Electrodynamics

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To Claire

Thanks for your patience

Contents

Introduction

These notes are based on the course "Electrodynamics" given by Dr. M. J. Perry in Cambridge in the Michælmas Term 1997. These typeset notes have been produced mainly for my own benefit but seem to be officially supported. The recommended books for this course are discussed in the bibliography.

A word or two about the philosophy of these notes seem in order. They are based in content on the lectures given, but I have felt free to expand and contract various details, as well as to clarify explanations and improve the narrative flow. Errors in content are (hopefully) mine and mine alone but I accept no responsibility for your use of these notes.

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Richard Cameron	*Analysis*	Hugh Osborn	*Proofreading*
Claire Gough	*Proofreading*	Malcolm Perry	*Accomodation*
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Although these notes are free of charge anyone who wishes to express their thanks could send a couple of bottles of interesting beer to Y1 Burrell's Field, Grange Road.

Paul Metcalfe
4th December 1997

Chapter 1

Point of departure

This is a review of terminology and results from Special Relativity and Electromagnetism (possibly rewritten in a more grown-up way).

1.1 Maxwell's Equations

These are :

$$\begin{aligned} \operatorname{div}\mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \operatorname{curl}\mathbf{E} &= -\dot{\mathbf{B}} \\ \operatorname{div}\mathbf{B} &= 0 \\ \operatorname{curl}\mathbf{B} &= \mu_0\mathbf{j} + \mu_0\epsilon_0\dot{\mathbf{E}}. \end{aligned}$$

ρ is the charge density. ϵ_0 is called the permittivity of free space. It is not a fundamental constant but merely determines units. Similarly, μ_0 is the permeability of free space and merely determines units. μ_0 and ϵ_0 satisfy $\mu_0\epsilon_0 = c^{-2}$, where c is the speed of light (and a fundamental constant). In familiar units $c \approx 2.997 \times 10^8\ ms^{-1}$, but we will choose units such that $c = 1$.[1] Dimensional analysis can replace c in any derived formulae.

1.2 Electrostatics

This is the case where there is no current and a time independent charge distribution. Then Maxwell's equations reduce to $\operatorname{div}\mathbf{E} = \frac{\rho}{\epsilon_0}$ and $\operatorname{curl}\mathbf{E} = 0$. We will assume $\mathbf{B} = 0$, but it does not affect the equations for the electric field.

The electric field due to a point charge q_1 is $\mathbf{E} = \frac{q_1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$. To measure the electric field we can take a charge q_2 and measure the force $\mathbf{F} = q_2\mathbf{E}$ on it.

[1] Despite the fact that the Schedules mandate SI units. Exam questions will be set such that $c = 1$.

1.2.1 Coulomb's Law

The force between two point charges is

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{1}{r^2}$$

directed on the line between the centres. It is repulsive for two like charges.

A point charge can be regarded as a charge distribution which is a delta function: $\rho = q_1\delta(\mathbf{r})$. To find the electric field due to a distribution of charges we can use linear superposition to find $\mathbf{E}$ everywhere. As $\operatorname{curl}\mathbf{E} = 0$ we can introduce the electrostatic potential ϕ such that $\mathbf{E} = -\nabla\phi$. Then $\nabla^2\phi = \frac{-\rho}{\epsilon_0}$. We can solve this using a Green's function, that is a function $G(\mathbf{r},\mathbf{r}')$ such that $\nabla_r^2 G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r},\mathbf{r}')$. We can see that $G(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|}$. Then

$$\phi(\mathbf{r}) = -\frac{1}{\epsilon_0}\int \mathrm{d}^3 r'\, G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}').$$

Proof. Firstly, we see that

$$\begin{aligned}\nabla_r^2\phi &= -\frac{1}{\epsilon_0}\int \mathrm{d}^3 r'\, \nabla_r^2 G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\\ &= -\frac{1}{\epsilon_0}\int \mathrm{d}^3 r'\, \delta(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\\ &= \frac{-\rho(\mathbf{r})}{\epsilon_0}.\end{aligned}$$

Then we merely note that solutions to Poisson's equation are unique. □

1.2.2 Multipole expansion

Suppose we have a charge distribution in a region B as shown.

The multipole expansion of the potential is what happens to a general expression for ϕ if $|\mathbf{r}| \gg |\mathbf{r}'|$. We expand $\frac{1}{|\mathbf{r}-\mathbf{r}'|}$ using the binomial expansion.

$$\begin{aligned}\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= (r^2 - 2r_i r_i' + r'^2)^{-\frac{1}{2}}\\ &= \frac{1}{r}\left(1 - \frac{2r_i r_i}{r^2} + \frac{r'^2}{r^2}\right)^{-\frac{1}{2}}\\ &= \frac{1}{r}\left(1 + \frac{r_i r_i'}{r^2} - \frac{1}{2}\frac{r'^2}{r^2} + \frac{3}{2}\frac{(r_i r_i')^2}{r^4} + \dots\right).\end{aligned}$$

We substitute into the general expression for ϕ to obtain

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r}\int_B \mathrm{d}^3 r'\,\rho(\mathbf{r}')\left(1 + \frac{r_i r_i'}{r^2} + \frac{1}{2r^4}\left(3r_i r_j r_i' r_j' - r^2\delta_{ij} r_i' r_j'\right) + \dots\right).$$

This is an expansion of ϕ in inverse powers of r. The term in r^{-l} is referred to as the 2^l-pole term.

When $l = 0$ we have a monopole. If $Q = \int \mathrm{d}^3 r' \, \rho(\mathbf{r}')$ then $\phi = \frac{Q}{4\pi\epsilon_0 r}$, which is usually called the Coulomb term.

When $l = 1$ we have a dipole. Let $d_i = \int d^3 r' \, \rho(\mathbf{r}') r'_i$ (the dipole moment). Then $\phi = \frac{1}{4\pi\epsilon_0 r^3} r_i d_i$.

When $l = 2$ we have a quadrupole and the contribution to ϕ is $\frac{1}{4\pi\epsilon_0 r^5} r_i r_j Q_{ij}$, where Q_{ij} is the quadrupole moment and $Q_{ij} = \frac{1}{2} \int \mathrm{d}^3 r' \, \rho(\mathbf{r}') \left(3 r'_i r'_j - \delta_{ij} r'^2 \right)$. Q_{ij} is a symmetric tracefree tensor, as

$$Q_{ij}\delta_{ij} = \frac{1}{2} \int \mathrm{d}^3 r' \, \rho(\mathbf{r}') \left(3 r'_i r'_j \delta_{ij} - \delta_{ij}\delta_{ij} r'^2 \right) = 0.$$

It has 5 independent components. In general the r^{-l} term has $2l + 1$ independent components. When $l = 3$ we have the octopole moment and when $l = 4$ the hexadecapole moment, but these become increasingly cumbersome.

1.3 Special Relativity

Special relativity has two postulates:

1. The laws of nature are the same in any inertial frame.

2. The speed of light is independent of the speed of its source.

This leads us to consider Minkowski space, viz. $x^\mu = (t, \mathbf{x}) = (t, x^i)$. μ runs from 0 to 3 and i runs from 1 to 3. These are inertial co-ordinates. If a particle is at rest at $\mathbf{x} = 0$ at $t = 0$ then in remains at rest at $\mathbf{x} = 0$ for all time. t is then the proper time for that particle — what a clock sitting on the particle would measure.

We can relate the physics in one inertial frame to another by Lorentz transformations. Suppose that one has a second frame moving with velocity v in the x direction relative to the first frame. Then we have new inertial co-ordinates

$$\begin{aligned} t' &= \gamma(v)(t - vx) \\ x' &= \gamma(v)(x - vt) \\ y' &= y \\ z' &= z, \end{aligned}$$

where $\gamma(v) = \frac{1}{\sqrt{1-v^2}}$. This can be written $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, where $\Lambda^\mu{}_\nu$ is the matrix form of the Lorentz transformation, in this case:

$$\begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

To find the Lorentz transformation of arbitrary motion described by a unit vector $\mathbf{n}$ then we consider $R^{-1}(\mathbf{n})\Lambda R(\mathbf{n})$, where $R(\mathbf{n})$ is a rotation to move $\mathbf{n}$ into the x

direction.

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R_3 & \\ 0 & & & \end{pmatrix}$$

with R_3 an ordinary spatial rotation. We can make Lorentz transformations look like rotations — if we put $\gamma(v) = \cosh\phi$, then $v\gamma v = \sinh\phi$, which implies that $v = \tanh\phi$. This is sometimes called a hyperbolic rotation.

We define a distance $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ between two infinitesimally separated points. This can be shown to be invariant under Lorentz transforms. If $\mathrm{d}t = 0$ then it reduces to the ordinary Euclidean metric on $\mathbb{R}^3$.

If $\mathrm{d}s^2 > 0$ we say the two points are spacelike separated, if $\mathrm{d}s^2 < 0$ they are timelike separated and if $\mathrm{d}s^2 = 0$ they are null, or lightlike separated. The interior of the light cone has $\mathrm{d}s^2 < 0$ and the exterior $\mathrm{d}s^2 > 0$.

Proper distance is defined for spacelike separated events to be $\mathrm{d}s$ and proper time $\mathrm{d}\tau$ for timelike separated events by $\mathrm{d}\tau^2 = -\mathrm{d}s^2$. The invariance of $\mathrm{d}\tau$ means that the time seen by a clock sitting on some object can be computed in any inertial frame.

It is easier to write $\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$, where $\eta_{\mu\nu}$ is called the metric tensor and is

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We also have the inverse metric $\eta^{\mu\nu}$ defined by $\eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_\rho$. The matrix form of $\eta^{\mu\nu}$ is (obviously?)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Given two vectors x^μ and y^μ we want a scalar product that should be invariant under Lorentz transformations, and $S = \eta_{\mu\nu}x^\mu y^\nu$ will do the trick. This gives us the idea of defining covectors by $x_\nu = \eta_{\nu\mu}x^\mu$ and then $S = x_\nu y^\nu$. Then there is the inverse operation $\eta^{\mu\nu}x_\nu = \eta^{\mu\nu}\eta_{\nu\rho}x^\rho = x^\mu$.

