1. An infinite straight wire lies along the $z$-axis, and for $t < 0$ there is no current or field. For $t \geq 0$ a uniform current $I$ flows in the wire. Show that for $t > 0$ the vector potential $A(t, x, y) = A\hat{z}$ in the Lorenz gauge is

$$A = \begin{cases} \frac{\mu_0 I}{2\pi} \ln(\theta + \sqrt{\theta^2 - 1}) & \text{for } \theta > 1, \\ 0 & \text{for } \theta \leq 1, \end{cases}$$

where $\theta = ct/r$ and $r = \sqrt{x^2 + y^2}$. Obtain $\mathbf{E}$ and $\mathbf{B}$ and discuss the behaviour of the fields as $t \to \infty$.

2. For a localised charge density $\rho(x)e^{-i\omega t}$ and current density $J(x)e^{-i\omega t}$ use current conservation to show that

$$\int x_i J_j(x) d^3x = \epsilon_{ijk}m_k - \frac{1}{6}i\omega Q'_{ij},$$

where

$$m = \frac{1}{2} \int x \times J(x) d^3x, \quad Q'_{ij} = 3 \int x_i x_j \rho(x) d^3x.$$

Hence show that if $\int \rho(x) d^3x = \int x \rho(x) d^3x = 0$ then at distances $r \gg c/\omega \gg a$, where $a$ is the extent of the charge and current distribution, the leading contributions to the scalar $\phi(x)e^{-i\omega t}$ and vector potentials $A(x)e^{-i\omega t}$ are

$$\phi(x) \approx -\frac{1}{6} \frac{1}{4\pi \varepsilon_0 r} e^{ikr} k^2 \hat{x}_i \hat{x}_j Q'_{ij},$$

and

$$A_i(x) \approx \frac{\mu_0}{4\pi r} e^{ikr} \hat{x}_i \times (\hat{x} \times \mathbf{m})_i - \frac{1}{6\pi r} \frac{\mu_0}{4\pi r} e^{ikr} k \hat{x}_j Q'_{ij},$$

where $r = |x|$, $\hat{x} = x/r$ and $k = \omega/c$. Writing $Q'_{ij} = Q_{ij} + P\delta_{ij}$, where $Q_{ii} = 0$, show that the terms involving $P$ may be removed by a gauge transformation, at least at large distances. These results represent magnetic dipole and electric quadrupole radiation.
3. A small loop of wire lies in a plane with unit normal \( \hat{N} \), and encloses an area \( S \). A current \( I_0 \cos \omega t \) flows around the loop, with \( c/\omega \) much larger than the size of the loop. Using results from Question 2, show that in the far-field at displacement \( x \) from the centre of the loop, the magnetic vector potential is
\[
A(t, x) = \hat{x} \times \hat{N} \frac{\mu_0 I_0 S \omega}{4\pi rc} \sin(\omega t - kr) + O \left( \frac{1}{r^2} \right),
\]
where \( k = \omega/c \) and \( r = |x| \).

[You may use the result \( \oint x_i dx_j = S \epsilon_{ijk} N_k \).]

Find the leading-order magnetic field in the far-field and show that the average radiated power \( dE/dt \) is
\[
\frac{dE}{dt} = \frac{\mu_0}{12\pi} \frac{S^2 I_0^2 \omega^4}{c^3}.
\]

4. (Optional, for enthusiasts.) Let \( \phi \) be the retarded scalar potential given by
\[
\phi(t, x) = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(t_{\text{ret}}, y)}{R} d^3y,
\]
where \( R = |x - y| \), the retarded time \( t_{\text{ret}} = t - R/c \), and set \( \hat{R} = (x - y)/R \). Show that
\[
\frac{\partial}{\partial t} \phi(t, x) = \frac{1}{4\pi \epsilon_0} \int \frac{\dot{\rho}(t_{\text{ret}}, y)}{R} d^3y,
\]
where \( \dot{\rho}(t_{\text{ret}}, y) \) is \( \partial \rho(t, y)/\partial t \) evaluated at \( t_{\text{ret}} \). Show further that
\[
\nabla \phi(t, x) = -\frac{1}{4\pi \epsilon_0} \hat{R} \left( \frac{1}{R^2} \rho(t_{\text{ret}}, y) + \frac{1}{cR} \dot{\rho}(t_{\text{ret}}, y) \right) d^3y.
\]
Hence verify, using \( \nabla^2(1/R) = -4\pi \delta^{(3)}(x - y) \), that \( \phi \) satisfies the wave equation
\[
\Box \phi(t, x) = -\frac{1}{\epsilon_0} \rho(t, x).
\]

Write down a similar retarded solution for the vector potential \( A \) in terms of the current density \( J \).

