Mathematical Tripos Part II

Electrodynamics

Problems 1.

SI units are used. The signature is \(- + + +\).

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1. A particle of rest mass \(m\) and charge \(q\) moves in constant uniform fields \(\mathbf{E} = (0, E, 0)\) and \(\mathbf{B} = (0, 0, E/c)\), starting from rest at the origin. Show that \(\frac{dt}{d\tau} - \frac{1}{c} \frac{dx}{d\tau} = 1\) and that

\[
t = \tau + \frac{1}{6c^2} \alpha^2 \tau^3, \quad x = \frac{1}{6c} \alpha^2 \tau^3, \quad y = \frac{1}{2} \alpha \tau^2, \quad z = 0,
\]

where \(\alpha = qE/m\). By projecting the orbit in the \(t-x\), \(t-y\) and \(x-y\) planes, give a qualitative description of the motion.

2. Show that the Lagrangian density \(\mathcal{L}\), where \(S = \int \mathcal{L} \, dt d^3 x\), for the electromagnetic action

\[
S = -\frac{1}{4 \mu_0 c} \int F_{\mu\nu} F^{\mu\nu} \, d^4 x + \frac{1}{c} \int A_\mu J^\mu \, d^4 x,
\]

can be written as

\[
\mathcal{L} = \frac{\varepsilon_0}{2} |\nabla \phi + \partial \mathbf{A}/\partial t|^2 - \frac{1}{2\mu_0} |\nabla \times \mathbf{A}|^2 - \rho \phi + \mathbf{J} \cdot \mathbf{A},
\]

where \(A^\mu = (\phi/c, \mathbf{A})\) and \(J^\mu = (\rho c, \mathbf{J})\). Vary the action with respect to \(\phi\) and \(\mathbf{A}\) directly to obtain the sourced Maxwell equations

\[
\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}.
\]
3. Consider the following action for the electromagnetic field,

\[ S = \frac{1}{c} \int \left( -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu \right) d^4x , \]

for a prescribed 4-current \( J^\mu \) with \( \partial_\mu J^\mu = 0 \). Assuming

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \]

show that requiring \( \delta S = 0 \) for arbitrary variations \( \delta A_\mu \) that vanish at infinity implies one half of the Maxwell equations:

\[ \partial_\mu F^{\mu\nu} = -\mu_0 J^\nu . \]

Show also that \( S \) is gauge-invariant.

Next consider

\[ S_P = \frac{1}{c} \int \left( \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\mu_0} F^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) + J^\mu A_\mu \right) d^4x , \]

which reduces to \( S \) if \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). Regarding \( A_\mu \) and \( F_{\mu\nu} \) as independent quantities, show that requiring \( \delta S_P = 0 \) for arbitrary variations \( \delta F_{\mu\nu} \) implies Eq. (2) as before. Show also that requiring \( \delta S_P = 0 \) for arbitrary variations \( \delta A_\mu \) that vanish at infinity implies Eq. (1) and hence the other half of the Maxwell equations.

4. The fields on either side of a physical boundary \( S \) with unit normal \( \mathbf{n} \), pointing from region 1 to 2, are \( (E_1, B_1) \) and \( (E_2, B_2) \). The discontinuities across \( S \) of the electromagnetic field are \( B_2 - B_1 = \mu_0 J_S \times \mathbf{n} \) and \( E_2 - E_1 = \sigma_S \mathbf{n}/\epsilon_0 \) where \( J_S \) and \( \sigma_S \) are the surface current density and surface charge density respectively. Verify that the net rate at which electromagnetic momentum flows into the discontinuity per unit area, \( f_{S,i} = \sigma_1^{ij} \hat{n}_j - \sigma_2^{ij} \hat{n}_j \), is given by

\[ f_S = \frac{1}{2} \left[ J_S \times (B_1 + B_2) + \sigma_S (E_1 + E_2) \right] , \]

so that \( f_S \) is the force per area acting on the surface.

[Hint: You may find it easier to consider the electric and magnetic parts, and the parallel and perpendicular components, separately.]

5. Show that the equation \( \epsilon^{\mu\nu\rho\sigma} \partial_\sigma F_{\rho\sigma} = 0 \) is equivalent to \( \partial_\nu F_{\rho\sigma} + \partial_\rho F_{\sigma\nu} + \partial_\sigma F_{\nu\rho} = 0 \). Using this, and \( \partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \), show that the electromagnetic stress-energy tensor

\[ T^{\mu\nu} = \frac{1}{\mu_0} \left( F^{\mu\rho} F_{\nu\rho} - \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \]

satisfies \( \eta^{\mu\nu} T^{\mu\nu} = 0 \) and \( \partial_\mu T^{\mu\nu} = -F^{\nu \rho} J_\rho \). Verify that \( T^{00} = \frac{1}{2\mu_0} (|E|^2/c^2 + |B|^2) \), \( T^{0i} = \frac{1}{\mu_0 c} (E \times B)_i \), and construct the components of the Maxwell stress tensor \( \sigma_{ij} \).

[Hint: You may wish to use \( \epsilon_{\mu\rho\sigma} \epsilon^{\alpha\beta\gamma\sigma} = -6\delta_\mu^\alpha \delta_\rho^\beta \delta_\sigma^\gamma \) where, recall, square brackets denote antisymmetrisation on the enclosed indices.]


6. If $J^\mu$ is a conserved current, i.e., $\partial_\mu J^\mu = 0$, verify that the corresponding charge $Q = \int (J^0 / c) \, d^3x$ is conserved. If $T^{\mu\nu} = T^{\nu\mu}$ is the conserved stress-energy tensor, i.e., $\partial_\nu T^{\mu\nu} = 0$ verify, by considering $S^{\mu\nu\rho} = T^{\mu\rho}x^\nu - T^{\mu\nu}x^\rho$ or otherwise, that

$$M^{\mu\nu} = \int (x^{\mu} T^{0\nu} - x^{\nu} T^{0\mu}) \, d^3x$$

is conserved.

Let $M_{ij} = \varepsilon_{ijk} J_{em,k}$. Show that for the electromagnetic field

$$J_{em} = \varepsilon_0 \int x \times (E \times B) \, d^3x.$$  

By expressing the rate of change of $J_{em}$ in terms of the charge and current densities, show that $J_{em}$ may be interpreted as the angular momentum of the electromagnetic field.

7. A hypothetical magnetic monopole is regarded as fixed at the origin and has a magnetic field $B(x) = g \mu_0 x / (4\pi |x|^3)$. A particle of charge $q$ is situated at $r$. Show that the angular momentum of the electromagnetic field can be written as

$$J_{em} = \int x \times \left( \frac{g \mu_0 x}{4\pi |x|^3} \times \nabla \left( \frac{q}{4\pi |x - r|} \right) \right) \, d^3x$$

$$= -\frac{gq\mu_0}{4\pi} \int \frac{\partial}{\partial x_i} \left( \frac{x}{|x|} \right) \frac{\partial}{\partial x_i} \left( \frac{1}{4\pi |x - r|} \right) \, d^3x = -\frac{gq\mu_0}{4\pi} \frac{r}{|r|^3},$$

after integrating by parts and neglecting a surface integral.

For non-relativistic motion of the electric charge, treat its electric field as that due to a charge at rest at its current location and ignore its magnetic field. Show directly that the total angular momentum $\mathbf{J} = r \times \mathbf{p} + J_{em}$ is constant using $\mathbf{p} = q \dot{r} \times \mathbf{B}(r)$. 