We ask ourselves how a covector transforms, and we obtain $x_\nu \mapsto x'_\nu = \Lambda_\nu{}^\mu x_\mu$, where $\Lambda_\nu{}^\mu = \eta_{\nu\rho}\Lambda^\rho{}_\sigma\eta^{\sigma\mu}$, or in matrix form

$$\begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for the transform we looked at earlier. We see that $(\Lambda^\mu{}_\nu)^{-1} = \Lambda_\nu{}^\mu$ and

$$\begin{aligned} x_\mu y^\mu \mapsto x'_\mu y'^\mu &= \Lambda_\mu{}^\rho x_\rho \Lambda^\mu{}_\sigma y^\sigma \\ &= \Lambda_\mu{}^\rho \Lambda^\mu{}_\sigma x_\rho y^\sigma \\ &= \delta^\rho_\sigma x_\rho y^\sigma \\ &= x_\sigma y^\sigma, \end{aligned}$$

which is, on the whole, a good thing. This is analogous to rotations in $\mathbb{R}^3$ preserving the metric, the rotations being classified by $R^T R = 1$ and $R_{ij}\delta_{jk}R_{kl} = \delta_{il}$ excluding reflections. A Lorentz transform preserves the Minkowski space metric and $\Lambda_\sigma{}^\mu \Lambda_\tau{}^\nu \eta_{\mu\nu} = \eta_{\sigma\tau}$. To see the equivalence multiply both sides by $\eta^{\tau\lambda}$ to get (eventually) $\delta_\sigma^\lambda = \delta_\sigma^\lambda$.

The Lorentz transforms are defined by $\Lambda_\mu{}^\nu \Lambda_\sigma{}^\tau \eta_{\nu\tau} = \eta_{\mu\sigma}$ (the group $SO(3,1)$) with no spatial reflections and preservation of time, giving the Lorentz group.

Chapter 2

The relativistic theory of electromagnetism

We start with the Lorentz force law, $\mathbf{F} = e\mathbf{E} + e\mathbf{v} \wedge \mathbf{B}$ and seek to generalise it. Non-relativistically we have

$$m\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = e\left(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}\right)^i,$$

and we already know the equation of motion for a free relativistic particle,

$$m\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = 0.$$

We also recall the non-relativistic velocity 4-vector,

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = u^\mu = (1, \mathbf{u}),$$

and we know that $\mathrm{d}\tau^2$ is Lorentz invariant and hence $-1 = u^\mu u^\nu \eta_{\mu\nu}$. We guess a force law

$$m\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = eF^\mu{}_\nu \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}.$$

We see immediately that by the quotient theorem, $F^\mu{}_\nu$ must be a tensor. We also know that this equation must be true in any inertial frame, and so is always true. We take the non-relativistic case, where $\frac{\mathrm{d}}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}t}$ to find what $F^\mu{}_\nu$ must be.

$$eE_x + e(v_y B_z - v_z B_y) = eF^1{}_0\frac{\mathrm{d}t}{\mathrm{d}\tau} + eF^1{}_1\frac{\mathrm{d}x}{\mathrm{d}\tau} + eF^1{}_2\frac{\mathrm{d}y}{\mathrm{d}\tau} + eF^1{}_3\frac{\mathrm{d}z}{\mathrm{d}\tau}$$

We thus identify $F^1_0 = E_x$, $F^1{}_1 = 0$, $F^1{}_2 = B_z$ and $F^1{}_3 = -B_y$. We repeat this process with y and z, and lower the index, giving us

$$\eta_{\mu\lambda}F^\lambda{}_\nu = F_{\mu\nu} = \begin{pmatrix} E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}.$$

We note that the spatial part of this is antisymmetric, and since we treat space and time equivalently, we can finally define

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}.$$

This is the electromagnetic field (strength) tensor, or Maxwell tensor. The only puzzle is what the time component of the relativistic Lorentz equation represents. It is $m\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = eF^0{}_i \frac{\mathrm{d}x^i}{\mathrm{d}\tau}$. We note that $u^\mu = \gamma(1, \mathbf{v})$ and that γ is a kind of relativistic energy, giving $\frac{\mathrm{d}}{\mathrm{d}\tau}(m\gamma) = e\mathbf{v}.\mathbf{E}$, which we know as "the rate of change of energy equals the rate of doing work by the electric field".

2.0.1 Relativistic motion in constant electric field

We consider a constant electric field in the x direction. We have a particle with charge e which starts at rest at the origin. The non-relativistic case is $m\ddot{x} = eE$, which gives $x = \frac{eE}{2m}t^2$ and $\dot{x} = \frac{eE}{m}t$, which eventually exceeds 1!

In the relativistic case, $\dot{f} \equiv \frac{\mathrm{d}f}{\mathrm{d}\tau}$.

Relativistically $m\ddot{y} = m\ddot{z} = 0$, which are trivial. We also have the equations $m\ddot{x} = eF^x{}_0\dot{t} = eE\dot{t}$ and $m\ddot{t} = eE\dot{x}$.

We integrate these once and use the initial conditions to get $m\dot{t} = eEx + C_!$ and $m\dot{x} = eEt$. (Note that we set $\tau = 0$ at $t = 0$.) Integrating again we get

$$x(\tau) = A\sinh\frac{eE\tau}{m} + B\cosh\frac{eE\tau}{m} - \frac{mC_1}{eE}$$
$$t(\tau) = A\cosh\frac{eE\tau}{m} + B\sinh\frac{eE\tau}{m} + \tilde{C}.$$

We have the boundary conditions $x = \dot{x} = 0$ at $\tau = 0$ and $t = 0$ at $\tau = 0$. There is a temptation to put $\dot{t} = 0$ at $\tau = 0$ — but this is inconsistent, as $-1 = u^\mu u_\mu = (-\dot{t}^2 + \dot{x}^2)$. Thus we put $\dot{t} = 1$ at $\tau = 0$ — which we could have guessed, we know that at rest, co-ordinate time is the same as proper time. Finally, we get

$$x(\tau) = \frac{m}{eE}\left(\cosh\frac{eE\tau}{m} - 1\right)$$
$$t(\tau) = \frac{m}{eE}\sinh\frac{eE\tau}{m}.$$

To find the velocity we can write $x(t)$ by eliminating τ

$$x(t) = \frac{m}{eE}\left(\left(1 + \frac{e^2E^2t^2}{m^2}\right)^{\frac{1}{2}} - 1\right).$$

We can now find the velocity

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{eE}{m}\frac{t}{\sqrt{1 + \frac{e^2E^2t^2}{m}}}.$$

For small t we have the reassuring $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{eEt}{m}$, but in the large t case we find only that $v \to 1$ from below, and is always less than 1.

2.1 Transformation of $F_{\mu\nu}$

We Lorentz transform (in the x direction with velocity v) the tensor $F_{\mu\nu}$ to see how the electric and magnetic fields change under Lorentz transformations.

We know that $F_{\mu\nu} \mapsto F'_{\mu\nu} = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma}$, and we just perform these sums to get:

$$\begin{aligned} E'_x &= E_x \\ E'_y &= \gamma(E_y - vB_z) \\ E'_z &= \gamma(E_z + vB_y) \\ B'_x &= B_x \\ B'_y &= \gamma(B_y + vE_z) \\ B'_z &= \gamma(B_z - vE_y). \end{aligned}$$

These are radically different from what we would expect if there were two electric and magnetic 4-vectors.

2.2 Lorentz invariant scalars

We know that $F_{\mu\nu}$ and $\eta_{\mu\nu}$ are Lorentz invariant, and we can derive some Lorentz scalars from them. The most obvious one is $F_{\mu\nu}\eta^{\mu\nu}$, but as F is antisymmetric and η symmetric this evaluates to zero. A more useful Lorentz scalar is $F_{\mu\nu}F^{\mu\nu} = 2\left(\mathbf{B}^2 - \mathbf{E}^2\right)$.

We can get another Lorentz scalar by introducing the *alternating tensor*, which is defined as

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of } 0123 \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of } 0123 \\ 0 & \text{otherwise.} \end{cases}$$

We can now define the dual field strength tensor, $G^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. We can evaluate this

$$G_{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{pmatrix}.$$

The dual tensor G can be found from F by $\mathbf{E} \mapsto \mathbf{B}$ and $\mathbf{B} \mapsto -\mathbf{E}$, which is sometimes called a duality rotation. We can now define a further Lorentz scalar $F_{\mu\nu}G^{\mu\nu} = -4\,\mathbf{E}.\mathbf{B}$.

2.3 Tensorial form of Maxwell's equations

$\partial_i \equiv \frac{\partial}{\partial x^i}$ and $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$

We start from Maxwell's equations to see what they turn into in tensor notation. We take them in an slightly unusual order, $\operatorname{div}\mathbf{E} = \mu_0\rho$ and $\operatorname{curl}\mathbf{B} = \mu_0\mathbf{j} + \dot{\mathbf{E}}$ and seek to write them as tensor equations.

We note that $E_i = F_{i0} = -F^{0i}$ and $B_i = \frac{1}{2}\epsilon_{ijk}F^{jk}$. The divergence equation becomes $\partial_i F_{i0} = \mu_0 \rho$. The curl equation is

$$\begin{aligned}\frac{1}{2}\epsilon_{ijk}\partial_j \epsilon_{klm} F^{lm} &= \frac{1}{2}\left(\partial_j F_{ij} - \partial_j F_{ji}\right) \\ &= \partial_j F_{ij} = \mu_0 j_i + \partial_0 F_{i0}.\end{aligned}$$

We hope that this is consistent with a tensor equation $\partial_\mu F^{\mu\nu} = X^\nu$. If we study this we see that this works if $X^\mu = -\mu_0(\rho, \mathbf{j})$. Thus two of the Maxwell equations become

$$\partial_\mu F^{\mu\nu} = -\mu_0 j^\nu.$$

where $j^\mu = (\rho, \mathbf{j})$ is the 4-vector (electric) current. This illustrates that a moving charge and a current are just the same thing as j^ν is a four-vector and so is consistent with Lorentz transforms.

