1. A Wigner crystal is a triangular lattice of electrons in a two-dimensional plane. The longitudinal vibration modes of this crystal are bosons with dispersion relation $\omega = \alpha \sqrt{|k|}$ for small $|k|$. Show that, at low temperatures, these modes provide a contribution to the heat capacity that scales as $C \sim T^4$.

2. Use the fact that the density of states is constant in $d = 2$ dimensions to show that Bose-Einstein condensation does not occur no matter how low the temperature.

3. Consider $N$ non-interacting, non-relativistic bosons, each of mass $m$, in a cubic box of side $L$. Show that the transition temperature scales as $T_c \sim N^{2/3}/mL^2$ and the 1-particle energy levels scale as $E_n \propto 1/mL^2$. Show that when $T < T_c$, the mean occupancy of the first few excited 1-particle states is large, but not as large as $O(N)$.

4. Consider an ideal gas of bosons whose density of states is given by $g(E) = CE^{\alpha-1}$ for some constants $C$ and $\alpha > 1$. Derive an expression for the critical temperature $T_c$, below which the gas experiences Bose-Einstein condensation.

In BEC experiments, atoms are confined in magnetic traps which can be modelled by a quadratic potential of the type discussed in Question 10 of Example Sheet 2. Determine $T_c$ for bosons in a three-dimensional trap. Show that bosons in a two-dimensional trap will condense at suitably low temperatures. In each case, calculate the number of particles in the condensate as a function of $T < T_c$.

5. A system has two energy levels with energies 0 and $\epsilon$. These can be occupied by (spinless) fermions from a particle and heat bath with temperature $T$ and chemical potential $\mu$. The fermions are non-interacting. Show that there are four possible microstates, and show that the grand partition function is

$$Z(\mu, V, T) = 1 + z + ze^{-\beta\epsilon} + z^2 e^{-2\beta\epsilon}$$

where $z = e^{\beta\mu}$. Verify that $Z$ factorises into a product of partition functions for the two energy levels separately. Evaluate the mean occupation number of the state of energy $\epsilon$, and show that this is compatible with the result of the calculation of the mean energy of the system using the Fermi-Dirac distribution. How could you take account of fermion interactions?
6. In an ideal Fermi gas the mean occupation number of the single particle state \( |r\rangle \) is \( n_r \). Show that the entropy

\[
S = \frac{\partial}{\partial T} (k_B T \ln Z)_{\mu,V}
\]

can be written as

\[
S = -k_B \sum_r [(1 - n_r) \ln(1 - n_r) + n_r \ln n_r].
\]

Find the corresponding expression for an ideal Bose gas.

Show that \((\Delta n_r)^2 = n_r(1 - n_r)\) for the ideal Fermi gas. Comment on this result, especially for very low \(T\). What is the corresponding result for an ideal Bose gas?

7. As a simple model of a semiconductor, suppose that there are \(N\) bound electron states, each having energy \(-\Delta < 0\), which are filled at zero temperature. At non-zero temperature some electrons are excited into the conduction band, which is a continuum of positive energy states. The density of these states is given by

\[
g(E) dE = A \sqrt{E} dE
\]

where \(A\) is a constant. Show that at temperature \(T\) the mean number \(n_c\) of excited electrons is determined by the pair of equations

\[
n_c = \frac{N}{e^{(\mu+\Delta)/k_B T} + 1} = \int_0^\infty \frac{g(E) dE}{e^{(E-\mu)/k_B T + 1}}.
\]

Show also that, if \(n_c \ll N\), \(k_B T \ll \Delta\) and \(e^{\mu/k_B T} \ll 1\), then

\[
2\mu \approx -\Delta + k_B T \ln \left[ \frac{2N}{A \sqrt{\pi(k_B T)^3}} \right].
\]

8. * Let \(f(E)\) be a smooth function, independent of \(T\), bounded at \(E = 0\), and not growing too fast for large \(E\). Establish the (asymptotic) expansion for \(\mu > 0\) and small \(T\)

\[
\int_0^\infty \frac{f(E) dE}{e^{(E-\mu)/T + 1}} \sim \int_0^\mu f(E) dE + \frac{\pi^2}{6} T^2 f'(\mu) + \ldots.
\]

[Hint: Split integral into ranges \(E < \mu\) and \(E > \mu\). For \(E < \mu\), separate off the integral of \(f\). Make change of variable \(E - \mu = Tx\), use binomial expansion for denominator, exploit values of Gamma function and Riemann zeta function.]
9. Consider an almost degenerate Fermi gas of electrons with spin degeneracy $g_s = 2$. At high temperatures, show that the equation of state is given by

$$pV = Nk_B T \left( 1 + \frac{\lambda^3 N}{4\sqrt{2} g_s V} + \ldots \right)$$

where $\lambda$ is the thermal wavelength of the electrons. At low temperatures, show that the chemical potential is

$$\mu = E_F \left( 1 - \frac{\pi^2}{12} \left( \frac{k_B T}{E_F} \right)^2 + \ldots \right),$$

and the mean energy is

$$E = \frac{3NE_F}{5} \left( 1 + \frac{5\pi^2}{12} \left( \frac{k_B T}{E_F} \right)^2 + \ldots \right).$$

10. Consider a gas of non-interacting ultra-relativistic electrons, whose mass may be neglected. Find an integral for the grand potential $\Phi$. Show that $3pV = E$. Show that at zero temperature $pV^{4/3} = \text{const}$. Show that at high temperatures $E = 3Nk_B T$, and the equation of state coincides with that of a classical ultra-relativistic gas.

11. A crude non-relativistic model of a white dwarf star consists of a sphere of radius $R$ of free electrons at zero temperature together with a sufficient number of protons to make the star electrically neutral. Determine the energy $E_{el}$ of all the electrons. Assuming the gravitational energy of the star is given by $E_{grav} = -\gamma M^2 / R$, where $M$ is the total mass of the star, show that if the state of equilibrium of the star is given by minimising the total energy $E_{grav} + E_{el}$ then $R$ is proportional to $M^{-\frac{1}{3}}$. What justification can be given for neglecting the proton zero-point energy?

Comments and corrections to c.e.thomas@damtp.cam.ac.uk.