

Problems 2.

SI units are used. The signature is  $-+++$ .

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1. An infinite straight wire lies along the  $z$ -axis, and for  $t < 0$  there is no current or field. For  $t \geq 0$  a uniform current  $I$  flows in the wire. Show that for  $t > 0$  the vector potential  $\mathbf{A}(t, x, y) = A\hat{\mathbf{z}}$  in the Lorenz gauge is

$$A = \begin{cases} \frac{\mu_0 I}{2\pi} \ln(\theta + \sqrt{\theta^2 - 1}) & \text{for } \theta > 1, \\ 0 & \text{for } \theta \leq 1, \end{cases}$$

where  $\theta = ct/r$  and  $r = \sqrt{x^2 + y^2}$ . Obtain  $\mathbf{E}$  and  $\mathbf{B}$  and discuss the behaviour of the fields as  $t \rightarrow \infty$ .

2. For a localised charge density  $\rho(\mathbf{x})e^{-i\omega t}$  and current density  $\mathbf{J}(\mathbf{x})e^{-i\omega t}$  use current conservation to show that

$$\int x_i J_j(\mathbf{x}) d^3\mathbf{x} = \epsilon_{ijk} m_k - \frac{1}{6} i\omega Q'_{ij},$$

where

$$\mathbf{m} = \frac{1}{2} \int \mathbf{x} \times \mathbf{J}(\mathbf{x}) d^3\mathbf{x}, \quad Q'_{ij} = 3 \int x_i x_j \rho(\mathbf{x}) d^3\mathbf{x}.$$

Hence show that if  $\int \rho(\mathbf{x}) d^3\mathbf{x} = \int \mathbf{x}\rho(\mathbf{x}) d^3\mathbf{x} = 0$  then at distances  $r \gg c/\omega \gg a$ , where  $a$  is the extent of the charge and current distribution, the leading contributions to the scalar  $\phi(\mathbf{x})e^{-i\omega t}$  and vector potentials  $\mathbf{A}(\mathbf{x})e^{-i\omega t}$  are

$$\phi(\mathbf{x}) \approx -\frac{1}{6} \frac{1}{4\pi\epsilon_0 r} e^{ikr} k^2 \hat{x}_i \hat{x}_j Q'_{ij},$$

and

$$A_i(\mathbf{x}) \approx \frac{\mu_0}{4\pi r} e^{ikr} ik(\hat{\mathbf{x}} \times \mathbf{m})_i - \frac{1}{6} \frac{\mu_0}{4\pi r} e^{ikr} k\omega \hat{x}_j Q'_{ij},$$

where  $r = |\mathbf{x}|$ ,  $\hat{\mathbf{x}} = \mathbf{x}/r$  and  $k = \omega/c$ . Writing  $Q'_{ij} = Q_{ij} + P\delta_{ij}$ , where  $Q_{ii} = 0$ , show that the terms involving  $P$  may be removed by a gauge transformation, at least at large distances. These results represent magnetic dipole and electric quadrupole radiation.

3. A small loop of wire lies in a plane with unit normal  $\hat{\mathbf{N}}$ , and encloses an area  $S$ . A current  $I_0 \cos \omega t$  flows around the loop, with  $c/\omega$  much larger than the size of the loop. Using results from Question 2, show that in the far-field at displacement  $\mathbf{x}$  from the centre of the loop, the magnetic vector potential is

$$\mathbf{A}(t, \mathbf{x}) = \hat{\mathbf{x}} \times \hat{\mathbf{N}} \frac{\mu_0 I_0 S \omega}{4\pi r c} \sin(\omega t - kr) + O\left(\frac{1}{r^2}\right),$$

where  $k = \omega/c$  and  $r = |\mathbf{x}|$ .

[You may use the result  $\oint x_i dx_j = S \epsilon_{ijk} N_k$ .]

Find the leading-order magnetic field in the far-field and show that the average radiated power  $dE/dt$  is

$$\frac{dE}{dt} = \frac{\mu_0}{12\pi} \frac{S^2 I_0^2 \omega^4}{c^3}.$$

4. (Optional, for enthusiasts.) Let  $\phi$  be the retarded scalar potential given by

$$\phi(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(t_{\text{ret}}, \mathbf{y})}{R} d^3\mathbf{y},$$

where  $R = |\mathbf{x} - \mathbf{y}|$ , the retarded time  $t_{\text{ret}} = t - R/c$ , and set  $\hat{\mathbf{R}} = (\mathbf{x} - \mathbf{y})/R$ . Show that

$$\frac{\partial}{\partial t} \phi(t, \mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\dot{\rho}(t_{\text{ret}}, \mathbf{y})}{R} d^3\mathbf{y},$$

where  $\dot{\rho}(t_{\text{ret}}, \mathbf{y})$  is  $\partial\rho(t, \mathbf{y})/\partial t$  evaluated at  $t_{\text{ret}}$ . Show further that

$$\nabla\phi(t, \mathbf{x}) = -\frac{1}{4\pi\epsilon_0} \int \hat{\mathbf{R}} \left( \frac{1}{R^2} \rho(t_{\text{ret}}, \mathbf{y}) + \frac{1}{cR} \dot{\rho}(t_{\text{ret}}, \mathbf{y}) \right) d^3\mathbf{y}.$$

Hence verify, using  $\nabla^2(1/R) = -4\pi\delta^{(3)}(\mathbf{x} - \mathbf{y})$ , that  $\phi$  satisfies the wave equation

$$\square\phi(t, \mathbf{x}) = -\frac{1}{\epsilon_0} \rho(t, \mathbf{x}).$$

Write down a similar retarded solution for the vector potential  $\mathbf{A}$  in terms of the current density  $\mathbf{J}$ .