Incidentally, we have not lost conservation of charge as $0 \equiv \partial_\nu \partial_\mu F^{\mu\nu} = -\mu_0 \partial_\nu j^\nu$.

We now go after the next two Maxwell equations, $\operatorname{div}\mathbf{B} = 0$ and $\operatorname{curl}\mathbf{E} = -\dot{\mathbf{B}}$. The first is easy; it gives $\partial_i \epsilon_{ijk} F_{jk} = 0$. We guess that this is a component of $\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$, and indeed if we evaluate the spatial components we reproduce the last Maxwell equation.

We can rewrite this in terms of the dual field-strength tensor to get $\partial_\mu G^{\mu\nu} = 0$ — this says that there is no magnetic current. It turns out to be more useful to explicitly antisymmetrize our equation to get

$$\partial_\nu F_{\rho\sigma} + \partial_\rho F_{\sigma\nu} + \partial_\sigma F_{\nu\rho} = 0.$$

2.3.1 Potentials

In the non-relativistic case we know that $\mathbf{E}$ and $\mathbf{B}$ can be derived from potentials:

$$\begin{aligned}\mathbf{E} &= -\operatorname{grad}\phi - \dot{\mathbf{A}} \\ \mathbf{B} &= \operatorname{curl}\mathbf{A}.\end{aligned}$$

It turns out that we can come up with an electromagnetic 4-vector potential, $A^\mu = (\phi, \mathbf{A})$ such that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (simply expand this to see).

We know that $\mathbf{A}$ is not unique in non-relativistic electromagnetism; we can add on the gradient of any scalar function — called a gauge transformation. Similarly, we see that the 4-vector potential A_μ is unique up to $\partial_\mu \Lambda$ for any scalar function Λ. This means we can try to impose extra conditions on A_μ which (partially) prevents gauge transformations (called gauge fixing). The point is to ensure that a given $F_{\mu\nu}$ is the product of a moderately unique A_μ.[1]

A useful covariant gauge is to impose $\partial_\mu A^\mu = 0$. Thus if we have A^μ such that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and we set $A'^\mu = A^\mu + \partial^\mu \Lambda$, then we can have $0 = \partial_\mu A'^\mu = \partial_\mu A^\mu + \partial_\mu \partial^\mu \Lambda$. In principle we can solve $\partial_\mu \partial^\mu \Lambda = -\partial_\mu A^\mu$, so this gauge condition is possible.

If we consider this a little more, we see that A^μ is still non-unique, but only up to a solution of the wave equation, $\partial_\mu \partial^\mu \Lambda = 0$. Solutions of the wave equation are a combination of plane waves, $e^{\imath k_\mu x^\mu}$, where $k^\mu = (\omega, \mathbf{k})$ and $\omega = |\mathbf{k}|$. k^μ is the wave 4-vector or the momentum 4-vector.

[1] What "moderately unique" means has yet to be defined.

If we wish to find $F^{\mu\nu}$ given a collection of charges and currents we solve the equation $\partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu = -\mu_0 j^\nu$. This is where our gauge condition comes in useful, we get $\partial_\mu\partial^\mu A^\nu = -\mu_0 j^\nu$, or $\Box A^\nu = -\mu_0 j^\nu$, where $\partial_\mu\partial^\mu \equiv \Box$ (pronounced "box"). We can now see that $\partial_\nu A^\nu = 0$ is sensible — it is compatible with current conservation.

2.4 Least action principles

2.4.1 Particle motion

For a single free particle in special relativity we have the action

$$I = \int \mathrm{d}\tau\, m\sqrt{-\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu}}.$$

The usual Euler-Lagrange equations $\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) - \frac{\partial\mathcal{L}}{\partial x^\mu} = 0$ give $-m\ddot{x}_\mu = 0$.

For a particle with charge e in a potential A^μ we can generalise this to

$$I = \int \mathrm{d}\tau\, m\sqrt{-\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu}} - eA_\mu \dot{x}^\mu.$$

But can this possibly be gauge invariant — the quantity A_μ appears explicitly? But we see a current resulting from the motion of $j^\mu = e\dot{x}^\mu$. So suppose we make a small gauge transformation such that $\delta A_\mu = \partial_\mu \Lambda$. Then

$$\begin{aligned}\delta \int \mathrm{d}\tau\, j^\mu A_\mu &= \int \mathrm{d}\tau\, j^\mu \partial_\mu \Lambda \qquad \text{and integrating by parts we get}\\ &= -\int \mathrm{d}\tau\, \partial_\mu j^\mu \Lambda = 0 \qquad \text{by conservation of charge.}\end{aligned}$$

This is all somewhat academic if varying x^μ does not give us the Lorentz force law. Using the Euler-Lagrange equations we get

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{-m\dot{x}^\nu \eta_{\mu\nu}}{\sqrt{-\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu}}}\right) - \frac{\mathrm{d}}{\mathrm{d}\tau}(eA_\mu) + e\dot{x}^\nu \partial_\mu A_\nu$$

τ is the proper time so $\dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} = -1$ on the worldline

$$\begin{aligned}&= \frac{\mathrm{d}}{\mathrm{d}\tau}(-m\dot{x}_\mu) - e\dot{x}^\nu \partial_\nu A_\mu + e\dot{x}^\nu \partial_\mu A_\nu\\ &= -m\ddot{x}^\nu + e\dot{x}^\nu F_{\mu\nu}.\end{aligned}$$

This is the Lorentz force law. One can use the action as a quick way of finding the motion of a particle. For a constant electric field in the x direction we get $A^x = -Et$ and all other components of A^μ are zero. To get the motion of the particle we vary the action

$$I = \int \mathrm{d}\tau\, m\sqrt{\dot{t}^2 - \dot{x}^2} + eEt\dot{x}.$$

The Euler-Lagrange equations give the same differential equations for the motion of the particle as before, but more easily.

2.4.2 Field action

We also want to find a Lagrangian $\mathcal{L}$ that reproduces Maxwell's equations, that is if we take

$$I = \int \mathrm{d}^4x\, \mathcal{L},$$

$\delta I = 0$ under variations of something or other must reproduce Maxwell's equations.

$\mathcal{L}$ must be a Lorentz scalar, built out of $F_{\mu\nu}$ (or A_μ) and it must be gauge invariant. The Maxwell equations involve first derivatives of $F_{\mu\nu}$ (or second derivatives of A_μ). So the only real possibilities are varying A_μ and $\mathcal{L}$ must be quadratic in $F_{\mu\nu}$. $\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ gives nothing. The usual choice is

$$\mathcal{L} = -\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} + j_\mu A^\mu.$$

This is gauge invariant. If $\delta A_\mu = \partial_\mu \Lambda$, then if we assume $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ we get $\delta F_{\mu\nu} = \partial_\mu\partial_\nu\Lambda - \partial_\nu\partial_\mu\Lambda = 0$. The $\int j_\nu A^\nu$ part gives 0 as before.

$$\begin{aligned}
I &= -\int \mathrm{d}^4x \left(\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu\right) \qquad \text{and} \\
\delta I &= -\int \mathrm{d}^4x \left(\frac{1}{4\mu_0}\left(\delta F_{\mu\nu}F^{\mu\nu} + F_{\mu\nu}\delta F^{\mu\nu}\right) - j^\mu \delta A_\mu\right) \\
&= -\int \mathrm{d}^4x \left(\frac{1}{2\mu_0}\delta F_{\mu\nu}F^{\mu\nu} - j^\nu \delta A_\nu\right) \\
&= -\int \mathrm{d}^4x \left(\frac{1}{2\mu_0}\left(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu\right)F^{\mu\nu} - j^\nu \delta A_\nu\right) \\
&= -\int \mathrm{d}^4x \left(\frac{1}{\mu_0}\left(\partial_\mu \delta A_\nu\right)F^{\mu\nu} - \delta A_\nu j^\nu\right) \\
&= \int \mathrm{d}^4x\, \delta A_\nu \left(\tfrac{1}{\mu_0}\partial_\mu F^{\mu\nu} + j^\nu\right).
\end{aligned}$$

We perform the last line by integrating by parts and assuming that boundary conditions are all zero. As we can arbitrarily vary A_μ we must have $\frac{1}{\mu_0}\partial_\mu F^{\mu\nu} = -j^\nu$. The other Maxwell equation is automatic as we have assumed that $F_{\mu\nu}$ is derived from a potential A_μ. This least action principle requires that there is no magnetic current.

Chapter 3

Energy - Momentum Tensor

3.1 Definition

We seek a relativistic form of the field energy. We define the (stress-)energy tensor

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu\sigma}F^{\nu}{}_{\sigma} - \tfrac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right).$$

We note in passing that this is a symmetric tensor. Note that the T^{00} component is $\frac{1}{2\mu_0}\left(E^2+B^2\right)$, which reproduces the non-relativistic energy density. To interpret the rest of the components of $T^{\mu\nu}$ we recall Poynting's theorem.

Poynting's Theorem. Let D be a region in space. Then the rate of change of energy in D is

$$\begin{aligned}
\frac{1}{2\mu_0}\frac{\partial}{\partial t}\int_D \mathbf{E}^2 + \mathbf{B}^2 \,\mathrm{d}V &= \frac{1}{\mu_0}\int_D \mathbf{E}.\dot{\mathbf{E}} + \mathbf{B}.\dot{\mathbf{B}}\,\mathrm{d}V \\
&= \frac{1}{\mu_0}\int_D \mathbf{E}.\left(\operatorname{curl}\mathbf{B} - \mu_0\mathbf{j}\right) - \mathbf{B}.\operatorname{curl}\mathbf{E}\,\mathrm{d}V \\
&= -\int_D \mathbf{j}.\mathbf{E}\,\mathrm{d}V + \frac{1}{\mu_0}\int_D \mathbf{E}.\operatorname{curl}\mathbf{B} - \mathbf{B}.\operatorname{curl}\mathbf{E}\,\mathrm{d}V \\
&= -\int_D \mathbf{j}.\mathbf{E}\,\mathrm{d}V - \int_{\partial D}\mathbf{N}.\mathrm{d}\mathbf{S},
\end{aligned}$$

where we have introduced the Poynting vector $\mathbf{N} = \frac{1}{\mu_0}\mathbf{E}\wedge\mathbf{B}$. The Poynting vector is the energy flux.

