1. An infinite straight wire lies along the $z$-axis, and for $t < 0$ there is no current or field. For $t \geq 0$ a uniform current $I$ flows in the wire. Show that for $t > 0$ the vector potential $A(t, x, y) = A\hat{z}$ in the Lorenz gauge is

$$A = \begin{cases} \frac{\mu_0 I}{2\pi} \ln(\theta + \sqrt{\theta^2 - 1}) & \text{for } \theta > 1, \\ 0 & \text{for } \theta \leq 1, \end{cases}$$

where $\theta = ct/r$ and $r = \sqrt{x^2 + y^2}$. Obtain $E$ and $B$ and discuss the behaviour of the fields as $t \to \infty$.

2. For a localised charge density $\rho(x)e^{-i\omega t}$ and current density $J(x)e^{-i\omega t}$ use current conservation to show that

$$\int x_i J_j(x) \, d^3x = \epsilon_{ijk} m_k - \frac{1}{6} i\omega Q'_{ij},$$

where

$$m = \frac{1}{2} \int x \times J(x) \, d^3x,$$  

$$Q'_{ij} = 3 \int x_i x_j \rho(x) \, d^3x.$$ 

Hence show that if $\int \rho(x) \, d^3x = \int x \rho(x) \, d^3x = 0$ then at distances $r \gg c/\omega \gg a$, where $a$ is the extent of the charge and current distribution, the leading contributions to the scalar $\phi(x)e^{-i\omega t}$ and vector potentials $A(x)e^{-i\omega t}$ are

$$\phi(x) \approx -\frac{1}{6} \frac{1}{\epsilon_0 r} e^{ikr} k^2 \hat{x}_i \hat{x}_j Q'_{ij},$$

and

$$A_i(x) \approx \frac{\mu_0}{4\pi r} e^{ikr} i k (\hat{x} \times m)_i - \frac{1}{6} \frac{\mu_0}{4\pi r} e^{ikr} i \omega \hat{x}_j Q'_{ij},$$

where $r = |x|$, $\hat{x} = x/r$ and $k = \omega/c$. Writing $Q'_{ij} = Q_{ij} + P\delta_{ij}$, where $Q_{ii} = 0$, show that the terms involving $P$ may be removed by a gauge transformation, at least at large distances. These results represent magnetic dipole and electric quadrupole radiation.
3. A small loop of wire lies in a plane with unit normal $\hat{N}$, and encloses an area $S$. A current $I_0 \cos \omega t$ flows around the loop, with $c/\omega$ much larger than the size of the loop. Using results from Question 2, show that in the far-field at displacement $x$ from the centre of the loop, the magnetic vector potential is

$$A(t,x) = \hat{x} \times \hat{N} \frac{\mu_0 I_0 S \omega}{4\pi rc} \sin(\omega t - kr) + O \left( \frac{1}{r^2} \right),$$

where $k = \omega / c$ and $r = |x|$.

[You may use the result $\oint x_i dx_j = S \epsilon_{ijk} N_k$.]

Find the leading-order magnetic field in the far-field and show that the average radiated power $dE/dt$ is

$$\frac{dE}{dt} = \frac{\mu_0}{12\pi} \frac{S^2 I_0^2 \omega^4}{c^3}.$$ 

4. (Optional, for enthusiasts.) Let $\phi$ be the retarded scalar potential given by

$$\phi(t,x) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(t_{\text{ret}}, y)}{R} d^3 y,$$

where $R = |x - y|$, the retarded time $t_{\text{ret}} = t - R/c$, and set $\hat{R} = (x - y)/R$. Show that

$$\frac{\partial}{\partial t} \phi(t,x) = \frac{1}{4\pi\varepsilon_0} \int \frac{\dot{\rho}(t_{\text{ret}}, y)}{R} d^3 y,$$

where $\dot{\rho}(t_{\text{ret}}, y)$ is $\partial \rho(t,y)/\partial t$ evaluated at $t_{\text{ret}}$. Show further that

$$\nabla \phi(t,x) = -\frac{1}{4\pi\varepsilon_0} \int \hat{R} \left( \frac{1}{R^2} \rho(t_{\text{ret}}, y) + \frac{1}{cR} \dot{\rho}(t_{\text{ret}}, y) \right) d^3 y.$$

Hence verify, using $\nabla^2 (1/R) = -4\pi\delta^3(x - y)$, that $\phi$ satisfies the wave equation

$$\Box \phi(t,x) = -\frac{1}{\varepsilon_0} \rho(t,x).$$

Write down a similar retarded solution for the vector potential $A$ in terms of the current density $J$. 

