1. Show that the Lagrangian density $\mathcal{L}$, where $S = \int S \, dt \, d^3 x$, for the electromagnetic action

$$S = -\frac{1}{4\mu_0 c} \int F_{\mu\nu} F^{\mu\nu} \, d^4 x + \frac{1}{c} \int A_\mu J^\mu \, d^4 x ,$$

can be written as

$$\mathcal{L} = \frac{\epsilon_0}{2} | \nabla \phi + \partial A / \partial t |^2 - \frac{1}{2\mu_0} | \nabla \times A |^2 - \rho \phi + J \cdot A ,$$

where $A^\mu = (\phi/c, A)$ and $J^\mu = (\rho c, J)$. Vary the action with respect to $\phi$ and $A$ directly to obtain the sourced Maxwell equations

$$\nabla \cdot E = \frac{\rho}{\epsilon_0} , \quad \nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} .$$

2. Consider the following action for the electromagnetic field,

$$S = \frac{1}{c} \int \left( -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu \right) \, d^4 x ,$$

for a prescribed 4-current $J^\mu$ with $\partial_\mu J^\mu = 0$. Assuming

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu , \quad (1)$$

show that requiring $\delta S = 0$ for arbitrary variations $\delta A_\mu$ that vanish at infinity implies one half of the Maxwell equations:

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu . \quad (2)$$

Show also that $S$ is gauge-invariant.

Next consider

$$S_P = \frac{1}{c} \int \left( \frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\mu_0} F_{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) + J^\mu A_\mu \right) \, d^4 x ,$$

which reduces to $S$ if $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Regarding $A_\mu$ and $F_{\mu\nu}$ as independent quantities, show that requiring $\delta S_P = 0$ for arbitrary variations $\delta A_\mu$ that vanish at infinity implies Eq. (2) as before. Show also that requiring $\delta S_P = 0$ for arbitrary variations $\delta F_{\mu\nu}$ implies Eq. (1) and hence the other half of the Maxwell equations.
3. A particle of rest mass $m$ and charge $q$ moves in constant uniform fields $\mathbf{E} = (0, E, 0)$ and $\mathbf{B} = (0, 0, E/c)$, starting from rest at the origin. Show that $\frac{dE}{d\tau} - \frac{dx}{d\tau} = 1$ and that

$$t = \tau + \frac{1}{6c^2} \alpha^2 \tau^3, \quad x = \frac{1}{6c} \alpha^2 \tau^3, \quad y = \frac{1}{2} \alpha \tau^2, \quad z = 0,$$

where $\alpha = qE/m$. By projecting the orbit in the $t-x$, $t-y$ and $x-y$ planes, give a qualitative description of the motion.

4. The fields on either side of a physical boundary $S$ with unit normal $\hat{n}$, pointing from region 1 to 2, are $(\mathbf{E}_1, \mathbf{B}_1)$ and $(\mathbf{E}_2, \mathbf{B}_2)$. The discontinuities across $S$ of the electromagnetic field are $\mathbf{B}_2 - \mathbf{B}_1 = \mu_0 \mathbf{J}_S \times \hat{n}$ and $\mathbf{E}_2 - \mathbf{E}_1 = \sigma_S \hat{n}/\epsilon_0$ where $\mathbf{J}_S$ and $\sigma_S$ are the surface current density and surface charge density respectively. Verify that the net rate at which electromagnetic momentum flows into the discontinuity per unit area, $f_{S,i} = \sigma_{ij} \hat{n}_j - \sigma_{ij}^2 \hat{n}_j$, is given by

$$f_S = \frac{1}{2} \left[ \mathbf{J}_S \times (\mathbf{B}_1 + \mathbf{B}_2) + \sigma_S (\mathbf{E}_1 + \mathbf{E}_2) \right],$$

so that $f_S$ is the force per area acting on the surface.

[Hint: You may find it easier to consider the electric and magnetic parts, and the parallel and perpendicular components, separately.]

5. Show that the equation $\epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0$ is equivalent to $\partial_\nu F_{\rho\sigma} + \partial_\rho F_{\sigma\nu} + \partial_\sigma F_{\nu\rho} = 0$. Using this, and $\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu$, show that the electromagnetic stress-energy tensor

$$T^{\mu\nu} = \frac{1}{\mu_0} \left( F^{\mu}_{\rho} F^{\nu}_{\rho} - \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right)$$

satisfies $\eta_{\mu\nu} T^{\mu\nu} = 0$ and $\partial_\mu T^{\mu\nu} = -F^{\nu}_{\rho} J^\rho$. Verify that $T^{00} = \frac{1}{2\mu_0} (|\mathbf{E}|^2/c^2 + |\mathbf{B}|^2)$, $T^{0i} = \frac{1}{\mu_0 c} (\mathbf{E} \times \mathbf{B})_i$, and construct the components of the Maxwell stress tensor $\sigma_{ij}$.

[Hint: You may wish to use $\epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} = -6 \delta^{[\alpha}_{[\mu} \delta^{\beta}_{\nu} \delta^{\gamma}_{\rho]} \delta^{]}_{\sigma]}$ where, recall, square brackets denote antisymmetrisation on the enclosed indices.]

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6. If \( J^\mu \) is a conserved current, i.e., \( \partial_\mu J^\mu = 0 \), verify that the corresponding charge \( Q = \int (J^0/c) \, d^3x \) is conserved. If \( T^{\mu \nu} = T^{\nu \mu} \) is the conserved stress-energy tensor, i.e., \( \partial_\nu T^{\mu \nu} = 0 \) verify, by considering \( S^{\mu \rho \nu} = T^{\nu \rho} x^\nu - T^{\mu \nu} x^\rho \) or otherwise, that
\[
M^{\mu \nu} = \int (x^\mu T^{0 \nu} - x^\nu T^{0 \mu}) \, d^3x
\]
is conserved.

Let \( M_{ij} = \epsilon_{ijk} J_{em,k} \). Show that for the electromagnetic field
\[
J_{em} = \epsilon_0 \int x \times (E \times B) \, d^3x.
\]

By expressing the rate of change of \( J_{em} \) in terms of the charge and current densities, show that \( J_{em} \) may be interpreted as the angular momentum of the electromagnetic field.

7. A hypothetical magnetic monopole is regarded as fixed at the origin and has a magnetic field \( B(x) = g\mu_0 x/(4\pi |x|^3) \). A particle of charge \( q \) is situated at \( r \). Show that the angular momentum of the electromagnetic field can be written as
\[
J_{em} = \int x \times \left( \frac{g\mu_0 x}{4\pi |x|^3} \times \nabla \frac{q}{4\pi |x-r|} \right) \, d^3x
\]
\[
= -\frac{gq\mu_0}{4\pi} \int \frac{\partial}{\partial x_i} \left( \frac{x}{|x|} \right) \frac{\partial}{\partial x_i} \left( \frac{1}{4\pi |x-r|} \right) \, d^3x = -\frac{gq\mu_0}{4\pi} \frac{r}{|r|^3}.
\]

after integrating by parts and neglecting a surface integral.

For non-relativistic motion of the electric charge, treat its electric field as that due to a charge at rest at its current location and ignore its magnetic field. Show directly that the total angular momentum \( J = r \times p + J_{em} \) is constant using \( \dot{p} = q \dot{r} \times B(r) \).