By performing the sum we find that $T^{0k} = N^k$. We find that

$$T^{\mu\nu} = \left(\begin{array}{c|c} \text{Energy density} & \text{Energy flux} \\ \hline \text{Energy flux} & \text{“Stress”} \end{array}\right).$$

We want to evaluate the "stress" part of this tensor.

$$
\begin{aligned}
T_{ij} &= \frac{1}{\mu_0}\left(F_{i\mu}F_j{}^{\mu} - \frac{1}{4}\eta_{ij}F^{\rho\sigma}F_{\rho\sigma}\right) \\
&= \frac{1}{\mu_0}\left(F_{i0}F_j{}^0 + F_{ik}F_j{}^k - \frac{1}{4}\delta_{ij}\left(-2E^2 + 2B^2\right)\right) \\
&= \frac{1}{\mu_0}\left(-E_iE_j + \epsilon_{ikl}B_l\epsilon_{jkm}B_m + \frac{1}{2}\delta_{ij}E^2 - \frac{1}{2}\delta_{ij}B^2\right) \\
&= \frac{1}{\mu_0}\left(-E_iE_j + \frac{1}{2}\delta_{ij}E^2 - B_iB_j + \frac{1}{2}\delta_{ij}B^2\right).
\end{aligned}
$$

This is the Maxwell stress tensor, and can be thought of as the pressure of the electromagnetic field.

3.1.1 Conservation of energy-momentum

We compute the divergence of $T^{\mu\nu}$.

$$
\begin{aligned}
\partial_\nu T^{\mu\nu} &= \frac{1}{\mu_0}\partial_\nu\left(F^{\mu\sigma}F^{\nu}{}_{\sigma} - \frac{1}{4}\eta^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}\right) \\
&= \frac{1}{\mu_0}\left(\left(\partial_\nu F^{\mu\sigma}\right)F^{\nu}{}_{\sigma} + F^{\mu\sigma}\left(\partial_\nu F^{\nu}{}_{\sigma}\right) - \frac{1}{2}\left(\partial^\mu F^{\rho\sigma}\right)F_{\rho\sigma}\right) \\
&= -F^{\mu\sigma}j_\sigma + \frac{1}{\mu_0}F_{\rho\sigma}\left(-\frac{1}{2}\partial^\mu F^{\rho\sigma} + \partial_\rho F^{\mu\sigma}\right) \\
&= -F^{\mu\sigma}j_\sigma - \frac{1}{2\mu_0}F_{\rho\sigma}\left(\partial^\mu F^{\rho\sigma} + \partial^\sigma F^{\mu\rho} + \partial^\rho F^{\sigma\mu}\right) \\
&= -F^{\mu\sigma}j_\sigma \qquad \text{by Maxwell's equations.}
\end{aligned}
$$

Thus in the absence of charges/currents, the energy-momentum tensor is conserved. We can evaluate the right hand side of this equation to get

$$\partial_\nu T^{\mu\nu} = \left(-\mathbf{j}.\mathbf{E}, \rho\mathbf{E} + \mathbf{j}\wedge\mathbf{B}\right).$$

The time component of this is the work done by the electromagnetic field and the spatial components give the electric force on a current $\mathbf{j}$ due to $\mathbf{B}$ and on a charge density ρ due to $\mathbf{E}$.

3.2 Plane waves

This has the equation $\Box A^\mu = 0$ in the gauge $\partial_\mu A^\mu = 0$, which has solutions

$$A_\mu = A\epsilon_\mu \exp\left(\imath k_\sigma x^\sigma\right),$$

Note that this is a complex solution, so when we work with $\mathbf{E}$ and $\mathbf{B}$ we must take the real part of the *field*, which corresponds to the imaginary part of A_μ.

ϵ_μ is the polarisation vector, and $k_\sigma = (-\omega, \mathbf{k})$ is the wave 4-vector. ω is the angular frequency, $\mathbf{k}$ is the wave vector and $|\mathbf{k}| = \omega$. A is the amplitude. Imposing the gauge condition requires $\epsilon_\mu k^\mu = 0$, so we have the transversality of the wave.

This does not completely specify the gauge. If we let $\delta A_\mu = \partial_\mu \Lambda$ with $\Lambda = -\imath C e^{\imath k.x}$ then $\epsilon_\mu \mapsto \epsilon_\mu + Ck_\mu$. Since $k_\mu k^\mu = 0$ this preserves the gauge condition. This freedom is usually exploited to put $A^0 = 0$. In this case,

$$A_\mu = (0, A\epsilon e^{\imath k.x}) = (0, A\epsilon e^{\imath(\mathbf{k}.\mathbf{x} - \omega t)}),$$

where ϵ is a spatial vector. The gauge condition gives $\mathbf{k}.\epsilon = 0$.

Finding $\mathbf{E}$ and $\mathbf{B}$ from A_μ is easy:

$$E_i = -F_{0i} = -\partial_0 A_i + \partial_i A_0 = \imath\omega A_i.$$

and

$$B_i = \tfrac{1}{2}\epsilon_{ijk}F_{jk} = \epsilon_{ijk}\partial_j A_k = -\imath\mathbf{k}\wedge\mathbf{A}.$$

The physical fields correspond to the real parts of these quantities.

3.3 Radiation pressure

Suppose that we have a situation as drawn above, with a plane wave propagating in the z direction. Then the electric and magnetic fields are in the y and x directions respectively, with $E_y = \omega A \sin\omega(t-z)$ and $B_x = \omega A\cos\omega(t-z)$.

The rate of flow of momentum per unit area is $|\mathbf{N}|$, where $\mathbf{N}$ is the Poynting vector and this has a time average $\frac{1}{2\mu_0}\omega^2 A^2$. This is coincidentally the same as the energy density. We also evaluate the stress-energy tensor

$$T_{ij} = \frac{1}{\mu_0}\left(\tfrac{1}{2}\delta_{ij}\left(E^2+B^2\right) - E_iE_j - B_iB_j\right).$$

This is clearly diagonal, and evaluating the diagonal components we get $\langle T_{xx}\rangle = 0$, $\langle T_{yy}\rangle = 0$ and $\langle T_{zz}\rangle = \frac{1}{2\mu_0}\omega^2 A^2$. There is a pressure due to the wave, but importantly *it is not isotropic*.

Chapter 4

Solving Maxwell's Equations

4.1 A Green's Function

We hope to find a general expression for A_μ given a time dependent distribution of charges and currents. We will work in the $\partial_\mu A^\mu = 0$ gauge, and so we have to solve the equation $\Box A^\mu = -\mu_0 j^\mu$. We proceed naively and see what happens.

We hope to find a Green's function $G(x, x')$ such that $\Box G = \delta^{(4)}(x, x')$ and so

$$A^\mu(x) = -\mu_0 \int \mathrm{d}^4x'\, G(x, x') j^\mu(x')$$

is a solution of $\Box A^\mu = -\mu_0 j^\mu$. One problem is that $\Box$ is a hyperbolic operator so there exist non-trivial solutions to $\Box\phi = 0$ with $\phi \to 0$ at infinity.

The four dimension Fourier transform is defined by

$$\hat{f}(k) = \int \mathrm{d}^4x\, f(x) e^{-\imath k.x}.$$

The minus sign in the exponential is not arbitrary. If f is a plane wave $f(x) \sim e^{\imath p.x}$ then $\hat{f}(k) = (2\pi)^4 \delta^{(4)}(p-k)$, which is what we want.

We will solve $\Box G(x, x') = \delta^{(4)}(x, x')$ using the Fourier transform. $\hat{G}(k, x') = -k^{-2} e^{-\imath k.x}$ and so defining $z_\mu = x_\mu - x'_\mu = (z^0, \mathbf{z})$ we find

$$G(x, x') = \frac{1}{(2\pi)^4} \int \mathrm{d}^3k \mathrm{d}k^0\, \frac{e^{\imath \mathbf{k}.\mathbf{z}} e^{-\imath k^0 z^0}}{k^{0 2} - \mathbf{k}^2}$$

and note that if we perform the k_0 integral we see that the integrand is singular at $k_0 = \pm |\mathbf{k}|$. We thus need to choose on which contour to perform the integral.

If we consider the retarded Green's function G_{ret}, which we get by integrating along Γ_1, we see that for $z^0 < 0$, $G(x, x') = 0$ as we can close the contour in the upper half plane and apply Cauchy's theorem. For $z^0 > 0$ we have to close the contour in the lower half plane. In doing this we pick up two poles at $\pm|\mathbf{k}|$ and can apply the residue theorem.

The advanced Green's function G_{adv} is obtained by integrating along Γ_2. In this case G is only non-zero for $z^0 > 0$.

