non-zero temperature some electrons are excited into the conduction band, which is a continuum of positive energy states. The density of these states is given by $g(\epsilon) \, d\epsilon =$
\[ A \sqrt{\epsilon} \, d\epsilon \text{ where } A \text{ is a constant.} \]

Show that at temperature \( T \) the mean number \( n_c \) of excited electrons is determined by the pair of equations

\[ n_c = \frac{N}{e^{(\mu+\Delta)/T} + 1} = \int_0^\infty \frac{g(\epsilon) \, d\epsilon}{e^{(\epsilon-\mu)/T} + 1}. \]

Show also that, if \( n_c \ll N \), \( T \ll \Delta \) and \( e^{\mu/T} \ll 1 \), then

\[ 2\mu \approx -\Delta + T \log \left[ \frac{2N}{A \sqrt{\pi T^3}} \right]. \]

6. Let \( f(\epsilon) \) be a smooth function, independent of \( T \), bounded at \( \epsilon = 0 \), and not growing too fast for large \( \epsilon \). Establish the (asymptotic) expansion for \( \mu > 0 \) and small \( T \)

\[ \int_0^\infty \frac{f(\epsilon) \, d\epsilon}{e^{(\epsilon-\mu)/T} + 1} \sim \int_0^{\mu} f(\epsilon) \, d\epsilon + \frac{\pi^2}{6} T^2 f'(\mu) + \ldots. \]

Hint: Split integral into ranges \( \epsilon < \mu \) and \( \epsilon > \mu \). For \( \epsilon < \mu \), separate off the integral of \( f \). Make change of variable \( \epsilon - \mu = Tx \), use binomial expansion for denominator, exploit values of Gamma function and Riemann zeta function.

7. Consider an almost degenerate Fermi gas of electrons with spin degeneracy \( g_s = 2 \).

At high temperatures, show that the equation of state is given by

\[ PV = NT \left( 1 + \frac{\lambda^3 N}{4\sqrt{2g_s}V} + \ldots \right) \]

where \( \lambda \) is the thermal length of the electrons. At low temperatures, show that the chemical potential is

\[ \mu = \epsilon_F \left( 1 - \frac{\pi^2}{12} \left( \frac{T}{\epsilon_F} \right)^2 + \ldots \right), \]

and the mean energy is

\[ E = \frac{3N\epsilon_F}{5} \left( 1 + \frac{5\pi^2}{12} \left( \frac{T}{\epsilon_F} \right)^2 + \ldots \right). \]

8. Consider a gas of non-interacting ultra-relativistic electrons, whose mass may be neglected. Find an integral for the grand potential \( \Phi \). Show that \( 3PV = E \). Show that at zero temperature \( PV^{4/3} = \text{const} \). Show that at high temperatures \( E = 3NT \), and the equation of state coincides with that of a classical ultra-relativistic gas.

9. A crude non-relativistic model of a white dwarf star consists of a sphere of radius \( R \) of free electrons at zero temperature together with a sufficient number of protons to make the star electrically neutral. Determine the energy \( E_{\text{el}} \) of all the electrons. Assuming the gravitational energy of the star is given by \( E_{\text{grav}} = -\gamma M^2/R \), where \( M \) is the total mass of the star, show that if the state of equilibrium of the star is given by minimising the total energy \( E_{\text{grav}} + E_{\text{el}} \) then \( R \) is proportional to \( M^{-1/4} \). What justification can be given for neglecting the proton zero-point energy?
10. Use the fact that the density of states is constant in $d = 2$ dimensions to show that Bose-Einstein condensation does not occur no matter how low the temperature.

11. Consider $N$ non-interacting, non-relativistic bosons, each of mass $m$, in a cubic box of side $L$. Show that the transition temperature scales as $T_c \sim N^{2/3}/mL^2$ and the 1-particle energy levels scale as $\epsilon_n \propto 1/mL^2$. Show that when $T < T_c$, the mean occupancy of the first few excited 1-particle states is large, but not as large as $O(N)$.

12. Consider an ideal gas of bosons whose density of states is given by $g(\epsilon) = Ce^{\alpha-1}$ for some constants $C$ and $\alpha > 1$. Derive an expression for the critical temperature $T_c$, below which the gas experiences Bose-Einstein condensation.