Mathematical Tripos Part II Christopher Thomas

STATISTICAL PHYSICS Examples 3 Lent 2024

1. A *Wigner crystal* is a triangular lattice of electrons in a two-dimensional plane. The longitudinal vibration modes of this crystal are bosons with dispersion relation $\omega = \alpha\sqrt{|\mathbf{k}|}$ for small $|\mathbf{k}|$. Show that, at low temperatures, these modes provide a contribution to the heat capacity that scales as $C \sim T^4$.

2. Use the fact that the density of states is constant in $d = 2$ dimensions to show that Bose-Einstein condensation does not occur no matter how low the temperature.

3. Consider N non-interacting, non-relativistic bosons, each of mass m, in a cubic box of side L. Show that the transition temperature scales as $T_c \sim N^{2/3}/mL^2$ and the 1-particle energy levels scale as $E_n \propto 1/mL^2$. Show that when $T < T_c$, the mean occupancy of the first few excited 1-particle states is large, but not as large as $\mathcal{O}(N)$.

4. Consider an ideal gas of bosons whose density of states is given by $g(E) = CE^{\alpha-1}$ for some constants C and $\alpha > 1$. Derive an expression for the critical temperature T_c, below which the gas experiences Bose-Einstein condensation.

   In BEC experiments, atoms are confined in magnetic traps which can be modelled by a quadratic potential of the type discussed in Question 10 of Example Sheet 2. Determine T_c for bosons in a three-dimensional trap. Show that bosons in a two-dimensional trap will condense at suitably low temperatures. In each case, calculate the number of particles in the condensate as a function of $T < T_c$.

5. A system has two energy levels with energies 0 and ϵ. These can be occupied by (spinless) fermions from a particle and heat bath with temperature T and chemical potential μ. The fermions are non-interacting. Show that there are four possible microstates, and show that the grand partition function is

$$\mathcal{Z}(\mu, V, T) \ = \ 1 + z + ze^{-\beta\epsilon} + z^2 e^{-\beta\epsilon}$$

   where $z = e^{\beta\mu}$. Verify that $\mathcal{Z}$ factorises into a product of partition functions for the two energy levels separately. Evaluate the mean occupation number of the state of energy ϵ, and show that this is compatible with the result of the calculation of the mean energy of the system using the Fermi-Dirac distribution. How could you take account of fermion interactions?

6. In an ideal Fermi gas the mean occupation number of the single particle state $|r\rangle$ is n_r. Show that the entropy

$$S = \frac{\partial}{\partial T}(k_B T \ln \mathcal{Z})_{\mu,V}$$

can be written as

$$S = -k_B \sum_r \left[(1-n_r)\ln(1-n_r) + n_r \ln n_r\right].$$

Find the corresponding expression for an ideal Bose gas.

Show that $(\Delta n_r)^2 = n_r(1-n_r)$ for the ideal Fermi gas. Comment on this result, especially for very low T. What is the corresponding result for an ideal Bose gas?

7. As a simple model of a semiconductor, suppose that there are N bound electron states, each having energy $-\Delta < 0$, which are filled at zero temperature. At non-zero temperature some electrons are excited into the conduction band, which is a continuum of positive energy states. The density of these states is given by $g(E)\,dE = A\sqrt{E}\,dE$ where A is a constant. Show that at temperature T the mean number n_c of excited electrons is determined by the pair of equations

$$n_c = \frac{N}{e^{(\mu+\Delta)/k_B T} + 1} = \int_0^\infty \frac{g(E)\,dE}{e^{(E-\mu)/k_B T} + 1}.$$

Show also that, if $n_c \ll N$, $k_B T \ll \Delta$ and $e^{\mu/k_B T} \ll 1$, then

$$2\mu \approx -\Delta + k_B T \ln\left[\frac{2N}{A\sqrt{\pi(k_B T)^3}}\right].$$

8. * Let $f(E)$ be a smooth function, independent of T, bounded at $E = 0$, and not growing too fast for large E. Establish the (asymptotic) expansion for $\mu > 0$ and small T

$$\int_0^\infty \frac{f(E)\,dE}{e^{(E-\mu)/T} + 1} \sim \int_0^\mu f(E)\,dE + \frac{\pi^2}{6} T^2 f'(\mu) + \dots .$$

[Hint: Split integral into ranges $E < \mu$ and $E > \mu$. For $E < \mu$, separate off the integral of f. Make change of variable $E - \mu = Tx$, use binomial expansion for denominator, exploit values of Gamma function and Riemann zeta function.]

9. Consider an almost degenerate Fermi gas of electrons with spin degeneracy $g_s = 2$. At high temperatures, show that the equation of state is given by

$$pV = Nk_BT\left(1 + \frac{\lambda^3 N}{4\sqrt{2}g_sV} + \dots\right)$$

where λ is the thermal wavelength of the electrons. At low temperatures, show that the chemical potential is

$$\mu = E_F\left(1 - \frac{\pi^2}{12}\left(\frac{k_BT}{E_F}\right)^2 + \dots\right),$$

and the mean energy is

$$E = \frac{3NE_F}{5}\left(1 + \frac{5\pi^2}{12}\left(\frac{k_BT}{E_F}\right)^2 + \dots\right).$$

10. Consider a gas of non-interacting ultra-relativistic electrons, whose mass may be neglected. Find an integral for the grand potential Φ. Show that $3pV = E$. Show that at zero temperature $pV^{4/3} = \text{const}$. Show that at high temperatures $E = 3Nk_BT$, and the equation of state coincides with that of a classical ultra-relativistic gas.

11. A crude non-relativistic model of a white dwarf star consists of a sphere of radius R of free electrons at zero temperature together with a sufficient number of protons to make the star electrically neutral. Determine the energy E_{el} of all the electrons. Assuming the gravitational energy of the star is given by $E_{\text{grav}} = -\gamma M^2/R$, where M is the total mass of the star, show that if the state of equilibrium of the star is given by minimising the total energy $E_{\text{grav}} + E_{\text{el}}$ then R is proportional to $M^{-\frac{1}{3}}$. What justification can be given for neglecting the proton zero-point energy?

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