The retarded Green's function agrees with intuitive ideas of causality so we use that. All we have to do now is evaluate it.

$$G_{\text{ret}}(x,x') = \theta(z^0)\frac{1}{(2\pi)^4}\int \mathrm{d}^3k\mathrm{d}k^0\,\frac{e^{\imath\mathbf{k}.\mathbf{z}}e^{-\imath k^0 z^0}}{k^{0^2}-\mathbf{k}^2}$$

We close the contour clockwise so we get $\{-2\pi\imath\sum$ residues$\}$ for the k^0 integral, thus

$$G_{\text{ret}}(x,x') = -\frac{2\pi\imath\theta(z^0)}{(2\pi)^4}\int \mathrm{d}^3k\,e^{\imath\mathbf{k}.\mathbf{z}}\left(\frac{e^{-\imath|\mathbf{k}|z^0}-e^{\imath|\mathbf{k}|z^0}}{2\,|\mathbf{k}|}\right).$$

We convert this into spherical polars in $\mathbf{k}$-space: $k_x = k\sin\theta\cos\phi$, $k_y = k\sin\theta\sin\phi$ and $k_z = k\cos\theta$ and so

$$\begin{aligned}G_{\text{ret}}(x,x') &= -\frac{\imath\theta(z^0)}{16\pi^3}\int k^2\mathrm{d}k\,\sin\theta\mathrm{d}\theta\,\mathrm{d}\phi\,\frac{e^{\imath kz\cos\theta}}{k}\left(e^{-\imath kz^0}-e^{\imath kz^0}\right)\\ &= -\frac{\imath\theta(z^0)}{8\pi^2}\int k\mathrm{d}k\,\sin\theta\mathrm{d}\theta\,\frac{e^{\imath kz\cos\theta}}{k}\left(e^{-\imath kz^0}-e^{\imath kz^0}\right)\\ &= \frac{\theta(z^0)}{8\pi^2 z}\int_0^\infty \mathrm{d}k\,\left[e^{\imath kz\cos\theta}\right]_{\theta=0}^{\theta=\pi}\left(e^{-\imath kz^0}-e^{\imath kz^0}\right).\end{aligned}$$

The integrand is even in k, so

$$G_{\text{ret}}(x,x') = \frac{\theta(z^0)}{16\pi^2 z}\int_{-\infty}^{\infty}\mathrm{d}k\,\left(e^{-\imath kz}-e^{\imath kz}\right)\left(e^{-\imath kz^0}-e^{\imath kz^0}\right).$$

Recall that $\int_{-\infty}^{\infty}\mathrm{d}ke^{\imath kx} = 2\pi\delta(x)$. Thus the integral is a combination of four delta functions, but the step function θ kills two of them off and we get

$$G_{\text{ret}}(x,x') = -\frac{\theta(z^0)}{4\pi z}\delta(z-z^0).$$

For the record, $G_{\text{adv}} = \frac{\theta(-z^0)}{4\pi z}\delta(z+z^0)$. We can make our result for G_{ret} look more covariant by recalling that

$$\delta(f(x)) = \sum_i \frac{\delta(x-a_i)}{|f'(a_i)|}$$

where $f(a_i) = 0$. Then as $\delta^{(3)}(z^2) = \delta(|\mathbf{z}|^2 - z^{0^2})$ (z is a four-vector) we have

$$\delta^{(3)}(z^2) = \frac{1}{2\,|\mathbf{z}|}\left(\delta^{(3)}(|\mathbf{z}|-z^0)+\delta^{(3)}(|\mathbf{z}|+z^0)\right)$$

and as the step function removes one of these delta functions we get

$$G_{\text{ret}}(x,x') = -\frac{1}{2\pi}\theta(z^0)\delta^{(3)}(z^2), \qquad z^\mu = x^\mu - x'^\mu.$$

Now suppose we wish to evaluate $A^\mu(x)$ for some current distribution j^μ as shown. We get

$$A^\mu(x) = \frac{\mu_0}{2\pi}\int \mathrm{d}^4x'\, j^\mu(x')\delta^{(3)}\big((x-x')^2\big)\,\theta(x^0 - x'^0).$$

This comes about because we chose G_{ret}. The advanced Green's function gives the reverse. Thus G_{ret} is consistent with our ideas of causality. Other choices of G are not. This choice goes beyond local physics. It is presumably solved by appealing to cosmology or quantum theory.

We also note that the only contributions to $A^\mu(x)$ come from points x' such that $(x-x')^2 = 0$ — that is only when x and x' can be joined by a light ray pointing towards the future of x'.

4.2 The field of a moving charge

Suppose we have a moving charge, with (non-relativistically) $\rho = e\delta^3(\mathbf{x} - (t))$ and therefore $\mathbf{j} = e\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t}\delta^3(\mathbf{x} - (t))$. In the relativistic case we replace $\mathbf{y}(t)$ with $y^\mu(\tau)$ and get $j^\mu = eu^\mu\delta^3(x^i - y^i(t))$. We can use a trick to make this look more covariant,

$$j^\mu = e\int \mathrm{d}\tau\, u^\mu \delta^4(x^\nu - y^\nu(\tau)).$$

Then

$$A^\mu = \frac{e\mu_0}{2\pi}\int \mathrm{d}^4x'\,\mathrm{d}\tau\,\delta\big((x-x')^2\big)\,\theta(x^0 - x'^0)\frac{\mathrm{d}y^\mu}{\mathrm{d}\tau}\delta^4(x^\nu - y^\nu(\tau)),$$

which has the effect of integrating over the backward light cone. This can be evaluated by carrying out the x' integral first. Note that

$$\delta\big((x-y)^2\big) = \frac{\delta\big(|\mathbf{x}-\mathbf{y}| - (x^0-y^0)\big) + \delta\big(|\mathbf{x}-\mathbf{y}| + (x^0-y^0)\big)}{-2(x-y)_\nu\frac{\mathrm{d}y^\nu}{\mathrm{d}\tau}}$$

The second delta function does not contribute (because we are using the retarded Green's function) and so

$$\begin{aligned} A^\mu &= \frac{e\mu_0}{2\pi}\int \mathrm{d}\tau\,\frac{\mathrm{d}y^\mu}{\mathrm{d}\tau}\frac{\delta\big(|\mathbf{x}-\mathbf{y}| - (x^0-y^0)\big)}{-2(x-y)^\nu\frac{\mathrm{d}y_\nu}{\mathrm{d}\tau}} \\ &= \frac{e\mu_0}{4\pi}\left.\frac{\frac{\mathrm{d}y^\mu}{\mathrm{d}\tau}}{\frac{\mathrm{d}y^\nu}{\mathrm{d}\tau}(x-y)_\nu}\right|_{\tau=\tau_0} \end{aligned}$$

where τ_0 is the value of τ on the world-line where the past light cone of x intersects the world line of the particle. This is usually referred to as evaluating at some instant of retarded time. These are the Lienard-Wiechert potentials, and are painful to use in arbitrary relativistic motion.

The result

$$A^\mu = \frac{e\mu_0}{2\pi} \int \mathrm{d}\tau \, \delta\big((x-y)^2\big) \, \theta(x^0 - y^0) \frac{\mathrm{d}y^\mu}{\mathrm{d}\tau}$$

is useful for relativistic motion. To evaluate the fields we need to calculate terms of the form

$$\begin{aligned} \partial^\nu A^\mu &= \frac{e\mu_0}{2\pi} \int \mathrm{d}\tau \, \theta(x^0 - y^0) \frac{\mathrm{d}y^\mu}{\mathrm{d}\tau} \partial^\nu \delta\big((x-y)^2\big) \\ &= \frac{e\mu_0}{2\pi} \int \mathrm{d}\tau \, \theta(x^0 - y^0(\tau)) \frac{\mathrm{d}y^\mu}{\mathrm{d}\tau} \frac{x^\nu - y^\nu(\tau)}{\frac{\mathrm{d}y^\rho}{\mathrm{d}\tau}(x-y)_\rho} \frac{\mathrm{d}}{\mathrm{d}\tau} \delta\big((x-y)^2\big) \end{aligned}$$

To evaluate this we integrate by parts. We take $x^0 \neq y^0$ and thus remove points on the world-line of the particle from consideration. The field is not well defined there. Thus

$$F^{\mu\nu} = -\frac{e\mu_0}{2\pi} \frac{\mathrm{d}}{\mathrm{d}\tau} \left[\frac{(x-y)^\mu \frac{\mathrm{d}y^\nu}{\mathrm{d}\tau} - (x-y)^\nu \frac{\mathrm{d}y^\mu}{\mathrm{d}\tau}}{\frac{\mathrm{d}y^\rho}{\mathrm{d}\tau}(x-y)_\rho} \right]_{\tau=\tau_0} .$$

We write $(x-y)^\mu = (+R, R\mathbf{n})$ where R is the spatial distance $|\mathbf{x} - \mathbf{y}|$ and $\mathbf{n}$ is a unit vector. The plus sign on R comes from the retarded Green's function. We also need the velocity $\mathbf{v} = (\gamma, \gamma\mathbf{v})$ where $\mathbf{v} = \frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t}$. After evaluating this we get

$$\begin{aligned} \mathbf{E} &= \frac{e\mu_0}{2\pi} \left[\frac{\mathbf{n} - \mathbf{v}}{\gamma^2(1 - \mathbf{n}.\mathbf{v})^3 R^2} + \frac{\mathbf{n} \wedge \{(\mathbf{n} - \mathbf{v}) \wedge \dot{\mathbf{v}}\}}{(1 - \mathbf{n}.\mathbf{v})^3 R} \right]_{\text{retarded time}} \\ \mathbf{B} &= \mathbf{n} \wedge \mathbf{E}. \end{aligned}$$

The first term in the expression for $\mathbf{E}$ is just the Coulomb field (put $\mathbf{v} = 0$ to see this). The second term only appears if $\dot{v} \neq 0$ — it depends on the acceleration. Then $\mathbf{E} \sim$ acceleration$R$ and $\mathbf{B} \sim$ acceleration$R$ and are perpendicular. Thus the Poynting vector is $\mathbf{N} \sim \frac{\text{acceleration}^2}{R^2}$. Thus the energy flux out of a large radius sphere $\sim$ acceleration^2 — accelerating particles radiate energy.

For a non-relativistic particle it is somewhat easier. We use the Lienard-Wiechert potentials

$$A^\mu = \frac{e\mu_0}{4\pi} \left. \frac{\frac{\mathrm{d}y^\mu}{\mathrm{d}\tau}}{\frac{\mathrm{d}y^\nu}{\mathrm{d}\tau}(x-y)_\nu} \right|_{\text{retarded time}}$$

and put $x^\mu = (t, \mathbf{x})$, $y^\mu = (t', \mathbf{y})$ with $\mathbf{x} - \mathbf{y} = R\mathbf{n}$ where $\mathbf{n}$ is a unit vector. For non-relativistic motion $\frac{\mathrm{d}y^\mu}{\mathrm{d}\tau} = (1, \mathbf{v})$ and

$$A^\mu = \frac{e\mu_0}{4\pi} \left. \frac{(1, \mathbf{v})}{|\mathbf{x} - \mathbf{y}|} \right|_{\text{at } t' = t - R} .$$

It is straightforward to calculate $\mathbf{E}$ and $\mathbf{B}$ from $\mathbf{B} = \operatorname{curl} \mathbf{A}$ and $\mathbf{E} = -\operatorname{grad} A^0 - \dot{\mathbf{A}}$. We first evaluate $\mathbf{B}$, and note that we are only interested in the $\mathcal{O}(R^{-1})$ terms — to get the radiation at infinity. Thus

$$\mathbf{B} \sim -\frac{e\mu_0}{4\pi R} \mathbf{n} \wedge \dot{\mathbf{v}} \quad \text{and} \quad \mathbf{E} \sim -\frac{e\mu_0}{4\pi R} \mathbf{n} \wedge [\mathbf{n}\dot{\mathbf{v}}] .$$

The Poynting vector $\mathbf{N} = \frac{1}{\mu_0}\mathbf{E} \wedge \mathbf{B}$ evaluates as $\mathbf{N} = \frac{e^2\mu_0}{16\pi^2 R^2}\left(\ddot{\mathbf{y}}\right)^2 \sin^2\theta\, \mathbf{n}$.