Now assume that $\rho$ and $J$ are non-zero only in a finite region. Setting $\hat{x} = x/|x|$, show that the leading terms in the far-field expansion are

$$E(t,x) \approx \frac{\mu_0}{4\pi|x|} \int \left( \hat{x} \dot{\rho}(t_{\text{ret}}, y) c - \hat{x} \times \dot{J}(t_{\text{ret}}, y) \right) d^3 y$$

$$= \frac{\mu_0}{4\pi|x|} \hat{x} \times \left( \hat{x} \times \int \dot{J}(t_{\text{ret}}, y) d^3 y \right).$$
where conservation of current, integration by parts and the discarding of a surface integral has been used, and

\[ B(t, x) \approx -\frac{\mu_0}{4\pi c|\mathbf{x}|} \mathbf{\dot{x}} \times \int \mathbf{J}(t_{ret}, y) \, d^3y = \frac{1}{c} \mathbf{\dot{x}} \times \mathbf{E}(t, x). \]

Note that these results do not assume the dipole approximation. Determine the Poynting vector.

*Hint: When using current conservation, and integration by parts, be careful with the y-dependence of \( t_{ret} \).*

5. Starting from the power radiated in the electric-dipole approximation, derive Larmor’s formula for the rate at which radiation is produced by a non-relativistic particle of charge \( q \) moving along a trajectory \( \mathbf{x}(t) \).

A non-relativistic particle of mass \( m \), charge \( q \) and energy \( E \) incident along a radial line in a central potential \( V(r) \). The potential is vanishingly small for \( r \) very large, but increases without bound as \( r \to 0 \). Show that the total amount of energy \( \mathcal{E} \) radiated by the particle is

\[ \mathcal{E} = \frac{\mu_0 q^2}{3\pi cm^2} \sqrt{\frac{m}{2}} \int_{r_0}^{\infty} \frac{1}{\sqrt{E - V(r)}} \left( \frac{dV}{dr} \right)^2 \, dr, \]

where \( V(r_0) = E \), assuming \( \mathcal{E} \ll E \).

Suppose that \( V \) is a Coulomb potential \( C/r \). Evaluate \( \mathcal{E} \).

6. For a relativistic particle of charge \( q \) on a trajectory \( y^\mu(\tau) \), where \( \tau \) is proper time, the current density 4-vector is

\[ J^\mu(x) = q c \int \delta^{(4)}(x - y(\tau)) \dot{y}^\mu(\tau) \, d\tau, \]

with \( \dot{y}^\mu \dot{y}_\mu = -c^2 \) and \( \dot{y}^0 > 0 \). Show that the 4-vector potential is given by

\[ A^\mu(x) = \frac{\mu_0}{2\pi} \int \Theta(x^0 - z^0) \delta(\eta_{\alpha\beta}(x^\alpha - z^\alpha)(x^\beta - z^\beta)) J^\mu(z) \, d^4z \]

\[ = -\frac{\mu_0 q c}{4\pi} \frac{\dot{y}^\mu(\tau_*)}{R^\nu(\tau_*) \dot{y}_\nu(\tau_*)}, \]

where \( R^\nu(\tau) = x^\nu - y^\nu(\tau) \) and \( \tau_* \) is determined by \( R^\mu(\tau_*) R_\mu(\tau_*) = 0 \) and \( R^0(\tau_*) > 0 \).

Verify that the Lorenz gauge condition \( \partial_\mu A^\mu = 0 \) holds and show that

\[ F_{\mu\nu} = -\frac{\mu_0 q c}{4\pi} \frac{1}{(R^\rho \dot{y}_\rho)^2} (R^\rho S_\nu - R_\nu S_\rho), \]

where \( S_\nu = \frac{\dot{y}_\nu}{R^\rho \dot{y}_\rho} (c^2 + R^2 \dot{y}_\tau) \)

and all quantities on the right are evaluated at \( \tau_* \). Check this result for the case of a stationary charge at the origin.