1. Show that the equation $\epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0$ is equivalent to $\partial_\nu F_{\rho\sigma} + \partial_\rho F_{\sigma\nu} + \partial_\sigma F_{\nu\rho} = 0$. Using this, and $\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu$, show that the electromagnetic stress-energy tensor

$$T^{\mu\nu} = \frac{1}{\mu_0} \left( F^{\mu}_\rho F^{\nu\rho} - \frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right)$$

satisfies $\eta^{\mu\nu} T^{\mu\nu} = 0$ and $\partial_\mu T^{\mu\nu} = -F^{\nu}_\rho J^\rho$. Verify that $T^{00} = \frac{1}{2\mu_0} (|E|^2/c^2 + |B|^2)$, $T^{0i} = \frac{1}{\mu_0 c} (E \times B)_i$, and construct the components of the Maxwell stress tensor $\sigma_{ij}$.

[Hint: You may wish to use $\epsilon^{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma}\sigma = -6 \delta^{[\alpha}_{[\mu} \delta^{\beta}_{\nu]} \delta^{\gamma}_{\rho]}$ where, recall, square brackets denote antisymmetrisation on the enclosed indices.]

2. The fields on either side of a physical boundary $S$ with unit normal $\hat{n}$, pointing from region 1 to 2, are $(E_1, B_1)$ and $(E_2, B_2)$. The discontinuities across $S$ of the electromagnetic field are $B_2 - B_1 = \mu_0 J_S \times \hat{n}$ and $E_2 - E_1 = \sigma_S \hat{n}/\epsilon_0$ where $J_S$ and $\sigma_S$ are the surface current density and surface charge density respectively. Verify that the net rate at which electromagnetic momentum flows into the discontinuity per unit area, $f_{S,i} = \sigma_{ij} \hat{n}_j - \sigma_{ij}^2 \hat{n}_j$, is given by

$$f_S = \frac{1}{2} \left[ J_S \times (B_1 + B_2) + \sigma_S (E_1 + E_2) \right],$$

so that $f_S$ is the force per area acting on the surface.

[You may find it easier to consider the electric and magnetic parts, and the parallel and perpendicular components, separately.]
3. If \( J^\mu \) is a conserved current, i.e., \( \partial_\mu J^\mu = 0 \), verify that the corresponding charge \( Q = \int (J^0/c) \, d^3x \) is conserved. If \( T^{\mu\nu} = T^{\nu\mu} \) is the conserved stress-energy tensor, i.e., \( \partial_\nu T^{\mu\nu} = 0 \) verify, by considering \( S^{\mu\nu\rho} = T^{\mu\rho x^\nu} - T^{\mu\nu x^\rho} \) or otherwise, that
\[
M^{\mu\nu} = \int (x^\mu T^{0\nu} - x^\nu T^{0\mu}) \, d^3x
\]
is conserved.

Let \( M_{ij} = c\epsilon_{ijk} J_{em,k} \). Show that for the electromagnetic field
\[
J_{em} = \epsilon_0 \int \mathbf{x} \times (\mathbf{E} \times \mathbf{B}) \, d^3x.
\]

By expressing the rate of change of \( J_{em} \) in terms of the charge and current densities, show that \( J_{em} \) may be interpreted as the angular momentum of the electromagnetic field.

4. A hypothetical magnetic monopole is regarded as fixed at the origin and has a magnetic field \( \mathbf{B}(\mathbf{x}) = g\mu_0 \mathbf{x}/(4\pi |\mathbf{x}|^3) \). A particle of charge \( q \) is situated at \( \mathbf{r} \). Show that the angular momentum of the electromagnetic field can be written as
\[
J_{em} = \int \mathbf{x} \times \left( \frac{g\mu_0 \mathbf{x}}{4\pi |\mathbf{x}|^3} \times \nabla \frac{q}{4\pi |\mathbf{x} - \mathbf{r}|} \right) \, d^3x
\]
\[
= - \frac{gq\mu_0}{4\pi} \int \frac{\partial}{\partial x_i} \left( \frac{x}{|x|} \right) \frac{1}{4\pi |\mathbf{x} - \mathbf{r}|} \, d^3x
\]
\[
= - \frac{gq\mu_0}{4\pi} \frac{\mathbf{r}}{|\mathbf{r}|^3}
\]
after integrating by parts and neglecting a surface integral.

For non-relativistic motion of the electric charge, treat its electric field as that due to a charge at rest at its current location and ignore its magnetic field. Show directly that the total angular momentum \( \mathbf{J} = \mathbf{r} \times \mathbf{p} + J_{em} \) is constant using \( \dot{\mathbf{p}} = q\dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r}) \).