The radiation is mainly perpendicular to the direction of the acceleration and is axisymmetric about that axis. $\dot{v}$ determines the time dependence of the radiation and thus the frequency can be found by Fourier transforming $\dot{v}$.

The power radiated (or the total flux of radiation) is $\int \mathbf{N}.\mathbf{dS}$ over a sphere at infinity which we assume is at a large distance from the particle (the celestial sphere). Converting to polars we get

$$\text{flux of radiation} = \frac{e^2\mu_0}{16\pi^2}\left(\ddot{\mathbf{y}}\right)^2 \int \sin^3\theta \,\mathrm{d}\theta\mathrm{d}\phi = \frac{e^2\mu_0}{6\pi}\left(\ddot{\mathbf{y}}\right)^2.$$

This is Larmor's formula.

4.2.1 Radiation reaction

We consider a particle with mass m and charge e moving under an external force $\mathbf{F}_{\text{ext}}$. Assume (naively) Newton's Law $m\ddot{\mathbf{y}} = \mathbf{F}_{\text{ext}}$ and dot this with $\dot{y}$ and integrate to get that the change in kinetic energy equals the work done by the applied force. But we know that this is not true — there are radiative losses at infinity. We therefore guess another force $\mathbf{F}_R$ and propose $m\ddot{\mathbf{y}} = \mathbf{F}_{\text{ext}} + \mathbf{F}_R$. Dotting this with $\dot{y}$ and integrating we see that $\mathbf{F}_R.\dot{\mathbf{y}}$ is the radiative energy loss. Using Larmor's formula and assuming that there is no acceleration at the endpoints of the motion we get $\mathbf{F}_R = \frac{e^2\mu_0}{6\pi}\dddot{\mathbf{y}}$ and derive the Abraham-Lorentz equation

$$m\left(\ddot{\mathbf{y}} - \frac{e^2\mu_0}{6\pi m}\dddot{\mathbf{y}}\right) = \mathbf{F}_{\text{ext}}.$$

This is very odd and leads to embarrassing difficulties. To solve such an equation three initial conditions are needed, position, velocity and acceleration. But if we take $\mathbf{F}_{\text{ext}} = \mathbf{0}$ we see that the solutions are $A + Bt + Ce^{\frac{t}{\tau}}$, where τ is the timescale $\frac{e^2\mu_0}{6\pi\mu}$. This exponential runaway solution is presumed to be unphysical.

If $\mathbf{F}_{\text{ext}}$ is a delta function and we assume the initial conditions $x = \dot{x} = 0$ we have to adjust $\ddot{x}$ to suppress the runaway solution. We see that the particle (if charged) must start accelerating *before* the force is applied. This acausal behaviour is called pre-acceleration and is governed by the timescale τ, which for an electron is approximately 6×10^{-24}s, into the realm of quantum mechanical effects.

4.3 Oscillating Fields

Assume that $j^\mu(\mathbf{x}, t) = j^\mu(\mathbf{x})e^{\imath\omega t}$ with j^μ non-zero only in some domain D.

Using our result for A^μ we get

$$A^\mu(\mathbf{x},t) = \frac{\mu_0}{4\pi} e^{\imath\omega t} \int \mathrm{d}^3x'\, j^\mu(\mathbf{x}') \frac{1}{|\mathbf{x}-\mathbf{x}'|} e^{-\imath\omega|\mathbf{x}-\mathbf{x}'|}.$$

If $R \gg d$ we can expand $\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ in the usual way as

$$\begin{aligned}\frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \frac{1}{R}\left(1 - \frac{2\mathbf{x}.\mathbf{x}'}{R^2} + \frac{|\mathbf{x}'|^2}{R^2}\right)^{-\frac{1}{2}} \\ &= \frac{1}{R}\left(1 + \frac{\mathbf{x}.\mathbf{x}'}{R^2} - \frac{1}{2}\frac{|\mathbf{x}'|^2}{R^2} + \frac{3}{2}\frac{(\mathbf{x}.\mathbf{x}')^2}{R^4}\dots\right).\end{aligned}$$

The $\mathcal{O}(R^{-2})$ and lower terms do not contribute to the radiation and will be omitted. We get

$$A^\mu(\mathbf{x},t) = \frac{\mu_0}{4\pi R} e^{\imath\omega t} \int \mathrm{d}^3x' j^\mu(\mathbf{x}') e^{-\imath\omega|\mathbf{x}-\mathbf{x}'|}.$$

We can perform a similar expansion on $e^{-\imath\omega|\mathbf{x}-\mathbf{x}'|}$ and finally get

$$A^\mu(\mathbf{x},t) = \frac{\mu_0}{4\pi R} e^{\imath\omega(t-R)} \int \mathrm{d}^3x' j^\mu(\mathbf{x}') e^{\imath\omega\frac{\mathbf{x}.\mathbf{x}'}{R}}$$

provided $R \gg \lambda$, the wavelength. Thus the expansion we have derived is valid when $R \gg d, \lambda$. This is called the radiation zone.

Thus at large distances the system appears to be a source of spherical waves. To proceed further we can expand out the phase factor in powers of ω. We get

$$A^\mu(\mathbf{x},t) = \frac{\mu_0}{4\pi R} e^{\imath\omega(t-R)} \int \mathrm{d}^3x'\, j^\mu(\mathbf{x}') \left[1 + \imath\omega\frac{\mathbf{x}.\mathbf{x}'}{R} - \omega^2\frac{(\mathbf{x}.\mathbf{x}')^2}{2R^2} + \dots\right].$$

In the radiation zone when $\omega d \gg 1$ these terms are successively smaller.

Recall that $j^\mu = (\rho, \mathbf{j})$. Then

$$A^0(\mathbf{x},t) = \frac{\mu_0}{4\pi R} e^{\imath\omega(t-R)} \left[Q + \frac{\imath\omega}{R}\mathbf{x}.\mathbf{p} + \dots\right]$$

where $\mathbf{p}$ is the electric dipole moment of the system. Note that $Q = 0$ as the total charge cannot depend on time. For the vector potential,

$$A^i(\mathbf{x},t) = \frac{\mu_0}{4\pi R} e^{\imath\omega(t-R)} \left[\int \mathrm{d}^3x'\, j_i(\mathbf{x}') + \frac{\imath\omega}{R} x_j \int \mathrm{d}^3x'\, x'_j j_i(\mathbf{x}') + \dots\right]$$

We can simplify this by noting (integrate by parts) that

$$\int \mathrm{d}^3x'\, j_i(\mathbf{x}') = -\int \mathrm{d}^3x'\, x'_i \partial_j j_j(\mathbf{x}')$$

and applying the continuity equation, which in this case is $\imath\omega\rho + \operatorname{div}\mathbf{j} = 0$. Thus

$$\int \mathrm{d}^3x'\, j_i(\mathbf{x}') = \imath\omega\mathbf{p}$$

and we get

$$A^0 = \frac{\imath\omega\mu_0}{4\pi R^2} e^{\imath\omega(t-R)} \frac{\mathbf{p}.\mathbf{x}}{R} \quad \text{and} \quad \mathbf{A} = \frac{\imath\omega\mu_0}{4\pi R} e^{\imath\omega(t-R)} \mathbf{p}.$$

We can now calculate $\mathbf{E}$ and $\mathbf{B}$ as

$$\begin{aligned} \mathbf{E} &= \frac{\omega^2\mu_0}{4\pi R^3} e^{\imath\omega(t-R)} \left(R^2\mathbf{p} - (\mathbf{x}.\mathbf{p})\mathbf{x}\right) \\ &= \frac{\omega^2\mu_0}{4\pi R^3} e^{\imath\omega(t-R)} \mathbf{x} \wedge (\mathbf{x} \wedge \mathbf{p}) \quad \text{and} \\ \mathbf{B} &= \frac{\omega^2\mu_0}{4\pi R^2} e^{\imath\omega(t-R)} \mathbf{x} \wedge \mathbf{p}. \end{aligned}$$

The time averaged Poynting vector thus points radially outwards and has magnitude $N = \frac{\mu_0\omega^4|\mathbf{p}|^2}{32\pi^2R^2} \sin^2\theta$ and the average power radiated is therefore $\frac{\mu_0\omega^4|\mathbf{p}|^2}{12\pi}$.

The scattered light has ω^4 dependence times the spectrum of the light. Thus blue light is scattered preferentially to red and the sky appears blue. This also explains the red sun at sunset; since there is more scattering when the angle of the sun is low and the blue light is scattered more.

Chapter 5

Quantum mechanical effects

5.1 Minimal coupling

Consider a particle with charge e, worldline $x^\mu(\tau)$ in an electromagnetic field with potential A^μ. Recall we obtained an action

$$I = -\int \mathrm{d}\tau \, \left(m\sqrt{-\dot{x}^2} - eA_\mu \dot{x}^\mu \right) = \int \mathrm{d}\tau \, \mathcal{L}(x, \dot{x}),$$

where the minus sign is inessential; it just normalizes things nicely. The momentum π^μ conjugate to x^μ is $\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = m\dot{x}^\mu + eA^\mu$, consisting of the mechanical momentum and a modification due to the electromagnetic field. The Hamiltonian $H(x,\pi) = \pi^\mu \dot{x}_\mu - \mathcal{L} = \frac{(\pi - eA)^2}{m}$.

