Mathematical Tripos Part II Prof. N. Dorey

Electrodynamics Michaelmas Term 2019

Problems 1.

SI units are used. The signature is $-+++$.

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1. Show that the Lagrangian density $\mathcal{L}$, where $S = \int \mathcal{L}\, dt d^3\mathbf{x}$, for the electromagnetic action

$$S = -\frac{1}{4\mu_0 c}\int F_{\mu\nu}F^{\mu\nu}\, d^4x + \frac{1}{c}\int A_\mu J^\mu\, d^4x\,,$$

can be written as

$$\mathcal{L} = \frac{\epsilon_0}{2}|\boldsymbol{\nabla}\phi + \partial\mathbf{A}/\partial t|^2 - \frac{1}{2\mu_0}|\boldsymbol{\nabla}\times\mathbf{A}|^2 - \rho\phi + \mathbf{J}\cdot\mathbf{A}\,,$$

where $A^\mu = (\phi/c, \mathbf{A})$ and $J^\mu = (\rho c, \mathbf{J})$. Vary the action with respect to ϕ and $\mathbf{A}$ directly to obtain the sourced Maxwell equations

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}\,, \qquad \boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\,.$$

2. Consider the following action for the electromagnetic field,

$$S = \frac{1}{c}\int\left(-\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} + J^\mu A_\mu\right) d^4x\,,$$

for a prescribed 4-current J^μ with $\partial_\mu J^\mu = 0$. Assuming

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{1}$$

show that requiring $\delta S = 0$ for arbitrary variations δA_μ that vanish at infinity implies one half of the Maxwell equations:

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu. \tag{2}$$

Show also that S is gauge-invariant.

Next consider

$$S_P = \frac{1}{c}\int\left(\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\mu_0}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + J^\mu A_\mu\right) d^4x\,,$$

which reduces to S if $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Regarding A_μ and $F^{\mu\nu}$ as independent quantities, show that requiring $\delta S_P = 0$ for arbitrary variations δA_μ that vanish at infinity implies Eq. (2) as before. Show also that requiring $\delta S_P = 0$ for arbitrary variations $\delta F^{\mu\nu}$ implies Eq. (1) and hence the other half of the Maxwell equations.

3. A particle of rest mass m and charge q moves in constant uniform fields $\mathbf{E} = (0, E, 0)$ and $\mathbf{B} = (0, 0, E/c)$, starting from rest at the origin. Show that $\frac{dt}{d\tau} - \frac{1}{c}\frac{dx}{d\tau} = 1$ and that

$$t = \tau + \frac{1}{6c^2}\alpha^2\tau^3, \quad x = \frac{1}{6c}\alpha^2\tau^3, \quad y = \frac{1}{2}\alpha\tau^2, \quad z = 0,$$

where $\alpha = qE/m$. By projecting the orbit in the t–x, t–y and x–y planes, give a qualitative description of the motion.

4. The fields on either side of a physical boundary S with unit normal $\hat{\mathbf{n}}$, pointing from region 1 to 2, are $(\mathbf{E}_1, \mathbf{B}_1)$ and $(\mathbf{E}_2, \mathbf{B}_2)$. The discontinuities across S of the electromagnetic field are $\mathbf{B}_2 - \mathbf{B}_1 = \mu_0 \mathbf{J}_S \times \hat{\mathbf{n}}$ and $\mathbf{E}_2 - \mathbf{E}_1 = \sigma_S \hat{\mathbf{n}}/\epsilon_0$ where $\mathbf{J}_S$ and σ_S are the surface current density and surface charge density respectively. Verify that the net rate at which electromagnetic momentum flows into the discontinuity per unit area, $f_{S,i} = \sigma^1_{ij}\hat{n}_j - \sigma^2_{ij}\hat{n}_j$, is given by

$$\mathbf{f}_S = \frac{1}{2}\left[\mathbf{J}_S \times (\mathbf{B}_1 + \mathbf{B}_2) + \sigma_S(\mathbf{E}_1 + \mathbf{E}_2)\right],$$

so that $\mathbf{f}_S$ is the force per area acting on the surface.

