1. Show that the Lagrangian density $L$, where $S = \int L \, dt \, d^3x$, for the electromagnetic action

$$S = -\frac{1}{4\mu_0 c} \int F_{\mu\nu} F^{\mu\nu} \, d^4x + \frac{1}{c} \int A_{\mu} J^{\mu} \, d^4x,$$

can be written as

$$L = \frac{\epsilon_0}{2} |\nabla \phi + \partial A / \partial t|^2 - \frac{1}{2\mu_0} |\nabla \times A|^2 - \rho \phi + J \cdot A,$$

where $A^{\mu} = (\phi / c, A)$ and $J^{\mu} = (\rho c, J)$. Vary the action with respect to $\phi$ and $A$ directly to obtain the sourced Maxwell equations

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}, \quad \nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}.$$

2. Consider the following action for the electromagnetic field,

$$S = \frac{1}{c} \int \left( -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} + J^{\mu} A_{\mu} \right) \, d^4x,$$

for a prescribed 4-current $J^{\mu}$ with $\partial_{\mu} J^{\mu} = 0$. Assuming

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}, \quad (1)$$

show that requiring $\delta S = 0$ for arbitrary variations $\delta A_{\mu}$ that vanish at infinity implies one half of the Maxwell equations:

$$\partial_{\mu} F^{\mu\nu} = -\mu_0 J^{\nu}. \quad (2)$$

Show also that $S$ is gauge-invariant.

Next consider

$$S_P = \frac{1}{c} \int \left( \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\mu_0} F^{\mu\nu} (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}) + J^{\mu} A_{\mu} \right) \, d^4x,$$

which reduces to $S$ if $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$. Regarding $A_{\mu}$ and $F^{\mu\nu}$ as independent quantities, show that requiring $\delta S_P = 0$ for arbitrary variations $\delta A_{\mu}$ that vanish at infinity implies Eq. (2) as before. Show also that requiring $\delta S_P = 0$ for arbitrary variations $\delta F^{\mu\nu}$ implies Eq. (1) and hence the other half of the Maxwell equations.
3. A particle of rest mass \( m \) and charge \( q \) moves in constant uniform fields \( \mathbf{E} = (0, E, 0) \) and \( \mathbf{B} = (0, 0, E/c) \), starting from rest at the origin. Show that \( \frac{dt}{d\tau} - \frac{1}{c} \frac{dx}{d\tau} = 1 \) and that
\[
 t = \tau + \frac{1}{6c^2} \alpha^2 \tau^3, \quad x = \frac{1}{6c} \alpha^2 \tau^3, \quad y = \frac{1}{2} \alpha \tau^2, \quad z = 0,
\]
where \( \alpha = qE/m \). By projecting the orbit in the \( t-x \), \( t-y \) and \( x-y \) planes, give a qualitative description of the motion.

4. The fields on either side of a physical boundary \( S \) with unit normal \( \hat{n} \), pointing from region 1 to 2, are \( (\mathbf{E}_1, \mathbf{B}_1) \) and \( (\mathbf{E}_2, \mathbf{B}_2) \). The discontinuities across \( S \) of the electromagnetic field are \( \mathbf{B}_2 - \mathbf{B}_1 = \mu_0 \mathbf{J}_S \times \hat{n} \) and \( \mathbf{E}_2 - \mathbf{E}_1 = \sigma_S \hat{n} / \epsilon_0 \) where \( \mathbf{J}_S \) and \( \sigma_S \) are the surface current density and surface charge density respectively. Verify that the net rate at which electromagnetic momentum flows into the discontinuity per unit area, \( f_{S,i} = \sigma_{S,i} \hat{n}_i - \sigma_{S,j} \hat{n}_j \), is given by
\[
 f_S = \frac{1}{2} \left[ \mathbf{J}_S \times (\mathbf{B}_1 + \mathbf{B}_2) + \sigma_S (\mathbf{E}_1 + \mathbf{E}_2) \right],
\]
so that \( f_S \) is the force per area acting on the surface.

[Hint: You may find it easier to consider the electric and magnetic parts, and the parallel and perpendicular components, separately.]

5. Show that the equation \( \epsilon_{\mu\nu\rho\sigma} \partial_{\nu} F_{\rho\sigma} = 0 \) is equivalent to \( \partial_{\nu} F_{\rho\sigma} + \partial_{\rho} F_{\sigma\nu} + \partial_{\sigma} F_{\nu\rho} = 0 \). Using this, and \( \partial_{\mu} F^{\mu\nu} = -\mu_0 J^\nu \), show that the electromagnetic stress-energy tensor
\[
 T^{\mu\nu} = \frac{1}{\mu_0} \left( F^{\mu}_{\rho} F^{\nu\rho} - \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right)
\]
satisfies \( \eta_{\mu\nu} T^{\mu\nu} = 0 \) and \( \partial_{\mu} T^{\mu\nu} = -F^{\nu}_{\rho} J^\rho \). Verify that \( T^{00} = \frac{1}{2\mu_0} (|\mathbf{E}|^2/c^2 + |\mathbf{B}|^2) \), \( T^{0i} = \frac{1}{\mu_0 c} (\mathbf{E} \times \mathbf{B})_i \), and construct the components of the Maxwell stress tensor \( \sigma_{ij} \).

[Hint: You may wish to use \( \epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma} = -6 \delta^{[\beta}_{\mu} \delta^{\gamma]}_{\alpha} \delta^{\delta}_{\rho} \) where, recall, square brackets denote antisymmetrisation on the enclosed indices.]
6. If $J^\mu$ is a conserved current, i.e., $\partial_\mu J^\mu = 0$, verify that the corresponding charge $Q = \int (J^0/c) \, d^3x$ is conserved. If $T^{\mu\nu} = T^{\nu\mu}$ is the conserved stress-energy tensor, i.e., $\partial_\nu T^{\mu\nu} = 0$ verify, by considering $S^{\mu\nu\rho} = T^{\mu\rho}x^\nu - T^{\nu\mu}x^\rho$ or otherwise, that
\[
M^{\mu\nu} = \int (x^{\mu}T^{0\nu} - x^{\nu}T^{0\mu}) \, d^3x
\]
is conserved.

Let $M_{ij} = c\epsilon_{ijk}J_{em,k}$. Show that for the electromagnetic field
\[
J_{em} = \epsilon_0 \int x \times (E \times B) \, d^3x.
\]

By expressing the rate of change of $J_{em}$ in terms of the charge and current densities, show that $J_{em}$ may be interpreted as the angular momentum of the electromagnetic field.

7. A hypothetical magnetic monopole is regarded as fixed at the origin and has a magnetic field $B(x) = g\mu_0 x/(4\pi |x|^3)$. A particle of charge $q$ is situated at $r$. Show that the angular momentum of the electromagnetic field can be written as
\[
J_{em} = \int x \times \left( \frac{g\mu_0 x}{4\pi |x|^3} \times \nabla \frac{q}{4\pi |x - r|} \right) \, d^3x
\]
\[
= -\frac{gq\mu_0}{4\pi} \int \frac{\partial}{\partial x_i} \left( \frac{x}{|x|} \right) \frac{\partial}{\partial x_i} \left( \frac{1}{4\pi |x - r|} \right) \, d^3x = -\frac{gq\mu_0}{4\pi} \frac{r}{|r|^3}
\]
after integrating by parts and neglecting a surface integral.

For non-relativistic motion of the electric charge, treat its electric field as that due to a charge at rest at its current location and ignore its magnetic field. Show directly that the total angular momentum $J = r \times p + J_{em}$ is constant using $\dot{p} = q\dot{r} \times B(r)$. 