Now assume that  $\rho$  and  $\mathbf{J}$  are non-zero only in a finite region. Setting  $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ , show that the leading terms in the far-field expansion are

$$\begin{aligned} \mathbf{E}(t, \mathbf{x}) &\approx \frac{\mu_0}{4\pi|\mathbf{x}|} \int \left( \hat{\mathbf{x}} \dot{\rho}(t_{\text{ret}}, \mathbf{y}) c - \dot{\mathbf{J}}(t_{\text{ret}}, \mathbf{y}) \right) d^3\mathbf{y} \\ &= \frac{\mu_0}{4\pi|\mathbf{x}|} \hat{\mathbf{x}} \times \left( \hat{\mathbf{x}} \times \int \dot{\mathbf{J}}(t_{\text{ret}}, \mathbf{y}) d^3\mathbf{y} \right), \end{aligned}$$

where conservation of current, integration by parts and the discarding of a surface integral has been used, and

$$\mathbf{B}(t, \mathbf{x}) \approx -\frac{\mu_0}{4\pi c|\mathbf{x}|} \hat{\mathbf{x}} \times \int \mathbf{j}(t_{\text{ret}}, \mathbf{y}) d^3\mathbf{y} = \frac{1}{c} \hat{\mathbf{x}} \times \mathbf{E}(t, \mathbf{x}).$$

Note that these results do not assume the dipole approximation. Determine the Poynting vector.

[Hint: When using current conservation, and integration by parts, be careful with the  $\mathbf{y}$ -dependence of  $t_{\text{ret}}$ .]

5. Starting from the power radiated in the electric-dipole approximation, derive Larmor's formula for the rate at which radiation is produced by a non-relativistic particle of charge  $q$  moving along a trajectory  $\mathbf{x}(t)$ .

A non-relativistic particle of mass  $m$ , charge  $q$  and energy  $E$  is incident along a radial line in a central potential  $V(r)$ . The potential is vanishingly small for  $r$  very large, but increases without bound as  $r \rightarrow 0$ . Show that the total amount of energy  $\mathcal{E}$  radiated by the particle is

$$\mathcal{E} = \frac{\mu_0 q^2}{3\pi c m^2} \sqrt{\frac{m}{2}} \int_{r_0}^{\infty} \frac{1}{\sqrt{E - V(r)}} \left( \frac{dV}{dr} \right)^2 dr,$$

where  $V(r_0) = E$ , assuming  $\mathcal{E} \ll E$ .

Suppose that  $V$  is a Coulomb potential  $C/r$ . Evaluate  $\mathcal{E}$ .

6. For a relativistic particle of charge  $q$  on a trajectory  $y^\mu(\tau)$ , where  $\tau$  is proper time, the current density 4-vector is

$$J^\mu(x) = qc \int \delta^{(4)}(x - y(\tau)) \dot{y}^\mu(\tau) d\tau,$$

with  $\dot{y}^\mu \dot{y}_\mu = -c^2$  and  $\dot{y}^0 > 0$ . Show that the 4-vector potential is given by

$$\begin{aligned} A^\mu(x) &= \frac{\mu_0}{2\pi} \int \Theta(x^0 - z^0) \delta(\eta_{\alpha\beta}(x^\alpha - z^\alpha)(x^\beta - z^\beta)) J^\mu(z) d^4z \\ &= -\frac{\mu_0 qc}{4\pi} \frac{\dot{y}^\mu(\tau_*)}{R^\nu(\tau_*) \dot{y}_\nu(\tau_*)}, \end{aligned}$$

where  $R^\nu(\tau) = x^\nu - y^\nu(\tau)$  and  $\tau_*$  is determined by  $R^\mu(\tau_*) R_\mu(\tau_*) = 0$  and  $R^0(\tau_*) > 0$ .

Verify that the Lorenz gauge condition  $\partial_\mu A^\mu = 0$  holds and show that

$$F_{\mu\nu} = -\frac{\mu_0 qc}{4\pi} \frac{1}{(R^\rho \dot{y}_\rho)^2} (R_\mu S_\nu - R_\nu S_\mu), \quad \text{where} \quad S_\nu = \ddot{y}_\nu - \frac{\dot{y}_\nu}{R^\rho \dot{y}_\rho} (c^2 + R^\tau \ddot{y}_\tau)$$

and all quantities on the right are evaluated at  $\tau_*$ . Check this result for the case of a stationary charge at the origin.