Mathematical Tripos Part II Prof. M. J. Perry

Electrodynamics Michaelmas 2011

Problems 3.

SI units are used with $c = 1$. The signature is $-+++$. Please send comments/amendments etc. to `malcolm@damtp.cam.ac.uk`

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1. For a localised charge density $\rho(\mathbf{x})e^{-i\omega t}$ and current density $\mathbf{j}(\mathbf{x})e^{-i\omega t}$ use current conservation to show that

$$\int d^3x\, x_i j_j(\mathbf{x}) = \epsilon_{ijk} M_k - i\omega \mathcal{Q}_{ij}$$

where

$$\mathbf{M} = \tfrac{1}{2}\int d^3x\, \mathbf{x} \times \mathbf{j}(\mathbf{x}), \qquad \mathcal{Q}_{ij} = \tfrac{1}{2}\int d^3x\, x_i x_j \rho(\mathbf{x}).$$

Hence show that if $\int d^3x\, \rho(\mathbf{x}) = \int d^3x\, \mathbf{x}\rho(\mathbf{x}) = 0$ then at large distances and for large wavelengths the leading contributions to the scalar $\phi(\mathbf{x})e^{-i\omega t}$ and vector potentials $\mathbf{A}(\mathbf{x})e^{-i\omega t}$ are

$$\phi(\mathbf{x}) \sim -\frac{1}{4\pi\epsilon_0 r} e^{i\omega r}\omega^2 n_i n_j \mathcal{Q}_{ij},$$

and

$$A_i(\mathbf{x}) \sim \frac{\mu_0}{4\pi r} e^{i\omega r} i\omega(\mathbf{n}\times\mathbf{M})_i - \frac{\mu_0}{4\pi r} e^{i\omega r}\omega^2 n_j \mathcal{Q}_{ij},$$

where $r = |\mathbf{x}|$ and $\mathbf{n} = \mathbf{x}/r$. Writing $\mathcal{Q}_{ij} = Q_{ij} + P\delta_{ij}$ where $Q_{ii} = 0$ show that the terms involving P may be removed by a gauge transformation, at least at large distances. These results represent magnetic dipole and electric quadrupole radiation.

2. (2B02421) Derive Larmor's formula for the rate at which radiation is produced by a particle of charge q moving along a trajectory $\mathbf{x}(t)$.

A non-relativistic particle of mass m, charge q and energy E is incident along a radial line in a central potential $V(r)$. The potential is vanishingly small for r very large, but increases without bound as $r \to 0$. Show that the total amount of energy $\mathcal{E}$ radiated by the particle is

$$\mathcal{E} = \frac{\mu_0 q^2}{3\pi m^2}\sqrt{\frac{m}{2}}\int_{r_0}^{\infty} \frac{1}{\sqrt{E - V(r)}}\left(\frac{dV}{dr}\right)^2 dr\,,$$

where $V(r_0) = E$.

Suppose that V is the Coulomb potential $A(r)$. Evaluate $\mathcal{E}$.

3. (2B99421) The electromagnetic field strength tensor can be written in terms of the vector potential A_μ as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,.$$

Explain how this four-vector version is related to the three-vector formulation in terms of a vector potential $\mathbf{A}$ and a scalar potential ϕ.

In a gauge such that $\partial_\mu A^\mu = 0$, derive the field equation for A^μ in the presence of a four-current J^μ.

A small loop of wire lies in a plane with unit normal $\mathbf{n}$, and encloses an area S. A current $I_0 \cos \omega t$ flows around the loop. Show that in the radiation zone at displacement $\mathbf{r}$ from the centre of the loop

$$\mathbf{A}(t, \mathbf{r}) = \mathbf{r} \times \mathbf{n} \, \frac{\mu_0 I_0 S \omega}{4\pi r^2} \, \sin[\omega(t-r)] + O\left(\frac{1}{r^2}\right) .$$

[*You may use the result*

$$\oint dx_i \, x_j = -S\epsilon_{ijk} n_k \, .]$$

4. If V^μ is the 4-velocity of a particle show that

$$\frac{d^2V^\mu}{ds^2} + V^\mu \left(\frac{d^2V^\nu}{ds^2} V_\nu\right)$$

gives zero when contracted with V_μ.