The replacement of p with $\pi = p + eA$ is usually termed "minimal coupling" and corresponds to classical electrodynamics.

In quantum mechanics the momentum p is replaced with a momentum operator $\hat{p}$, and we assume that the momentum operator for charged particles is modified "like the classical momentum", that is $p \mapsto \pi = \hat{p} + eA \mapsto -\imath\hbar\nabla + eA$.

The Schrödinger equation for a static field $(0, \mathbf{A})$ is $\frac{p^2}{2m}\psi = E\psi$, which according to minimal coupling, and turning p into an operator, is $\frac{(-\imath\hbar\nabla + e\mathbf{A})}{2m}\psi = E\psi$.

Since gauge transformations are not supposed to have any physical effect solutions of the Schrödinger equation in one gauge must be solutions in another gauge. If we start with the universal combination $(-\imath\hbar\nabla + e\mathbf{A})\,\psi$, on sending $\mathbf{A} \mapsto \mathbf{A} + \nabla\Lambda$ the universal combination becomes $(-\imath\hbar\nabla + e\mathbf{A} + e\nabla\Lambda)\,\psi'$. This must be invariant (up to a phase factor), and so if $\psi' = \psi e^{-\frac{\imath e\Lambda}{\hbar}}$ we get $(-\imath\hbar\nabla + e\mathbf{A} + e\nabla\Lambda)\,\psi' = e^{-\frac{\imath e\Lambda}{\hbar}}\,(-\imath\hbar\nabla - e\nabla\Lambda + e\mathbf{A} + e\nabla\Lambda)\,\psi$ and the universal combination is invariant (up to a phase factor). Phase should not be too disturbing; the matrix element

$$\int \mathrm{d}^3x \, \psi_1^* \hat{O} \psi_2 \mapsto \int \mathrm{d}^3x \, \psi_1^* e^{\frac{\imath e\Lambda}{\hbar}} \hat{O} \psi_2 e^{-\frac{\imath e\Lambda}{\hbar}}$$

and under all normal circumstances the phase factors cancel; the matrix element is invariant.

This minimal coupling means that the vector potential can give rise to observable physical effects. One which you may have met before is the Aharonov - Bohm effect.

Consider the long, thin solenoid shown, with $\mathbf{B} \neq 0$ inside and $\mathbf{B} = 0$ outside. In classical mechanics, charged particles with be unaffected since $\mathbf{B} = 0$ outside the solenoid.

In quantum mechanics; consider eigenstates of π; states ψ with $(-\imath\hbar\nabla + e\mathbf{A})\,\psi = \pi\psi$. If the phases of the waves on the two paths differ then there will be destructive interference.

Now suppose that a neutral particle has a wavefunction $\psi_0(x)$. For a charged particle the corresponding wavefunction is $\psi(x) = \psi_0(x)\exp\left(-\frac{\imath e}{\hbar}\int_{x_0}^{x}\mathbf{A}.\mathrm{d}\mathbf{l}\right)$. Thus the phase factor (the difference in phase) between the two paths is

$$
\begin{aligned}
e^{-\frac{\imath e}{\hbar}\oint \mathbf{A}.\mathrm{d}\mathbf{l}} &= e^{-\frac{\imath e}{\hbar}\int \mathrm{curl}\,\mathbf{A}.\mathrm{d}\mathbf{S}} \\
&= e^{-\frac{\imath e}{\hbar}\int \mathbf{B}.\mathrm{d}\mathbf{S}} \\
&= e^{-\frac{\imath e}{\hbar}(\text{flux})}.
\end{aligned}
$$

By appropriate choice of the flux Φ we can get as much or as little interference as we want. If $\frac{e}{\hbar}\Phi = \pi$ then there is completely destructive interference; if $\frac{e}{\hbar}\Phi = 2\pi$ then the interference is completely constructive and the solenoid is undetectable. In general if $\Phi = \frac{2\pi n\hbar}{e}$ the solenoid is unobservable. This is an inherently quantum mechanical effect.

One might think that $\mathbf{B} = 0$ outside the solenoid implies that $\mathbf{A} = 0$ outside the solenoid. This is true only if the region is simply connected — which it isn't. We can make a gauge transformation to put $\mathbf{A} = 0$ at a point but because the region is not simply connected we cannot do this everywhere.

This was experimentally verified in the 1960's.

5.2 Conduction

An ordinary conductor looks something like a regular lattice of atoms, with the valence electrons forming an electron gas throughout the material.

An applied $\mathbf{E}$ field moves the gas, but electrons collide with atoms and stop. Suppose they move with an average velocity $\mathbf{v}$. Then the current density is the charge on an electron $\times$ the number density $\times\mathbf{v}$. The mean free path only depends on the geometry, so the current density is $\sigma\mathbf{E}$, with σ the conductivity.

Superconductivity is very different. It was first discovered by Kammerlingh-Onnes in 1905; he noticed that when some metals are cooled to $\approx 4\mathrm{K}$ the electric conductivity became infinite. Nowadays superconductivity is observed in certain materials up to about liquid nitrogen temperatures, $\approx 100\mathrm{K}$.

The fundamental description of superconductivity is due to Bardeen, Cooper and Schreiffer and is in detail beyond this course. The result is that the current is an inherently quantum mechanical effect in which bound states of pairs of electrons behave as bosons rather than as fermions. They have a charge $-2e$ and an effective mass of m (say).

We will examine the Landau-Ginzburg theory. Suppose the charge carriers have a wavefunction $\chi = Re^{\imath\phi}$. We can then interpret the probability current as the flux of these particles. We can evaluate

$$\mathbf{j}_{\text{prob}} = \frac{\hbar}{2\imath m}\left(\chi^*\nabla\chi - (\nabla\chi)^*\,\chi\right) = \frac{\hbar}{m}R^2\nabla\phi.$$

We interpret R^2 as a number density n_s and so we guess an electric current $\mathbf{j} = \frac{q\hbar}{m}\nabla\phi$. However this is not gauge invariant and as the electric current must stay the same under gauge transformations we fix up the equation to get the result (which can be *derived* from the BCS theory)

$$\mathbf{j}_s = \frac{q\hbar n_s}{m}\left(\nabla\phi - \frac{q}{\hbar}\mathbf{A}\right).$$

5.2.1 Meissner effect

Since $\operatorname{div}\mathbf{B} = 0$ lines of $\mathbf{B}$ cannot end. However if one takes a material in a magnetic field and cools it to its superconducting temperature one observes a change in the magnetic field.

We are led to guess that $\mathbf{B} = 0$ inside a superconductor. The above expression for $\mathbf{j}_s$ and the Maxwell equations give $\operatorname{curl}\mathbf{B} = \frac{\mu_0 q\hbar n_s}{m}\left\{\nabla\phi - \frac{q}{\hbar}\mathbf{A}\right\}$. Taking the curl of this we get a differential equation for $\mathbf{B}$:

$$\nabla^2\mathbf{B} = \frac{\mu_0 q^2 n_s}{m}\mathbf{B}.$$

In the region shown this simplifies to $\frac{\partial^2\mathbf{B}}{\partial z^2} = \frac{\mu_0 q^2 n_s}{m}\mathbf{B}$ and so we find that $\mathbf{B} = \mathbf{B}_0 \exp -\sqrt{\frac{\mu_0 q^2 n_s}{m}}z$, taking the negative root since the energy must be bounded.

$\mathbf{B}$ decays exponentially away from the surface on a distance scale $\sqrt{\frac{m}{n_s q^2 \mu_0}}$, which is of the order of atomic size. Thus in practice we have $\mathbf{B} = 0$ inside a superconductor and this is a better definition of a superconductor than saying that it has infinite conductivity.

$\mathbf{A}$ is not necessarily 0, but in order to get a superconducting current we must have $n_s \neq 0$. Landau and Ginzburg tried to construct an analog of the Schrödinger equation which gave this result.

It is easier (as always) to start from an action:

$$
\begin{aligned}
I &= \int \mathrm{d}^3x \ [\text{kinetic energy} - \text{potential energy}] \\
&= \int \mathrm{d}^3x \ \left[-\frac{\hbar^2}{2m} \psi^* \nabla^2 \psi - V \psi^* \psi \right] \\
&= \int \mathrm{d}^3x \ \left[\frac{1}{2m} \left(-\imath\hbar \nabla \psi \right)^* \left(-\imath\hbar \nabla \psi \right) + V \psi^* \psi \right]
\end{aligned}
$$

incorporating the magnetic field via minimal coupling

$$
= \int \mathrm{d}^3x \ \left[\left(-\imath\hbar \nabla \psi - q\mathbf{A}\psi \right)^* \left(-\imath\hbar \nabla \psi - q\mathbf{A}\psi \right) + V \psi^* \psi \right].
$$

This is gauge invariant.

It cannot depend on where we are in the superconductor and so V is constant. We get the familiar Schrödinger equation which has the obvious solution $\psi = 0$ and no other solution independent of $\mathbf{x}$.

Landau and Ginzburg proposed the addition of a term $\frac{1}{2} b \left| \psi \right|^4$ to this action to get

$$
I = \int \mathrm{d}^3x \, \frac{\hbar^2}{4m} \left| \left(\nabla - \frac{\imath q}{\hbar} \mathbf{A} \right) \psi \right|^2 + V \left| \psi \right|^2 + \frac{1}{2} b \left| \psi \right|^4 .
$$

This action can be derived from BCS theory and gives a nonlinear analog of the Schrödinger equation:

$$
-\frac{1}{4m} \left(-\imath\hbar - q\mathbf{A} \right)^2 \psi + V\psi + b \left| \psi \right|^2 \psi = 0.
$$

The current $\mathbf{j}_s = \frac{q\hbar}{2\imath m} \left(\psi^* \nabla \psi - \psi \left(\nabla \psi \right)^* \right) - \frac{2q^2}{m} \mathbf{A} \left| \psi \right|^2$.