Now assume that \( \rho \) and \( J \) are non-zero only in a finite region. Setting \( \hat{x} = x/|x| \), show that the leading terms in the far-field expansion are
\[
E(t, x) \approx \frac{\mu_0}{4\pi |x|} \int \left( \hat{x} \dot{\rho}(t_{\text{ret}}, y) c - \hat{J}(t_{\text{ret}}, y) \right) d^3y
\]
\[
= \frac{\mu_0}{4\pi |x|} \hat{\dot{x}} \times \left( \hat{x} \times \int \hat{J}(t_{\text{ret}}, y) d^3y \right),
\]
where conservation of current, integration by parts and the discarding of a surface integral has been used, and

\[
B(t, x) \approx -\frac{\mu_0}{4\pi c|x|} \hat{x} \times \int J(t_{\text{ret}}, y) \, d^3y = \frac{1}{c} \hat{x} \times E(t, x).
\]

Note that these results do not assume the dipole approximation. Determine the Poynting vector.

[Hint: When using current conservation, and integration by parts, be careful with the \(y\)-dependence of \(t_{\text{ret}}\).]

5. Starting from the power radiated in the electric-dipole approximation, derive Larmor’s formula for the rate at which radiation is produced by a non-relativistic particle of charge \(q\) moving along a trajectory \(x(t)\).

A non-relativistic particle of mass \(m\), charge \(q\) and energy \(E\) is incident along a radial line in a central potential \(V(r)\). The potential is vanishingly small for \(r\) very large, but increases without bound as \(r \to 0\). Show that the total amount of energy \(\mathcal{E}\) radiated by the particle is

\[
\mathcal{E} = \frac{\mu_0 q^2}{3\pi cm^2} \sqrt{\frac{m}{2}} \int_{r_0}^{\infty} \frac{1}{\sqrt{E - V(r)}} \left( \frac{dV}{dr} \right)^2 \, dr,
\]

where \(V(r_0) = E\), assuming \(\mathcal{E} \ll E\).

Suppose that \(V\) is a Coulomb potential \(C/r\). Evaluate \(\mathcal{E}\).

6. For a relativistic particle of charge \(q\) on a trajectory \(y^\mu(\tau)\), where \(\tau\) is proper time, the current density 4-vector is

\[
J^\mu(x) = qc \int \delta^{(4)}(x - y(\tau)) \dot{y}^\mu(\tau) \, d\tau,
\]

with \(\dot{y}^\mu = -c^2\) and \(\dot{y}^0 > 0\). Show that the 4-vector potential is given by

\[
A^\mu(x) = \frac{\mu_0}{2\pi} \int \Theta(x^0 - z^0) \delta(x^\alpha - z^\alpha)(x^\beta - z^\beta) J^\mu(z) \, d^4z
= -\frac{\mu_0 q c}{4\pi} \frac{\dot{y}^\mu(\tau_*)}{R^\nu(\tau_*) \dot{y}_\nu(\tau_*)},
\]

where \(R^\nu(\tau) = x^\nu - y^\nu(\tau)\) and \(\tau_*\) is determined by \(R^\mu(\tau_*) R_\mu(\tau_*) = 0\) and \(R^0(\tau_*) > 0\).

Verify that the Lorenz gauge condition \(\partial_\mu A^\mu = 0\) holds and show that

\[
F_{\mu\nu} = -\frac{\mu_0 q c}{4\pi} \frac{1}{(R^\rho \dot{y}_\rho)^2} (R_\mu S_\nu - R_\nu S_\mu), \quad \text{where} \quad S_\nu = \dot{y}_\nu - \frac{\dot{y}_\nu}{R^\rho \dot{y}_\rho} (c^2 + R^\tau \dot{y}_\tau)
\]

and all quantities on the right are evaluated at \(\tau_*\). Check this result for the case of a stationary charge at the origin.