1. The interior of an oblate ellipsoid

\[ \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1 \]

with \( a > c \) is occupied by material with a uniform charge density \( \rho = \rho_0 \). Obtain the total charge, dipole and quadrupole moments of this distribution. Now suppose that the interior of the ellipsoid is removed and its surface replaced by a conducting shell carrying the same total charge. Would the dipole moment and quadrupole moments change or remain the same? Justify your answer (without detailed calculation).

2. Assume that Coulomb's inverse square law is not completely accurate and that the electrostatic potential \( \phi(x) \) generated by a charge distribution \( \rho(x) \) is instead given by

\[ \phi(x) = \frac{1}{\epsilon_0} \int V(r) \rho(y) \, d^3y, \]

where \( r = |x - y| \), and \( V(r) \) is a more general potential. By expanding \( V(|x - y|) \) about the point \( x \) in a Taylor series in \( y \), obtain the multipole expansion for \( \phi(x) \) up to quadratic order in \( y \) in terms of \( V(|x|), V'(|x|) \) and \( V''(|x|) \). Show in particular that the extra moment \( \int d^3y \, y^2 \rho(y) \) is required. Specialise your results to the Coulomb potential \( V = \frac{1}{4\pi r} \) and the Yukawa potential \( V = \frac{e^{-\lambda r}}{4\pi r}, \lambda > 0 \). Show, contrary to the Coulomb case, that, in general, if the charge density \( \rho(y) \) is spherically symmetric about the origin \( y = 0 \), the field outside the charge distribution is not given solely by the total charge \( Q = \int \rho(y) \, d^3y \) but extra moments are required.
3. Show that for a static (time-independent) current distribution \( \mathbf{J}(\mathbf{x}) \) which is non-zero in a finite region \( V \),

\[-\nabla^2 \mathbf{A}(\mathbf{x}) = \mu_0 \mathbf{J}(\mathbf{x}),\]

if the gauge condition \( \nabla \cdot \mathbf{A} = 0 \) is imposed. (You should demonstrate explicitly that this is possible.) Hence obtain the solution

\[\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{1}{|\mathbf{x} - \mathbf{y}|} \mathbf{J}(\mathbf{y}) \, d^3\mathbf{y}.\]

From current conservation \( \nabla \cdot \mathbf{J} = 0 \) show that \( \partial \left[ y_j J_i (y) \right] / \partial y_i = J_j (y) \) and hence obtain \( \int \mathbf{J}(\mathbf{y}) \, d^3\mathbf{y} = 0 \) where the domain of integration includes all of \( V \). In a similar fashion derive \( \int [y_i J_j (y) + y_j J_i (y)] \, d^3\mathbf{y} = 0 \) and, using these results, show that for large \( |\mathbf{x}| \)

\[\mathbf{A}(\mathbf{x}) \approx \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{x}}{|\mathbf{x}|^3}, \quad \text{where} \quad \mathbf{m} = \frac{1}{2} \int \mathbf{y} \times \mathbf{J}(\mathbf{y}) \, d^3\mathbf{y}.\]

Verify that the magnetic field has the form

\[\mathbf{B}(\mathbf{x}) \approx -\frac{\mu_0}{4\pi} \nabla \left( \frac{\mathbf{m} \cdot \mathbf{x}}{|\mathbf{x}|^3} \right),\]

which corresponds to a magnetic dipole with moment \( \mathbf{m} \). If the current distribution \( \mathbf{J} \) represents a current \( I \) in a plane wire loop \( \gamma \), so that \( \mathbf{m} = \frac{1}{2} I \oint_{\gamma} \mathbf{y} \times d\mathbf{y} \), show that \( \mathbf{m} = I \mathbf{S} \) where \( \mathbf{S} \) is the vector area of the loop \( \gamma \).