The effects of the electromagnetic radiation which is produced by an accelerating charged particle can be described by including in the non-relativistic equation of motion a damping "self-force" equal to $\alpha\ddot{\mathbf{v}}$, where α is a constant and $\dot{}$ denotes d/dt,

$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \alpha\ddot{\mathbf{v}}.$$

Using the above results find a relativistic equation, which is equivalent to this in the non-relativistic limit, of the form

$$m\frac{dV^\mu}{ds} = f^\mu, \qquad \text{where} \quad f^\mu V_\mu = 0.$$

5. Suppose the current density vanishes, $\mathbf{j} = 0$. Consider the following Lagrangian density for the electromagnetic fields:

$$\mathcal{L} = \tfrac{1}{2}\epsilon_0|\boldsymbol{\nabla}\phi + \dot{\mathbf{A}}|^2 - \frac{1}{2\mu_0}|\boldsymbol{\nabla} \times \mathbf{A}|^2 - \rho\phi,$$

a function of ϕ, $\phi_{,i}$, A_i and $\dot{A}_i$. Obtain the conjugate momenta for ϕ and A_i and the Lagrange equations. Build the Hamiltonian density and show that it is the electromagnetic field energy density.

6. Consider the following action for the electromagnetic field

$$S = \int d^4x \left(-\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\nu \right).$$

Assuming

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu, \tag{1}$$

show that requiring $\delta S = 0$ for arbitrary variations δA_μ which vanish at infinity implies one half of the Maxwell equations

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu. \tag{2}$$

Show also that S is gauge-independent.

Next consider

$$S_P = \int d^4x \left(\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\mu_0} F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + J^\mu A_\nu \right).$$

Show, using (1) that $S_P = S$.

Now forget (1) and regard A_μ and $F^{\mu\nu}$ as independent quantities. Show that requiring $\delta S_P = 0$ for arbitrary variations δA_μ which vanish at infinity implies (2) as before. Show also that requiring $\delta S_P = 0$ for arbitrary variations $\delta F^{\mu\nu}$ implies (1) and hence the other half of the Maxwell equations.

7. A particle of mass m and charge q is subject to a potential $V(\mathbf{x})$ and a magnetic field $\mathbf{B}(\mathbf{x})$ described by a vector potential $\mathbf{A}(\mathbf{x})$. The wave function $\psi(\mathbf{x})$ of the particle satisfies the Schrödinger equation

$$\frac{1}{2m}(-i\hbar\boldsymbol{\nabla} - q\mathbf{A})^2\psi + V\psi = E\psi.$$

Consider the gauge change

$$\mathbf{A}(\mathbf{x}) \to \mathbf{A}'(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \frac{\hbar}{q}\boldsymbol{\nabla}\theta(\mathbf{x}).$$

Show that if the phase of the wave function is changed via

$$\psi(\mathbf{x}) \to \psi'(\mathbf{x}) = e^{i\theta(\mathbf{x})}\psi(\mathbf{x})$$

then the form of the Schrödinger equation is unaltered.

Verify also that the current density

$$\mathbf{j} = \frac{q}{2m}\left(-i\hbar\psi^*\boldsymbol{\nabla}\psi + (i\hbar\boldsymbol{\nabla}\psi^*)\psi - 2q\mathbf{A}\psi^*\psi\right)$$

is invariant under this change.

8. In the Ginzburg-Landau model for superconductivity the energy expressed in terms of the complex field $\psi(\mathbf{x})$ describing the superconducting charge carriers of charge q and mass m in a magnetic field $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ is given by

$$\mathcal{E} = \int d^3x \left(\frac{1}{2\mu_0}|\mathbf{B}|^2 + \frac{1}{2m}\left[(i\hbar\boldsymbol{\nabla} - q\mathbf{A})\psi^*\right]\left[(-i\hbar\boldsymbol{\nabla} - q\mathbf{A})\psi\right] + \alpha\psi^*\psi + \frac{1}{2}\beta(\psi^*\psi)^2\right),$$

where $\beta > 0$. Show that this is invariant under $\psi \to e^{i\theta}\psi$ and $\mathbf{A} \to \mathbf{A} + \hbar\boldsymbol{\nabla}\theta/q$. Setting $\psi = \sqrt{n}e^{i\phi}$ show that minimising $\mathcal{E}$ requires

$$\boldsymbol{\nabla}\times\mathbf{B} = -\mu_0\frac{q^2n}{m}\left(\mathbf{A} - \frac{\hbar}{q}\boldsymbol{\nabla}\phi\right),$$

and that if $\alpha < 0$, $n \approx n_0 = -\alpha/\beta$. Setting $n = n_0$ derive the equation

$$\nabla^2\mathbf{B} - \frac{1}{\lambda^2}\mathbf{B} = 0,$$

and describe briefly the interpretation of λ and the connection of this equation with the Meissner effect where in the interior of a superconductor $\mathbf{A} = \hbar\boldsymbol{\nabla}\phi/q$. How is this expression for $\mathbf{A}$ crucial for flux quantization through a superconducting ring?