[*Hint: You may find it easier to consider the electric and magnetic parts, and the parallel and perpendicular components, separately.*]

5. Show that the equation $\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$ is equivalent to $\partial_\nu F_{\rho\sigma} + \partial_\rho F_{\sigma\nu} + \partial_\sigma F_{\nu\rho} = 0$. Using this, and $\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu$, show that the electromagnetic stress-energy tensor

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)$$

satisfies $\eta_{\mu\nu}T^{\mu\nu} = 0$ and $\partial_\mu T^{\mu\nu} = -F^\nu{}_\rho J^\rho$. Verify that $T^{00} = \frac{1}{2\mu_0}(|\mathbf{E}|^2/c^2 + |\mathbf{B}|^2)$, $T^{0i} = \frac{1}{\mu_0 c}(\mathbf{E} \times \mathbf{B})_i$, and construct the components of the Maxwell stress tensor σ_{ij}.

[*Hint: You may wish to use $\epsilon_{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma\sigma} = -6\delta_{[\mu}{}^\alpha\delta_\nu{}^\beta\delta_{\rho]}{}^\gamma$ where, recall, square brackets denote antisymmetrisation on the enclosed indices.*]

6. If J^μ is a conserved current, i.e., $\partial_\mu J^\mu = 0$, verify that the corresponding charge $Q = \int (J^0/c)\, d^3\mathbf{x}$ is conserved. If $T^{\mu\nu} = T^{\nu\mu}$ is the conserved stress-energy tensor, i.e., $\partial_\nu T^{\mu\nu} = 0$ verify, by considering $S^{\mu\nu\rho} = T^{\mu\rho}x^\nu - T^{\mu\nu}x^\rho$ or otherwise, that

$$M^{\mu\nu} = \int (x^\mu T^{0\nu} - x^\nu T^{0\mu})\, d^3\mathbf{x}$$

is conserved.

Let $M_{ij} = c\epsilon_{ijk} J_{\mathrm{em},k}$. Show that for the electromagnetic field

$$\mathbf{J}_{\mathrm{em}} = \epsilon_0 \int \mathbf{x} \times (\mathbf{E} \times \mathbf{B})\, d^3\mathbf{x}\,.$$

By expressing the rate of change of $\mathbf{J}_{\mathrm{em}}$ in terms of the charge and current densities, show that $\mathbf{J}_{\mathrm{em}}$ may be interpreted as the angular momentum of the electromagnetic field.

7. A hypothetical magnetic monopole is regarded as fixed at the origin and has a magnetic field $\mathbf{B}(\mathbf{x}) = g\mu_0 \mathbf{x}/(4\pi|\mathbf{x}|^3)$. A particle of charge q is situated at $\mathbf{r}$. Show that the angular momentum of the electromagnetic field can be written as

$$\begin{aligned}\mathbf{J}_{\mathrm{em}} &= \int \mathbf{x} \times \left(\frac{g\mu_0 \mathbf{x}}{4\pi|\mathbf{x}|^3} \times \boldsymbol{\nabla} \frac{q}{4\pi|\mathbf{x}-\mathbf{r}|} \right) d^3\mathbf{x} \\ &= -\frac{gq\mu_0}{4\pi} \int \frac{\partial}{\partial x_i} \left(\frac{\mathbf{x}}{|\mathbf{x}|} \right) \frac{\partial}{\partial x_i} \left(\frac{1}{4\pi|\mathbf{x}-\mathbf{r}|} \right) d^3\mathbf{x} = -\frac{gq\mu_0}{4\pi} \frac{\mathbf{r}}{|\mathbf{r}|},\end{aligned}$$

after integrating by parts and neglecting a surface integral.

For non-relativistic motion of the electric charge, treat its electric field as that due to a charge at rest at its current location and ignore its magnetic field. Show directly that the total angular momentum $\mathbf{J} = \mathbf{r} \times \mathbf{p} + \mathbf{J}_{\mathrm{em}}$ is constant using $\dot{\mathbf{p}} = q\dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r})$.