4. The Lorentz transformation \( \Lambda(v)^\mu_\nu \) corresponding to a boost with velocity \( \mathbf{v} = c\beta \mathbf{\hat{v}} \) can be represented as a matrix in the form

\[\Lambda(v) = \begin{pmatrix} \gamma & -\gamma \beta^T \\ -\gamma \beta & I + (\gamma - I) \beta \mathbf{\hat{v}}^T \end{pmatrix}\]

where \( \gamma = \left(1 - \beta^2\right)^{-1/2} \) and \( I \) is the 3 \times 3 identity matrix. By computing \( \Lambda(v) \Lambda(-v) \) verify that \( \Lambda(v)^{-1} = \Lambda(-v) \). Let \( u^\mu = (c\gamma, \gamma \mathbf{v}) \) be the 4-velocity of a particle. By calculating \( u'^\mu = \Lambda(\delta v)^\mu_\nu u^\nu \) for small \( \delta v \), determine the addition law for 3-velocities in the form

\[\mathbf{v}' = \mathbf{v} - \delta' \mathbf{v}, \quad \text{where} \quad \delta' \mathbf{v} = \delta \mathbf{v} - \frac{1}{c^2} \mathbf{v} (\mathbf{v} \cdot \delta \mathbf{v}) + O(|\delta \mathbf{v}|^2).\]

5. For constant electric and magnetic fields, \( \mathbf{E} \) and \( \mathbf{B} \), show that if \( \mathbf{E} \cdot \mathbf{B} = 0 \) and \( \mathbf{E}^2 - c^2 \mathbf{B}^2 \neq 0 \) then there exist frames of reference where either \( \mathbf{E} \) or \( \mathbf{B} \) are zero, but not both. (It suffices to take just \( E_y \) and \( B_z \) non zero and consider Lorentz transformations along the \( x \)-direction with speed \( v < c \).)
6. An electromagnetic wave is reflected by a perfect conductor at $x = 0$. The electric field has the form

$$E(t, x) = \hat{y} [f(t_-) - f(t_+)], \quad \text{where} \quad ct_\pm = ct \pm x,$$

satisfying the boundary condition for a stationary perfect conductor that the tangential electric field should vanish. Find the corresponding magnetic field $B$. Show that under a Lorentz transformation to a frame moving with speed $v$ in the $x$-direction the electric field is transformed to

$$E'(t', x') = \hat{y} \left[ \rho f(\rho t'_-) - \frac{1}{\rho} f \left( \frac{t'_+}{\rho} \right) \right], \quad \text{where} \quad \rho = \left( \frac{c - v}{c + v} \right)^{1/2}.$$

Hence for an incident wave $E(t, x) = \hat{y} F(t_-)$ find the reflected wave after reflection by a perfectly conducting mirror moving with speed $v$ along the $x$-direction.

7. A particle of rest mass $m$ and charge $q$ moves in constant uniform fields $\mathbf{E} = (0, E, 0)$ and $\mathbf{B} = (0, 0, E/c)$, starting from rest at the origin. Show that $\frac{d\mathbf{r}}{d\tau} = 1$ and that

$$t = \tau + \frac{1}{6c^2} \alpha^2 \tau^3, \quad x = \frac{1}{6c} \alpha^2 \tau^3, \quad y = \frac{1}{2} \alpha \tau^2, \quad z = 0,$$

where $\alpha = qE/m$. By projecting the orbit in the $t$-$x$, $t$-$y$ and $x$-$y$ planes, give a qualitative description of the motion.

8. For a general 4-velocity, written as $u^\mu = \gamma(c, \mathbf{v})$, show that

$$F_{\mu\nu} u^\nu = \gamma \left( -\frac{1}{c} \mathbf{E} \cdot \mathbf{v}, \mathbf{E} + \mathbf{v} \times \mathbf{B} \right).$$

In the rest-frame of a conducting medium, Ohm’s law states that $\mathbf{J} = \sigma \mathbf{E}$ where $\sigma$ is the conductivity and $\mathbf{J}$ is the 3-current. Show that Ohm’s law can be written covariantly as

$$J^\mu + \frac{1}{c^2} (J^\nu u_\nu) u^\mu = \sigma F^{\mu\nu} u_\nu,$$

where $J^\mu$ is the 4-current and $u^\mu$ is the (uniform) 4-velocity of the medium. If the medium moves with 3-velocity $\mathbf{v}$ in some inertial frame, show that the current in that frame is

$$\mathbf{J} = \rho \mathbf{v} + \sigma \gamma \left( \mathbf{E} + \mathbf{v} \times \mathbf{B} - \frac{1}{c^2} (\mathbf{v} \cdot \mathbf{E}) \mathbf{v} \right),$$

where $\rho$ is the charge density.

Simplify this formula, given that the charge density vanishes in the rest-frame of the medium.