

1. A *Wigner crystal* is a triangular lattice of electrons in a two-dimensional plane. The longitudinal vibration modes of this crystal are bosons with dispersion relation $\omega = \alpha\sqrt{|\mathbf{k}|}$ for small $|\mathbf{k}|$. Show that, at low temperatures, these modes provide a contribution to the heat capacity that scales as $C \sim T^4$.
2. Use the fact that the density of states is constant in $d = 2$ dimensions to show that Bose-Einstein condensation does not occur no matter how low the temperature.
3. Consider N non-interacting, non-relativistic bosons, each of mass m , in a cubic box of side L . Show that the transition temperature scales as $T_c \sim N^{2/3}/mL^2$ and the 1-particle energy levels scale as $E_n \propto 1/mL^2$. Show that when $T < T_c$, the mean occupancy of the first few excited 1-particle states is large, but not as large as $\mathcal{O}(N)$.
4. Consider an ideal gas of bosons whose density of states is given by $g(E) = CE^{\alpha-1}$ for some constants C and $\alpha > 1$. Derive an expression for the critical temperature T_c , below which the gas experiences Bose-Einstein condensation.

In BEC experiments, atoms are confined in magnetic traps which can be modelled by a quadratic potential of the type discussed in Question 10 of Example Sheet 2. Determine T_c for bosons in a three-dimensional trap. Show that bosons in a two-dimensional trap will condense at suitably low temperatures. In each case, calculate the number of particles in the condensate as a function of $T < T_c$.

5. A system has two energy levels with energies 0 and ϵ . These can be occupied by (spinless) fermions from a particle and heat bath with temperature T and chemical potential μ . The fermions are non-interacting. Show that there are four possible microstates, and show that the grand partition function is

$$\mathcal{Z}(\mu, V, T) = 1 + z + ze^{-\beta\epsilon} + z^2e^{-\beta\epsilon}$$

where $z = e^{\beta\mu}$. Verify that \mathcal{Z} factorises into a product of partition functions for the two energy levels separately. Evaluate the mean occupation number of the state of energy ϵ , and show that this is compatible with the result of the calculation of the mean energy of the system using the Fermi-Dirac distribution. How could you take account of fermion interactions?

6. In an ideal Fermi gas the mean occupation number of the single particle state $|r\rangle$ is n_r . Show that the entropy

$$S = \frac{\partial}{\partial T} (k_B T \ln \mathcal{Z})_{\mu, V}$$

can be written as

$$S = -k_B \sum_r [(1 - n_r) \ln(1 - n_r) + n_r \ln n_r].$$

Find the corresponding expression for an ideal Bose gas.

Show that $(\Delta n_r)^2 = n_r(1 - n_r)$ for the ideal Fermi gas. Comment on this result, especially for very low T . What is the corresponding result for an ideal Bose gas?

7. As a simple model of a semiconductor, suppose that there are N bound electron states, each having energy $-\Delta < 0$, which are filled at zero temperature. At non-zero temperature some electrons are excited into the conduction band, which is a continuum of positive energy states. The density of these states is given by $g(E) dE = A\sqrt{E} dE$ where A is a constant. Show that at temperature T the mean number n_c of excited electrons is determined by the pair of equations

$$n_c = \frac{N}{e^{(\mu+\Delta)/k_B T} + 1} = \int_0^\infty \frac{g(E) dE}{e^{(E-\mu)/k_B T} + 1}.$$

Show also that, if $n_c \ll N$, $k_B T \ll \Delta$ and $e^{\mu/k_B T} \ll 1$, then

$$2\mu \approx -\Delta + k_B T \ln \left[\frac{2N}{A\sqrt{\pi}(k_B T)^3} \right].$$

8. * Let $f(E)$ be a smooth function, independent of T , bounded at $E = 0$, and not growing too fast for large E . Establish the (asymptotic) expansion for $\mu > 0$ and small T

$$\int_0^\infty \frac{f(E) dE}{e^{(E-\mu)/T} + 1} \sim \int_0^\mu f(E) dE + \frac{\pi^2}{6} T^2 f'(\mu) + \dots$$

[Hint: Split integral into ranges $E < \mu$ and $E > \mu$. For $E < \mu$, separate off the integral of f . Make change of variable $E - \mu = Tx$, use binomial expansion for denominator, exploit values of Gamma function and Riemann zeta function.]

9. Consider an almost degenerate Fermi gas of electrons with spin degeneracy $g_s = 2$. At high temperatures, show that the equation of state is given by

$$pV = Nk_B T \left(1 + \frac{\lambda^3 N}{4\sqrt{2}g_s V} + \dots \right)$$

where λ is the thermal wavelength of the electrons. At low temperatures, show that the chemical potential is

$$\mu = E_F \left(1 - \frac{\pi^2}{12} \left(\frac{k_B T}{E_F} \right)^2 + \dots \right),$$

and the mean energy is

$$E = \frac{3NE_F}{5} \left(1 + \frac{5\pi^2}{12} \left(\frac{k_B T}{E_F} \right)^2 + \dots \right).$$

10. Consider a gas of non-interacting ultra-relativistic electrons, whose mass may be neglected. Find an integral for the grand potential Φ . Show that $3pV = E$. Show that at zero temperature $pV^{4/3} = \text{const}$. Show that at high temperatures $E = 3Nk_B T$, and the equation of state coincides with that of a classical ultra-relativistic gas.
11. A crude non-relativistic model of a white dwarf star consists of a sphere of radius R of free electrons at zero temperature together with a sufficient number of protons to make the star electrically neutral. Determine the energy E_{el} of all the electrons. Assuming the gravitational energy of the star is given by $E_{\text{grav}} = -\gamma M^2/R$, where M is the total mass of the star, show that if the state of equilibrium of the star is given by minimising the total energy $E_{\text{grav}} + E_{\text{el}}$ then R is proportional to $M^{-1/3}$. What justification can be given for neglecting the proton zero-point energy?

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