5. An infinite straight wire lies along the \( z \)-axis, and for \( t < 0 \) there is no current or field. For \( t \geq 0 \) a uniform current \( I \) flows in the wire. Show that for \( t > 0 \) the vector potential \( \mathbf{A}(t, x, y) = A\mathbf{z} \) in the Lorenz gauge is
\[
A = \begin{cases} 
\frac{\mu_0 I}{2\pi} \ln(\theta + \sqrt{\theta^2 - 1}) & \text{for } \theta > 1, \\
0 & \text{for } \theta \leq 1,
\end{cases}
\]
where \( \theta = ct/r \) and \( r = \sqrt{x^2 + y^2} \). Obtain \( \mathbf{E} \) and \( \mathbf{B} \) and discuss the behaviour of the fields as \( t \to \infty \).
6. For a localised charge density $\rho(x)e^{-i\omega t}$ and current density $J(x)e^{-i\omega t}$ use current conservation to show that
\[
\int x_iJ_j(x)\,d^3x = \epsilon_{ijk}m_k - i\omega Q_{ij},
\]
where
\[
m = \frac{1}{2} \int x \times J(x)\,d^3x, \quad Q_{ij} = \frac{1}{2} \int x_i x_j \rho(x)\,d^3x.
\]
Hence show that if $\int \rho(x)\,d^3x = \int x \rho(x)\,d^3x = 0$ then at distances $r \gg c/\omega \gg a$, where $a$ is the extent of the charge and current distribution, the leading contributions to the scalar $\phi(x)e^{-i\omega t}$ and vector potentials $A(x)e^{-i\omega t}$ are
\[
\phi(x) \approx -\frac{1}{4\pi \epsilon_0} e^{ikr} k^2 \hat{x}_i \hat{x}_j Q_{ij},
\]
and
\[
A_i(x) \approx \frac{\mu_0}{4\pi r} e^{ikr} k(\hat{x} \times m)_i - \frac{\mu_0}{4\pi r} e^{ikr} k\omega \hat{x}_j Q_{ij},
\]
where $r = |x|$, $\hat{x} = x/r$ and $k = \omega/c$. Writing $Q_{ij} = Q_{ij} + P_{ij}$, where $Q_{ii} = 0$, show that the terms involving $P$ may be removed by a gauge transformation, at least at large distances. These results represent magnetic dipole and electric quadrupole radiation.

7. Optional, for enthusiasts. Let $\phi$ be the retarded scalar potential given by
\[
\phi(t, x) = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(t_{ret}, y)}{R} \,d^3y,
\]
where $R = |x - y|$, the retarded time $t_{ret} = t - R/c$, and set $\tilde{R} = (x - y)/R$. Show that
\[
\frac{\partial}{\partial t} \phi(t, x) = \frac{1}{4\pi \epsilon_0} \int \frac{\dot{\rho}(t_{ret}, y)}{R} \,d^3y,
\]
where $\dot{\rho}(t_{ret}, y)$ is $\partial \rho(t, y)/\partial t$ evaluated at $t_{ret}$. Show further that
\[
\nabla \phi(t, x) = -\frac{1}{4\pi \epsilon_0} \int \tilde{R} \left( \frac{1}{R^2} \rho(t_{ret}, y) + \frac{1}{cR} \dot{\rho}(t_{ret}, y) \right) \,d^3y.
\]
Hence verify, using $\nabla^2(1/R) = -4\pi \delta^{(3)}(x - y)$, that $\phi$ satisfies the wave equation
\[
\Box \phi(t, x) = -\frac{1}{\epsilon_0} \rho(t, x).
\]
Write down a similar retarded solution for the vector potential $A$ in terms of the current density $J$. 

3
Now assume that $\rho$ and $\mathbf{J}$ are non-zero only in a finite region. Setting $\hat{x} = x/|x|$, show that the leading terms in the far-field expansion are

$$
E(t, x) \approx \frac{\mu_0}{4\pi |x|} \int \left( \hat{x} \dot{\rho}(t_{\text{ret}}, y) c - \dot{\mathbf{J}}(t_{\text{ret}}, y) \right) d^3y
$$

$$
= \frac{\mu_0}{4\pi |x|} \hat{x} \times \left( \hat{x} \times \int \dot{\mathbf{J}}(t_{\text{ret}}, y) d^3y \right)
$$

where conservation of current, integration by parts and the discarding of a surface integral has been used, and

$$
B(t, x) \approx -\frac{\mu_0}{4\pi c |x|} \hat{x} \times \int \dot{\mathbf{J}}(t_{\text{ret}}, y) d^3y = \hat{x} \times E(t, x).
$$

Note that these results do not assume the dipole approximation. Determine the Poynting vector.

[Hint: When using current conservation, and integration by parts, be careful with the $y$-dependence of $t_{\text{ret}}$.]