We get a non-vanishing spatially independent solution of this "Schrödinger equation" when $V < 0$ and $b > 0$. This occurs when the temperature T is less than some critical temperature T_c; normal matter has $V > 0$.

BCS theory gives $b > 0$ and $V = V_0 \left(T - T_c \right)$.

5.3 Superconducting flux quantisation

Consider a ring of superconducting material as shown.

In the material $\mathbf{B} = 0$ and $\mathbf{j} = 0$. Since $\mathbf{j} \propto \left(\nabla \psi - \frac{q}{\hbar} \mathbf{A} \right)$ we must have $\mathbf{A} = \frac{\hbar}{q} \nabla \phi$ inside the ring.

The magnetic flux through the loop is

$$
\int_{\text{shaded surface}} \mathbf{B}.\mathrm{d}\mathbf{S} = \oint_{\text{boundary loop}} \mathbf{A}.\mathrm{d}\mathbf{l}.
$$

Evaluating this inside the superconductor we get $\frac{\hbar}{q}[\phi]$. As the wavefunction must be single valued this must be $n\frac{2\pi\hbar}{q}$ and since the charge carriers are electron pairs then the flux is quantised in units of $\frac{\pi\hbar}{|e|}$.

If we make a current I flow on the surface of the superconductor then as the flux through the loop is the inductance times the current, and so the flux is quantised we see that the current is quantised.

5.4 Magnetic monopoles

Suppose that a $\mathbf{B}$ field $\frac{\mu_0}{4\pi}P\frac{\hat{\mathbf{r}}}{r^2}$ is possible, by analogy with the Coulomb field in electrostatics.

Using Gauss' Law we have

$$P = \frac{1}{\mu_0}\int_{\text{closed surface}} \mathbf{B}.\mathrm{d}\mathbf{S} = \frac{1}{\mu_0}\int \operatorname{div}\mathbf{B}\,\mathrm{d}V.$$

Thus if $\operatorname{div}\mathbf{B} = 0$ everywhere then $P = 0$ and magnetic charges cannot arise. Thus Maxwell's equations must be modified in order to get this field.

A suitable vector potential $\mathbf{A}$ is $A_\phi = \frac{\mu_0 P}{4\pi}(1+\cos\theta)$ (in spherical polars). We have $|A| = \frac{\mu_0}{4\pi}\frac{(1+\cos\theta)}{r\sin\theta}$. There is a difficulty at $\theta = 0$ for all r.

This singularity on the North axis is called the Dirac string. It can be moved about by gauge transformations; if we have $\mathbf{A} \mapsto \mathbf{A} + \nabla\frac{-\mu_0 P}{2\pi}\phi$ we can put the Dirac string onto the South axis.

Since (by axiom) the observable physics should not depend on the gauge used the string singularity should be unobservable.

We showed earlier that the phase difference between two paths going in front of / behind the string is $\frac{e}{\hbar}\oint \mathbf{A}.\mathrm{d}\mathbf{l}$ and this must be an integer multiple of 2π. Evaluating the integral gives the Dirac quantisation condition $P = n\frac{2\pi\hbar}{\mu_0 e}$.

Chapter 6

Born-Infeld Theory

This material is starred and was included as a fill-in lecture.

Recall that Maxwell's theory is (in the absence of currents) governed by an action

$$I = \frac{1}{\mu_0} \int \mathrm{d}^4 x - \frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

giving the two Maxwell equations $\partial_\mu F_{\rho\sigma} + \partial_\rho F_{\sigma\mu} \partial_\sigma F_{\mu\rho} = 0$ and $\partial_\mu F^{\mu\nu} = 0$. There is a hidden duality symmetry under $F_{\mu\nu} \mapsto \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ of both the action and the equations of motion.

Recall also that electric charges have a radial component $E_r = \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2}$ and that the energy density in the electric field is $\frac{1}{2}\epsilon_0 \left|\mathbf{E}\right|^2$. We can see that the energy density blows up at the origin and also that the total energy in the electric field is infinite.

We also propose a similiar magnetic monopole field $B_r = \frac{P\mu_0}{4\pi} \frac{1}{r^2}$; the energy in this magnetic field is also infinite.

The Born-Infeld theory emerges from string theory. It depends on a parameter b with the dimensions of length. We take a new action;

$$\frac{1}{\mu_0 b^2} \int \mathrm{d}^4 x \left\{ 1 - \sqrt{\left|\det \eta_{\mu\nu} + b F_{\mu\nu}\right|} \right\}$$

and we suppose that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Since we have that, up to a Lorentz transform,

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & -\lambda_2 & 0 \end{pmatrix}$$

we can see that $\det \eta_{\mu\nu} + b\tilde{F}_{\mu\nu} = \det \eta_{\mu\nu} + bF_{\mu\nu}$ and so we can evaluate the action as

$$\frac{1}{\mu_0 b^2} \int \mathrm{d}^4 x \left\{ 1 - \sqrt{1 - \frac{1}{2} b^2 F_{\mu\nu} F^{\mu\nu} - \frac{1}{16} b^4 \left(\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}\right)^2} \right\}.$$

The limit $b \to 0$ (clearly) gives the Maxwell action. Since we are assuming that $F_{\mu\nu}$ is derived from a potential we still have the equation $\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$ and the other equation is $\partial_\mu \mathcal{G}^{\mu\nu} = 0$, the difference being that the equation for $\mathcal{G}^{\mu\nu}$ is a horrible mess:

$$\mathcal{G}^{\mu\nu} = \frac{F^{\mu\nu} - \frac{b^2}{4} F^{\lambda\tau}\varepsilon_{\mu\nu\lambda\tau}F_{\alpha\beta}F_{\gamma\delta}\varepsilon^{\alpha\beta\gamma\delta}}{\sqrt{1 - \frac{b^2}{2}F_{\xi\zeta}F^{\xi\zeta} - \frac{b^4}{16}\left(\varepsilon_{\xi\zeta\chi\varpi}F^{\xi\zeta}F^{\chi\varpi}\right)^2}}$$

(you have no idea how difficult it was to find that many different Greek letters).
We obtain

$$T_{\mu\nu} = \frac{1}{\mu_0}\left\{\mathcal{G}_\mu{}^\lambda F_{\nu\lambda} + \frac{1}{4}\eta_{\mu\nu}\mathcal{L}\right\}$$

where $\mathcal{L}$ is the Lagrangian; $\mathcal{L} = \frac{1}{b^2}\left\{1 - \sqrt{|\det \eta_{\mu\nu} + bF_{\mu\nu}|}\right\}$.

There is a symmetry in these equations under $F_{\mu\nu} \mapsto \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\mathcal{G}^{\rho\sigma}$ and $\mathcal{G}_{\mu\nu} \mapsto -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$: the Lagrangian and equations of motions are invariant. This has the effect of swapping $\mathbf{E}$ and $\mathbf{B}$.

The analog of the electric field of a point charge is an $\mathbf{E}$ field which is purely radial, defined by $A_0 = \phi$ and $E_r = -\nabla_r\phi$. In this case the action reduces to

$$\frac{1}{\mu_0 b^2}\int r^2 \mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\phi \left[1 - \sqrt{1 - b^2\phi_r^2}\right]$$

and variation of this yields

$$\frac{r^2\phi_r}{\sqrt{1 - b^2\phi_r^2}} = \text{const} = a.$$

Solving this for $\phi_r = -E_r$ yields

$$E_r = -\frac{a}{\sqrt{r^4 + a^2b^2}}$$

and so as $r \to \infty$, $E_r \sim -\frac{a}{r^2}$. Thus if we wish to reproduce the Maxwell field for large distances $a = -\frac{Q}{4\pi\epsilon_0}$. Thus as $r \to 0$ we see that $E_r \to \frac{1}{b}$.

The energy density in the electric field is

$$\frac{1}{\mu_0 b^2}\left\{\frac{1}{\sqrt{1 - b^2\mathbf{E}^2}} - 1\right\} = \frac{1}{\mu_0 b^2}\left\{\sqrt{1 + \frac{b^2Q^2}{16\pi^2\epsilon_0 r^4}} - 1\right\}$$

which is singular (but integrably so) at $r = 0$. Performing the integral to find the total energy in the electric field we obtain

$$\frac{4\pi}{3\sqrt{b}}\frac{1}{\Gamma(\frac{3}{4})^2}\left(\frac{Q}{4}\right)^{\frac{3}{2}}\frac{1}{\epsilon_0^{\frac{1}{2}}},$$

which is noticably finite.

This theory also has magnetic monopoles; an easy way is to see that the theory is invariant under swapping $\mathbf{E}$ and $\mathbf{B}$. The energy in a magnetic monopole field is

$$\frac{4\pi}{3\sqrt{b}}\frac{1}{\Gamma(\frac{3}{4})^2}\left(\frac{P}{4}\right)^{\frac{3}{2}}\sqrt{\mu_0}$$

if $\mathbf{B}_r \sim \frac{\mu_0 P}{4\pi r^2}$ as $r \to \infty$.

In fact $B_r = \frac{\mu_0 P}{4\pi r^2}$ which although it looks singular is perfectly reasonable.

References

- J.D. Jackson, *Classical Electrodynamics*, Second ed., Wiley, 1975.

  This is probably the best book for the course. It is very complete and reasonably easy to read. It has two major flaws; its price and its use of CGS units. Every college library should have at least one copy.

- Feynman, Leighton, Sands, *The Feynman Lectures on Physics Vol. 2*, Addison-Wesley, 1964.

  The Feynman lectures are always good reading. However Feynman II, the book on electromagnetism, is not really suitable as a textbook for this course. It is not complete enough although it is good for introductory reading and revision of earlier work. It's also fun...

- Landau and Lifschitz, *The Classical Theory of Fields*, Fourth ed., Butterworth-Heinemann, 1975.

  This is a good reference but isn't so good to learn by. It also uses CGS units.

The sign conventions for the tensor $F_{\mu\nu}$ are variable and it is essential to check which convention any given book is using.

There appears to be a gap in the market at about the level of this course. Most books on electromagnetism seem to be written either for a first course or for a postgraduate course. If you find a good one I haven't mentioned please send me a *brief* review and I will append it to this bibliography if I think it is suitable.